Introduction to lattice QCD: Exercises

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Lecture 1

1.1 Let ∂_μ and ∂_μ^* be the forward and backward lattice derivatives. Show that the generalized Leibniz rules

$$\partial_\mu\{f(x)g(x)\} = \partial_\mu f(x)g(x) + f(x)\partial_\mu g(x) + a\partial_\mu f(x)\partial_\mu g(x), \tag{1.1}$$

$$\partial_\mu^*\{f(x)g(x)\} = \partial_\mu^* f(x)g(x) + f(x)\partial_\mu^* g(x) + a\partial_\mu^* f(x)\partial_\mu^* g(x), \tag{1.2}$$

hold for any lattice functions f and g.

1.2 Verify the identities

$$[\partial_\mu, \partial_\nu] = [\partial_\mu, \partial_\nu^*] = [\partial_\mu^*, \partial_\nu^*] = 0, \tag{1.3}$$

$$a^4 \sum_x f(x)\partial_\mu g(x) = -a^4 \sum_x (\partial_\mu^* f)(x)g(x). \tag{1.4}$$

1.3 Derive an explicit expression for the spectral density $\varrho(\boldsymbol{p})$ of the free-quark two-point function and show that

$$\xi^\dagger \varrho(\boldsymbol{p})\gamma_0\xi \geq 0 \tag{1.5}$$

for all constant Dirac spinors ξ (together with the fact that the one-particle energies $\epsilon(\boldsymbol{p})$ are real, this property guarantees the unitarity of the free-quark theory).

Lecture 2

2.1 Work out the expansion

$$P_{\mu\nu}(x) = -\tfrac{1}{2}a^4\,\mathrm{tr}\,\{F_{\mu\nu}(x)F_{\mu\nu}(x)\}$$
$$\qquad -\tfrac{1}{2}a^5\,\mathrm{tr}\,\{F_{\mu\nu}(x)(D_\mu+D_\nu)F_{\mu\nu}(x)\}+\ldots \tag{2.6}$$

of the plaquette field $P_{\mu\nu}(x)$ in the classical continuum limit.

2.2 Find a local, gauge-invariant field that represents the topological density

$$q(x) = -\frac{1}{16\pi^2}\sum_{\mu,\nu,\rho,\sigma}\epsilon_{\mu\nu\rho\sigma}\,\mathrm{tr}\,\{F_{\mu\nu}(x)F_{\rho\sigma}(x)\} \tag{2.7}$$

on the lattice.

2.3 Consider the free-quark theory on a lattice of size $T\times L^3$. Impose the boundary conditions

$$\psi(x+T\hat{0}) = -\psi(x), \qquad \psi(x+L\hat{k}) = \psi(x) \quad (k=1,2,3), \tag{2.8}$$

$$\overline{\psi}(x+T\hat{0}) = -\overline{\psi}(x), \qquad \overline{\psi}(x+L\hat{k}) = \overline{\psi}(x) \quad (k=1,2,3), \tag{2.9}$$

on the quark fields and write down the quark propagator for this case.

Lecture 3

3.1 Consider the "correlation functions"

$$\langle c_{k_1} \dots c_{k_m} \bar{c}_{l_1} \dots \bar{c}_{l_m} \rangle_{\rm F} = \frac{1}{\mathcal{Z}_{\rm F}} \int {\rm D}[c]{\rm D}[\bar{c}]\, c_{k_1} \dots \bar{c}_{l_m} \exp\Bigl\{-\sum_{i,j} \bar{c}_i A_{ij} c_j\Bigr\} \qquad (3.10)$$

of the Grassmann variables $c_1, \dots, c_n, \bar{c}_1, \dots, \bar{c}_n$. Prove that

$$\langle c_k \bar{c}_l \rangle_{\rm F} = (A^{-1})_{kl} \qquad (3.11)$$

and more generally

$$\langle c_{k_1} \dots c_{k_m} \bar{c}_{l_1} \dots \bar{c}_{l_m} \rangle_{\rm F} = (-1)^{\frac{1}{2}m(m-1)} \det B, \qquad (3.12)$$

