Lectures on Effective Field Theory

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Problem 1

Consider fermionic atoms with a single spin state. Write down the most general Galilean-invariant effective lagrangian that can contribute to scattering through 2nd order in the momentum expansion. Use the anticommuting character of the fermionic field $\psi(\vec{r}, t)$ to show that the only 2-body interaction term at this order that does not vanish has the form

$$\mathcal{L}_{\rm int} = -\frac{g_P}{4m}\, \psi^\dagger \overleftrightarrow{\nabla} \psi^\dagger \cdot \psi \overleftrightarrow{\nabla} \psi,$$

where $\overleftrightarrow{\nabla} \equiv \overrightarrow{\nabla} - \overleftarrow{\nabla}$. Show that the resulting T-matrix element for scattering of atoms with momenta $+\vec{p}$ and $-\vec{p}$ into atoms with momenta $+\vec{p}\,'$ and $-\vec{p}\,'$ is

$$\mathcal{T} = \frac{4g_P}{m}\, \vec{p} \cdot \vec{p}\,'.$$

Show that this interaction term gives rise to P-wave scattering with differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{g_P^2}{\pi^2}\, p^4 \cos^2\theta.$$

Problem 2

Consider bosonic atoms with a single spin state. The most general Galilean-invariant effective lagrangian, including all terms that can contribute to scattering through 2nd order in the momentum expansion, is $\mathcal{L}_1 + \mathcal{L}_2$, where

$$\begin{aligned}
\mathcal{L}_1 &= \tfrac{1}{2}\left[\psi^\dagger \left(i\tfrac{\partial}{\partial t} + \nabla^2/2m\right)\psi + \text{h.c.}\right], \\
\mathcal{L}_2 &= -\frac{g_0}{4m}\,\psi^\dagger\psi^\dagger\psi\psi \\
&\quad + \frac{h_0}{4m}\left[\psi^\dagger\psi^\dagger\psi\,\overleftrightarrow{\nabla}^{\,2}\psi + \text{h.c.}\right] \\
&\quad - h_0'\left[\psi^\dagger\psi^\dagger\psi\left(i\tfrac{\partial}{\partial t} + \nabla^2/2m\right)\psi + \text{h.c.}\right].
\end{aligned}$$

The tree-level amplitude for the scattering of atoms with energy-momenta $(E_1, \vec{p}_1)$ and $E_2, \vec{p}_2)$ into atoms with energy-momenta $(E_1', \vec{p}_1{}')$ and $E_2', \vec{p}_2{}')$ satisfying energy-momentum conservation is

$$\begin{aligned}
&-i\frac{g_0}{m} - i\frac{h_0}{m}\left[(\vec{p}_1 - \vec{p}_2)^2 + (\vec{p}_1{}' - \vec{p}_2{}')^2\right] \\
&-ih_0'\left[(E_1 - p_1^2/2m) + (E_2 - p_2^2/2m) + (E_1' - p_1'^2/2m) + (E_2' - p_2'^2/2m)\right].
\end{aligned}$$

The T-matrix element for this scattering process is obtained by setting the external lines on their energy shells.

A. Note that the h_0' term does not contribute to the T-matrix element. Identify the field redefinition that can eliminate the h_0' term.

B. Calculate the one-loop amplitude for the propagation of a pair of atoms between successive contact interactions with g_0 vertices. Evaluate the diagram in the center-of-momentum frame with an ultraviolet momentum cutoff Λ. Use Galilean invariance to extend the result to an arbitrary frame in which the pair of atoms have total energy-momentum $(E, \vec{P})$:

$$-i\frac{m}{4\pi^2}\left(\Lambda - \frac{\pi}{2}\sqrt{-m(E - P^2/(4m) + i\epsilon)}\right).$$

C. Calculate the T-matrix element for the scattering of atoms with momenta $+\vec{p}$ and $-\vec{p}$ into atoms with momenta $+\vec{p}\,'$ and $-\vec{p}\,'$ to second order in the momentum expansion. Include the tree diagrams, the one-loop diagram with two g_0 vertices, and the two-loop diagram with three g_0 vertices.

D. The only contributions to the T-matrix element through 2nd order in the momentum expansion come from the S-wave channel. Unitarity requires the S-wave contribution to have the form

$$\mathcal{T} = \frac{8\pi}{m} \frac{1}{p \cot \delta(p) - ip},$$

where $\delta(p)$ is the S-wave phase shift. The low-momentum expansion of the S-wave phase shift is conventionally expressed in the form

$$p \cot \delta(p) = -1/a + \tfrac{1}{2} r_s p^2 + \ldots,$$

where a is the scattering length and r_s is the effective range. Expand the S-wave T-matrix element to second order in p. Show that the T-matrix element calculated from diagrams is consistent with this expansion. Determine the scattering length to 3rd order in g_0 and the effective range to leading order:

$$\begin{aligned} a &= \frac{g_0}{8\pi}\left[1 - \frac{g_0\Lambda}{4\pi^2} + \left(\frac{g_0\Lambda}{4\pi^2}\right)^2\right], \\ r_s &= \frac{128\pi h_0}{g_0^2}. \end{aligned}$$