# E. Braaten, Introd. to eff. field theories (EFT)

## Lecture I

Universal language for high energy physics
Also model independent ... beyond SM

Multiple scales:

low momentum scales $\lesssim p_{LO}$
high momentum scales $\gtrsim \Lambda_{HI}$

Separation of scales: $p_{LO} \ll \Lambda_{HI}$

EFT:
- describe low-momentum d.o.f.
- omit high- " "
  take into account through parameters
- systematically improvable
- error estimates $\sim \left(\frac{p_{LO}}{\Lambda_{HI}}\right)^N$

Non-relativistic particles with short range interactions

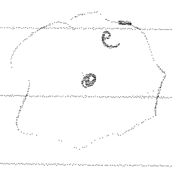
1. low-energy hadrons
   fundamental theory: QCD
   $p_{LO} \sim m_\pi \sim 140$ MeV
   $E_{LO} \sim \frac{m_\pi^2}{M} \sim 20$ MeV for n, p

2. ultracold atoms
   trapped with magnetic fields, or laser beams

Bose-Einstein condensates, super fluids

Alkali atoms: Li, Na, K, Rb, Cs

look like an H-atom




spin $s = \frac{1}{2}$

nucleus ..., spin-state

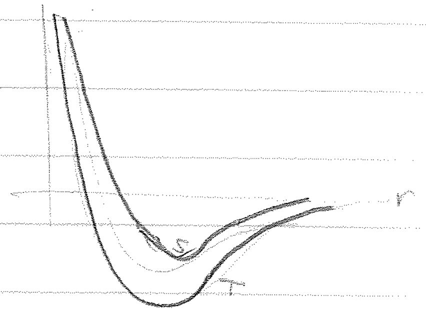
We know the fundamental theory, but can't solve it.

1-body terms

hyperfine: $\vec{I} \cdot \vec{S}$

magn. moment: $-\vec{\mu} \cdot \vec{B}$

2-body terms:




Born-Oppenheimer potentials

$V_T(r), V_S(r) \to -\frac{C_6}{r^6}$

Energy scales:

electronic $\sim Ry \simeq \frac{\hbar^2}{m_e a_0^2} \sim 10$ eV

Spin excitation: $E_{hf} \sim 10^{-3}$ eV

vibrational: $\frac{\hbar^3}{M} (m C_6)^{-1/2} \sim 10^{-6}$ eV

Ultracold: $\mu K \sim 10^{-10}$ eV $\sim \frac{p_{Lo}^2}{M}$

Applying EFT to uc atoms we can treat the atoms as indivisable, no substructure

Non-relativistic particles: fermions or bosons

quantum field: $\psi(\vec{r},t)$ annihilates atoms
$\psi^{+}(\vec{r},t)$ creates atoms

Lagrangian: $\mathcal{L} = \frac{1}{2}\left[\psi^{+} i \frac{d}{dt}\psi + h.c.\right] - \mathcal{H}$

Hamiltonian: $\mathcal{H} = \frac{1}{2m} \vec{\nabla}\psi^{+} \cdot \vec{\nabla}\psi$

Eq. of motion: $\left(i\frac{\partial}{\partial t} + \frac{1}{2m}\Delta\right)\psi = 0$

$$\psi(\vec{r},t) = e^{-iEt + i\vec{p}\cdot\vec{r}} \Rightarrow E = \frac{p^2}{2m}$$

EFT:

(1) identify low-energy d.o.f.
(2) identify the symmetries
(3) construct most general theory consistent with symmetries $\Rightarrow \infty$ many parameters
(4) determine relative importance of terms; "power-counting"
(5) choose desired accuracy: $\left(\frac{P_{LO}}{\Lambda_{HI}}\right)^N$
(6) determine parameters: "matching"

(1) Atoms
Non-rel point particles represented by quant. field $\psi$.

(2) (a) $U(1)$ phase symmetry $\psi(\vec{r},t) \to e^{i\theta}\psi(\vec{r},t)$
$\Rightarrow$ conservation of atoms

(b) Gallilean symmetry:

$$\psi(\vec{r},t) \to e^{iM\vec{v}\cdot\vec{r} - \frac{i}{2}Mv^2 t}\,\psi(\vec{r}-\vec{v}t, t)$$

(c) Translational symmetry
(d) Translation of time

$$\left.\begin{array}{l}\text{(c) Translational symmetry}\\ \text{(d) Translation of time}\end{array}\right\} L = \int d^3x \int dt\, \mathcal{L}$$

(e) Rotation symmetry $\to$ $\mathcal{L}(\psi, \frac{\partial}{\partial t}\psi, \nabla\psi, \dots)$