$$B_{ij} = (A^{-1})_{k_i l_j} \quad (i,j = 1, \dots m). \qquad (3.13)$$

3.2 Show that

$$\int_{{\rm SU}(3)} {\rm d}U\, U_{\alpha\beta} = \int_{{\rm SU}(3)} {\rm d}U\, U_{\alpha\beta} U_{\gamma\delta} = 0, \qquad (3.14)$$

$$\int_{{\rm SU}(3)} {\rm d}U\, U_{\alpha\beta} (U^{-1})_{\gamma\delta} = \tfrac{1}{3} \delta_{\alpha\delta} \delta_{\beta\gamma}, \qquad (3.15)$$

$$\int_{{\rm SU}(3)} {\rm d}U\, U_{\alpha\beta} U_{\gamma\delta} U_{\rho\sigma} = \tfrac{1}{6} \epsilon_{\alpha\gamma\rho} \epsilon_{\beta\delta\sigma}, \qquad (3.16)$$

assuming the invariant measure ${\rm d}U$ is normalized such that $\int_{{\rm SU}(3)} {\rm d}U = 1$.

3.3 Prove the PCAC relation

$$\sum_\mu \tfrac{1}{2}(\partial_\mu^* + \partial_\mu) \langle A_\mu(x) X(y) \rangle = (m_u + m_d) \langle P(x) X(y) \rangle + {\rm O}(a), \qquad (3.17)$$

$$A_\mu = \bar{u} \gamma_\mu \gamma_5 d, \quad P = \bar{u} \gamma_5 d, \quad X: \text{ any local lattice field}, \qquad (3.18)$$

in the free-quark theory at non-zero distances $|x - y|$. *Hint*: stay in position space and use the Leibniz rule for lattice functions.

Lecture 4

4.1 Show that the leading contribution to the expectation value of a $T \times L$ Wilson loop in the pure gauge theory is proportional to $(1/g_0^2)^{TL/a^2}$ when $g_0 \to \infty$.

4.2 A direct computation of the pion σ-term requires the three-point function

$$\langle P(x) S_0(y) \overline{P}(z) \rangle, \quad P = \bar{u}\gamma_5 d, \quad \overline{P} = \bar{d}\gamma_5 u, \quad S_0 = \bar{u}u + \bar{d}d \tag{4.19}$$

to be calculated. Write down the quark-line diagrams that contribute to this correlation function.

4.3 On the lattice the Fourier transform of the gauge potential $A_\mu(x)$ is defined by

$$\widetilde{A}_\mu(p) = a^4 \sum_x \mathrm{e}^{-i(px + \frac{1}{2}ap_\mu)} A_\mu(x) \tag{4.20}$$

(the extra phase $\frac{1}{2}ap_\mu$ is conventional and leads to some notational simplifications). Show that the gluon propagator is given by

$$\langle \widetilde{A}_\mu^a(p) \widetilde{A}_\nu^b(q) \rangle = (2\pi)^4 \delta_{\mathrm{P}}(p+q) \frac{\delta^{ab}}{\hat{p}^2} \Big\{ \delta_{\mu\nu} - (1 - \lambda_0^{-1}) \frac{\hat{p}_\mu \hat{p}_\nu}{\hat{p}^2} \Big\}, \tag{4.21}$$

$$\hat{p}_\mu = \frac{2}{a} \sin\Big(\frac{1}{2} a p_\mu\Big), \tag{4.22}$$

where $\delta_{\mathrm{P}}(k)$ denotes the 4-dimensional periodic Dirac δ-function with period $2\pi/a$.

4.4 Derive the asymptotic form

$$J(p) \underset{a \to 0}{=} -\frac{1}{16\pi^2} \ln(a^2 p^2) + c + \mathrm{O}(a^2), \qquad c = 0.0366\ldots, \tag{4.23}$$

of the scalar one-loop integral

$$J(p) = \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{\hat{k}_+^2 \hat{k}_-^2}, \quad k_\pm = k \pm \tfrac{1}{2} p. \tag{4.24}$$