Introduction to lattice QCD (1)

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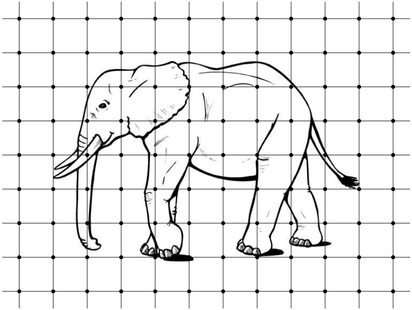


School on Flavour Physics, Benasque, 13.–25. July 2008

Why lattice QCD?

Would like to

- *verify or falsify QCD at low energies*
- *understand quark confinement*
- *be able to compute the basic hadron properties*
- *study exotic forms of matter*

Books

J. Smit, *Introduction to quantum fields on a lattice*,
Cambridge University Press 2002

H.J. Rothe, *Lattice gauge theories*, 3rd edition,
World Scientific 2005

T. DeGrand & C. DeTar, *Lattice methods for Quantum Chromodynamics*,
World Scientific 2006

C. Gattringer & C.B. Lang, *QCD on the lattice — an introduction for beginners*,
Springer Verlag 2009

Euclidean correlation functions

In Minkowski space

$$\langle 0|\phi(x)\phi(0)|0\rangle = \langle 0|\phi(0,\boldsymbol{x})\mathrm{e}^{-iHx_0}\phi(0)|0\rangle$$

Extends to an analytic function for $\mathrm{Im}\, x_0 < 0$ since $H \geq 0$

$\Rightarrow$ for $x_0 > 0$ we may define

$$\langle \phi(x)\phi(0)\rangle = \langle 0|\phi(x)\phi(0)|0\rangle|_{x_0\to -ix_0}$$

$$= \langle 0|\phi(0,\boldsymbol{x})\mathrm{e}^{-Hx_0}\phi(0)|0\rangle$$

Similarly, for ordered times we set

$$\langle \phi(x_1) \ldots \phi(x_n) \rangle =$$

$$\langle 0 | \phi(0, \boldsymbol{x_1}) \mathrm{e}^{-H(x_1 - x_2)_0} \phi(0, \boldsymbol{x_2}) \ldots \mathrm{e}^{-H(x_{n-1} - x_n)_0} \phi(0, \boldsymbol{x_n}) | 0 \rangle$$

Theorem

The euclidean n-point functions are real-analytic functions in $x_1, \ldots, x_n$ with power-singularities at coinciding points.

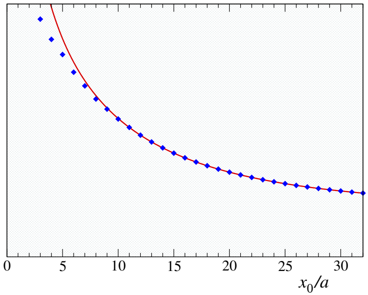
Pauli, Jost, Streater & Wightman, . . .

$\Rightarrow$ take these to be the primary objects in LQCD

Example: pion 2-point function

$$G(x_0) = \int \mathrm{d}^3\boldsymbol{x}\, \langle (\bar{u}\gamma_5 d)(x)(\bar{d}\gamma_5 u)(0)\rangle$$

On the lattice, $G(x_0)$ is obtained at $x_0 = a, 2a, 3a, \ldots$



Large-time behaviour

$$\int \mathrm{d}^3\boldsymbol{x}\,\langle(\bar{u}\gamma_5 d)(x)(\bar{d}\gamma_5 u)(0)\rangle = -\mathrm{e}^{-M_\pi x_0}\,|\langle 0|\bar{u}\gamma_5 d|\pi\rangle|^2 + \mathrm{O}(\mathrm{e}^{-3M_\pi x_0})$$

$\Rightarrow$ the computation of

- *hadron masses*
- *simple hadronic matrix elements*
- ...

does not require an analytic continuation back to Minkowski space!