(f) Parity

(g) Time reversal

3

(e) : tensor analysis $\delta_{ij}$ $\cancel{\epsilon_{ijk}}$

(f) : $\psi \to \psi^*$
$t \to -t$

(a): Same number of $\psi$ and $\psi^+$

(b): $\left(i\frac{\partial}{\partial t} - \frac{1}{2M}\nabla^2\right)\psi$ transforms like $\psi$

$\left(i\frac{\partial}{\partial t} - \frac{1}{4M}\nabla^2\right)\psi_1\psi_2$ like $\psi_1\psi_2$

$\psi_1 \overleftrightarrow{\nabla} \psi_2 = \psi_1 \vec{\nabla}\psi_2 - \psi_2\vec{\nabla}\psi_1$ like "

$\psi_1 \overleftrightarrow{\nabla}_1 \overleftrightarrow{\nabla}_2 \psi_2$ like "

4 Use free field theory as a guide

$$\psi(\vec{r},t) = e^{-iEt + i\vec{p}\cdot\vec{r}} \to E = \frac{p^2}{2M} \qquad i\frac{\partial}{\partial t} \sim E \sim \frac{p_0^2}{M}$$

$\psi^\dagger \psi \sim p_{Lo}^3$

General operator: $\left(M i \frac{\partial}{\partial t}\right)^l \nabla^m \psi^{\dagger n} \psi^n \sim p_{Lo}^{2l+m+3n}$

Appearing in the $\mathcal{L}$:

$$C_{lmn} \left(M i \frac{\partial}{\partial t}\right)^l \nabla^m \psi^{\dagger n} \psi^n$$

$$\Rightarrow \quad C_{lmn} \sim \frac{1}{M} \left(\frac{1}{\Lambda_{H}}\right)^{2l+m+3n-5}$$

$\Rightarrow$ effects are suppressed by $\left(\frac{p_{Lo}}{\Lambda_H}\right)^{2l+m+3n}$

Leading terms:

chemical potential

$d=3 : -(\mu)\psi^\dagger\psi$

$d=5 : \frac{1}{2}\psi^\dagger \left(i\frac{\partial}{\partial t} - \frac{\nabla^2}{2M}\right)\psi + \text{h.c.}$

$$\left.\begin{aligned} d=6 &: \psi^\dagger\psi^\dagger\psi\psi \\ d=7 &: \psi^\dagger\left(i\frac{\partial}{\partial t}+\nabla^2\right)^2\psi \\ d=8 &: \psi^\dagger \overleftrightarrow{\nabla} \psi^\dagger \cdot \psi \overleftrightarrow{\nabla} \psi = 0 \\ &\quad \psi^\dagger\psi^\dagger\psi \overleftrightarrow{\nabla}^2 \psi + \text{h.c.} \\ &\quad \ldots \end{aligned}\right\}$$

for bosonic field

perturbative

# Lecture II

## EFT for "weakly-interacting" atoms

high-mom. scales $\gtrsim \Lambda_{HI} \sim \frac{\hbar}{100\, a_0}$

low-mom. scales $\lesssim p_{LO}$

low-mom.: $p \ll \Lambda_{HI}$
low-temp.: $T \ll \frac{\Lambda_{HI}^2}{m} \sim 10^{-2}\,K$
low-density: $n \ll \frac{1}{\Lambda_{HI}^3} \sim 10^6 / (\mu m)^3$

Recipe to construct: (yesterday)

(3) Most general Lagr.:

$$\mathcal{L} = \frac{1}{2}\psi^\dagger i\frac{\partial}{\partial t}\psi + h.c. - \frac{1}{2m}\nabla\psi^\dagger\nabla\psi - \mu\,\psi^\dagger\psi + \sum_n c_n\, O_n$$

(4) Relative importance:

Dimension of operators: ("power counting dimension")

| | |
|---|---|
| $\nabla$ | 1 |
| $i\frac{\partial}{\partial t}$ | 2 |
| $\psi$ | $3/2$ |
| $O_n$ | $d_n$ |

$m$ has no dim in this power counting scheme. Just a conversion factor. $E \sim \frac{p^2}{m}$

Generally*: $c_n \sim \frac{1}{m}\left(\frac{1}{\Lambda_{HI}}\right)^{d_n - 5}$

$\Rightarrow$ Effects of $O_n$ are suppressed by $\left(\frac{p_{LO}}{\Lambda_{HI}}\right)^{d_n-5}$

* in absence of fine tuning

- perturbative: all interaction terms suppressed by powers of $\frac{p_{LO}}{\Lambda_{HI}}$

- theory that we get is renormalizable order by order in $p_{LO}/\Lambda_{HI}$

(5) choose desired accuracy to have errors $< \mathcal{O}\left(\left(\frac{p_{LO}}{\Lambda_{HI}}\right)^N\right)$

keep terms with $d_n - 5 \leq N$

$\Rightarrow$ finite # of parameters

(6) Simplify $\mathcal{L}_{eff}$:
Omit terms that (a) are derivatives* $\frac{\partial}{\partial t}(\ldots)$
$\nabla_i(\ldots)$