Lattice quark fields

Euclidean free-quark two-point function

$$\langle \psi(x)\overline{\psi}(0)\rangle = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\mathrm{e}^{ipx}}{i\gamma p + m}$$

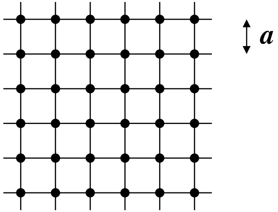
$$px = p_0 x_0 + \boldsymbol{p}\boldsymbol{x}, \qquad \gamma p = \gamma_0 p_0 + \boldsymbol{\gamma}\boldsymbol{p}$$

$$\gamma_\mu^\dagger = \gamma_\mu, \qquad \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$$

This is also the Green function of the Dirac operator

$$(\gamma\partial + m)\langle \psi(x)\overline{\psi}(0)\rangle = \delta(x)$$

Now replace space-time by a 4–dimensional hypercubic lattice



Dirac field

$$\psi(x), \qquad x = a\,(n_0, n_1, n_3, n_4)\,, \qquad n_\mu \in \mathbb{Z}$$

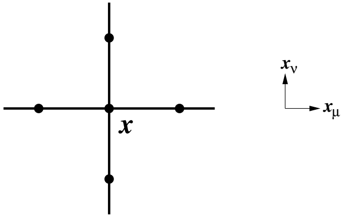
$$\widetilde{\psi}(p) = a^4 \sum_x \mathrm{e}^{-ipx}\psi(x) \quad \Leftrightarrow \quad \psi(x) = \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}^4 p}{(2\pi)^4}\,\mathrm{e}^{ipx}\,\widetilde{\psi}(p)$$

$\Rightarrow$ the lattice implies a momentum cutoff $|p_\mu| \leq \pi/a$

Forward & backward lattice "derivatives"

$$\partial_\mu \psi(x) = \left\{\psi(x + a\hat{\mu}) - \psi(x)\right\}/a$$

$$\partial^*_\mu \psi(x) = \left\{\psi(x) - \psi(x - a\hat{\mu})\right\}/a$$



In momentum space

$$\partial_\mu \to \frac{1}{a}\left\{\mathrm{e}^{iap_\mu} - 1\right\} = ip_\mu \left\{1 + \mathrm{O}(ap)\right\}$$

$$\tfrac{1}{2}(\partial^*_\mu + \partial_\mu) \to \frac{i}{a}\sin(ap_\mu) \equiv i\mathring{p}_\mu$$

$$\partial^*_\mu \partial_\mu \to -\hat{p}_\mu \hat{p}_\mu, \qquad \hat{p}_\mu \equiv \frac{2}{a}\sin\left(\frac{ap_\mu}{2}\right)$$

Wilson–Dirac operator

$$D_{\rm w} = \sum_{\mu=0}^{3} \tfrac{1}{2}\left\{\gamma_\mu(\partial_\mu^* + \partial_\mu) - a\partial_\mu^*\partial_\mu\right\}$$

$$\rightarrow i\gamma\mathring{p} + \tfrac{1}{2}a\hat{p}^2$$

$\Rightarrow$ free-quark two-point function on the lattice

$$(D_{\rm w} + m)\langle\psi(x)\overline{\psi}(0)\rangle = a^{-4}\delta_{x0}$$

$$\langle\psi(x)\overline{\psi}(0)\rangle = \int_{-\pi/a}^{\pi/a} \frac{{\rm d}^4p}{(2\pi)^4}\,\frac{{\rm e}^{ipx}}{i\gamma\mathring{p} + \frac{1}{2}a\hat{p}^2 + m}$$

Källén–Lehmann representation

In the complex p_0–plane, the integrand

$$\frac{1}{i\gamma\mathring{p} + \frac{1}{2}a\hat{p}^2 + m} = \frac{-i\gamma\mathring{p} + \frac{1}{2}a\hat{p}^2 + m}{\mathring{p}^2 + \left(\frac{1}{2}a\hat{p}^2 + m\right)^2}$$

has simple poles at $p_0 = \pm i\epsilon_{\boldsymbol{p}}$, where

$$\epsilon_{\boldsymbol{p}} = \frac{2}{a}\,\mathrm{asinh}\left\{\frac{a}{2}\sqrt{\frac{\mathring{\boldsymbol{p}}^2 + m_{\boldsymbol{p}}^2}{1 + am_{\boldsymbol{p}}}}\right\}$$