* be careful with topology

(b) can be removed by field redefinitions

$$\psi \to \psi + \underbrace{\Psi(\psi, \psi^+, i\frac{\partial}{\partial t}\psi, \nabla_i\psi, \ldots)}_{\text{transforms like } \psi}$$

change of integration variables.

$$\int D\psi\, e^{i\int dt \int d^3r\, \mathcal{L}}$$

Changes green's funct.
But leaves physical quant. invariant.
(e.g. S-Matrix elements)

$\rightarrow$ Can omit terms that vanish when using free field eq. of m.*

* $\mathcal{L}_{eff} = \frac{1}{2}\psi^\dagger\left[i\frac{\partial}{\partial t} - \frac{1}{2m}\nabla^2 - \mu\right]\psi + h.c. + \sum_n c_n O_n$

→ Free eq. of m. : $\left(i\partial_t + \frac{1}{2m}\Delta - \mu\right)\psi = 0$

$d = 6$ : $\psi^\dagger\psi^\dagger\psi\psi$

7 : $\cancel{\psi^\dagger\left(i\partial_t + \frac{1}{2m}\Delta - \mu\right)^2\psi}$

8 : $\psi^\dagger\overleftrightarrow{\nabla}\psi^\dagger \cdot \psi\overleftrightarrow{\nabla}\psi$

$\cancel{\psi^\dagger\psi^\dagger\psi\left(i\partial_t + \frac{1}{2m}\Delta - \mu\right)\psi}$

$\psi^\dagger\psi^\dagger\psi\overleftrightarrow{\nabla}^2\psi$

9 : $\psi^\dagger\psi^\dagger\psi^\dagger\psi\psi\psi$

(7) Determine parameters

Matching : Calculate same quantity
in fundamental theory
in effective "

Choose param. so they agree.

$2 \to 2$ scattering:

- Rotational symmetry! ⇒ partial wave decomposition

$$T(k) = \frac{8\pi}{m}\sum_{L=0}^{\infty}(2L+1)\,T_L(k)\,P_L(\cos\Theta)$$

T-Matrix

- Unitarity: $T_{L=0}(k) = \frac{1}{k\cot\delta(k) - ik}$

- Low-mom. expansion: $k\cot\delta(k) = -\frac{1}{a} + \frac{1}{2}r_s k^2 + \ldots$

$a$ : scattering length

$r_s$ : effective range

$$\mathcal{L}_{eff} = \frac{1}{2}\left[\psi^* \left(i\partial_t - \frac{1}{2m}\Delta - \mu\right)\psi + h.c.\right]$$

$$- \frac{4\pi a}{m} \psi^*\psi^*\psi\psi \quad + \text{higher dim. operators}$$

Weakly interacting Bose gas:

Number density: $\eta$

Temperature: $T = 0$

~~Bosons~~:

$a > 0$: stable

Bose-Einstein condensate

$\langle\psi\rangle \neq 0 \Rightarrow$ SSB of $U(1)$ sym.

$a < 0$: unstable $\rightarrow$ goldstone boson

Fermions:

1 spin-state: non-interacting Fermi gas

2 spin-state: $a > 0$: Fermi gas

$a < 0$: Cooper pairing

superfluidity

$\langle\psi_1\psi_2\rangle \neq 0 \Rightarrow$ SSB of $U(1)$

goldstone bosons

# Lecture III

~~$\mathcal{L}_{eff} = \mathcal{L}_{free} + \mathcal{L}_{int}$~~

$$\mathcal{L}_{free} = \frac{1}{2}\left[\psi^\dagger i \frac{\partial}{\partial t}\psi + h.c.\right] - \frac{1}{2m}\nabla\psi^\dagger\nabla\psi - \mu\psi^\dagger\psi$$

$$\mathcal{L}_{int} = -\frac{g}{m}\left(\psi^\dagger\psi\right)^2 - \frac{h}{m}\left(\psi^\dagger\psi^\dagger\psi\overleftrightarrow{\nabla}\psi + h.c.\right)$$

$$-\frac{g_3}{m}\left(\psi^\dagger\psi\right)^3 + \ldots$$

Power counting: $g \sim \frac{1}{\Lambda_{HI}}$

$$h \sim \frac{1}{\Lambda_{HI}^3}$$

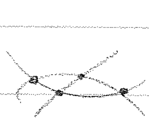
$$g_3 \sim \frac{1}{\Lambda_{HI}^4}.$$

Perturbative: interactions suppressed by powers of $\frac{p_{LO}}{\Lambda_{HI}}$

Renormalizable order by order: in $\frac{p_{LO}}{\Lambda_{HI}}$

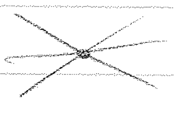
$$\overline{\quad(\omega, \vec{k})\quad} \sim \frac{i}{\omega - k^2/2m - \mu + i\epsilon} \sim \frac{1}{k^2}$$