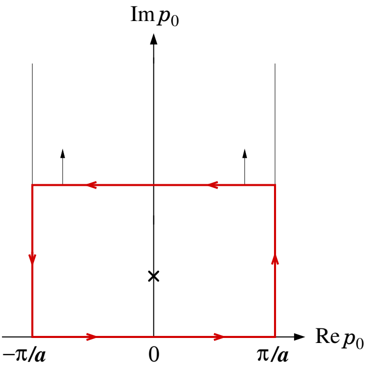
$$m_{\boldsymbol{p}} \equiv m + \tfrac{1}{2}a\hat{\boldsymbol{p}}^2$$

For $x_0 > 0$

$$\langle \psi(x)\overline{\psi}(0)\rangle$$

$$= \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\mathrm{e}^{ipx}}{i\gamma\mathring{p} + \frac{1}{2}a\hat{p}^2 + m}$$

$$= \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}^3 \boldsymbol{p}}{(2\pi)^3} \mathrm{e}^{-\epsilon_{\boldsymbol{p}} x_0 + i\boldsymbol{p}\boldsymbol{x}} \varrho_{\boldsymbol{p}}$$



$\Rightarrow$ $\epsilon_{\boldsymbol{p}}$ = energy of a lattice quark with 3-momentum $\boldsymbol{p}$

$\varrho_{\boldsymbol{p}}$ = associated spectral density

Continuum limit

The lattice spacing only sets the scale since

$$F(a, m, p, \ldots) = a^{d_F} F(1, am, ap, \ldots)$$

$\Rightarrow$ the continuum limit amounts to taking

$$m \ll 1/a, \quad p \ll 1/a, \quad |x| \gg a, \quad \ldots$$

Examples

$$\epsilon_{\boldsymbol{p}} = \sqrt{m^2 + \boldsymbol{p}^2} + \mathrm{O}(am, a\boldsymbol{p})$$

$$\varrho_{\boldsymbol{p}} = \left. \frac{i\gamma p - m}{2ip_0} \right|_{p_0 = i\sqrt{m^2 + \boldsymbol{p}^2}} + \mathrm{O}(am, a\boldsymbol{p})$$

At fixed x, the two-point function $\langle\psi(x)\overline{\psi}(0)\rangle$ converges too

$\Rightarrow$ the lattice theory has the expected continuum limit

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Remarks

- The size of the lattice effects also depends on the dynamical scales
- In presence of interactions, the couplings, etc., must be renormalized as $a \to 0$

Why do we need the Wilson term?

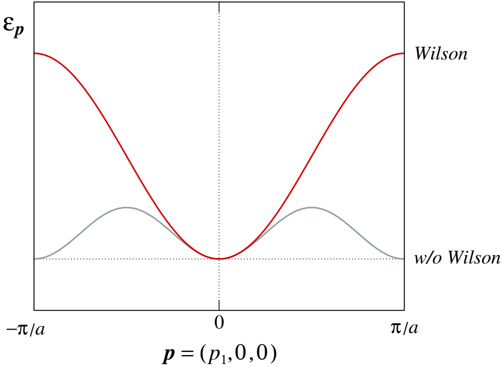
Recall

$$D_{\mathrm{w}} = \sum_{\mu=0}^{3} \tfrac{1}{2} \left\{ \gamma_\mu (\partial_\mu^* + \partial_\mu) - a \partial_\mu^* \partial_\mu \right\}$$

The Wilson term

- *is "irrelevant" in the continuum limit*
- *but breaks chiral symmetry at* $\mathrm{O}(a)$
- *which, in practice, complicates the situation*

However



$\Rightarrow$ *w/o Wilson term there are additional states with energy* $\ll \pi/a$

$\Rightarrow$ *wrong continuum limit!*