Loops: $\int \frac{d\omega}{2\pi}\int\frac{d^3k}{2\pi} \sim \int d^5k$

E.g.  $\sim g^4 \int \frac{(d^5k)^2}{(k^2)^5} \sim \int \frac{dk}{k}$

2 loops

$\Rightarrow$ log UV divergence



Now, take only

$$\mathcal{L}_{int} = -\frac{2\pi a}{m} (\psi^+ \psi)^2$$

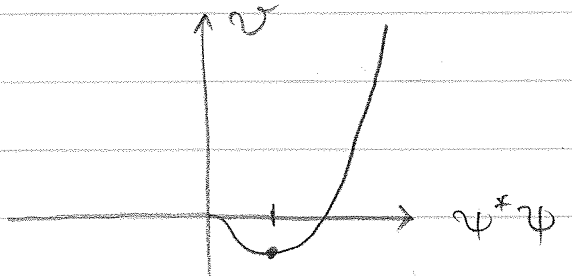
- Weakly interacting Bose gas, $a > 0$ :

Number density : $n$

Zero Temperature

Chemical potential: $\mu$, adjust so that $\langle \psi^+ \psi \rangle = n$

Potential energy density : $V(\psi^+\psi) = -\mu \psi^+\psi + \frac{2\pi a}{m} (\psi^+\psi)^2$



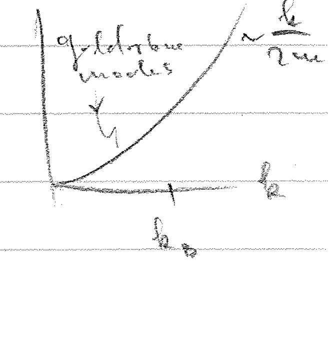

Minimum of $V$ : $-\mu + \frac{4\pi a}{m} \psi^+\psi = 0 \Rightarrow \mu = \frac{4\pi a}{m} n$

Notice: We have SSB of $U(1)$ symmetry:

$$\langle \psi \rangle \neq 0 , \quad \langle \psi \rangle = \sqrt{n}\, e^{i\Theta}$$

$\Rightarrow$ Bose-Einstein condensate

$\Rightarrow$ their must be Goldstone bosons




Homework: quasiparticles

$$\omega(k) = \frac{k\sqrt{k^2 + k_B^2}}{2m} , \quad k_B = \sqrt{16\pi a n}$$

Pressure: $P \overset{*}{=} -\mathcal{V}$

$$P(\mu) = \frac{m}{8\pi a}\mu^2 \quad \text{or} \quad P(\mu) = \frac{2\pi a}{m}\mu^2$$

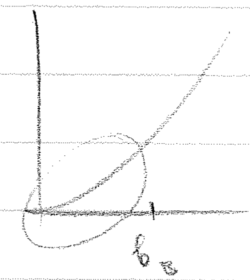
$*$ Reason: $e^{i\beta \mathcal{F} \mathcal{V}} = \int D\psi \, e^{i\int d^4x \mathcal{L}}$

$\uparrow$ free energy, determines thermodynamical properties

- Effective theory for Goldstone Bosons

Integrate out momenta above $k_B$

High mom. scale: $\Lambda_H = k_B = \ldots$



① d. o. f.: Goldstone boson field $\phi(\vec{r},t)$

Identify with phase of $\psi(\vec{r},t)$ : $\psi(\vec{r},t) = \langle\psi\rangle e^{i\phi(\vec{r},t)}$

② symmetries: $U(1)$ phase symmetry.

$$\psi(\vec{r},t) \to e^{i\Theta}\psi(\vec{r},t)$$

$$\phi(\vec{r},t) \to \phi(\vec{r},t) + \Theta$$

Galilean: $\psi(\vec{r},t) \to e^{\cdots}\psi(\vec{r}-\vec{v}t,t)$

$$\phi(\vec{r},t) \to \phi(\vec{r}-m\vec{v},t) + m\vec{v}\cdot\vec{r} - \frac{m}{2}v^2 t$$

Time reversal: $\psi(\vec{r},t) \to \psi^T(\vec{r},-t)$

$$\phi(\vec{r},t) \to -\phi(\vec{r},-t)$$

③ Most general $\mathcal{L}$:

$\mathcal{L}(\cancel{\phi}, \dot{\phi}, \nabla\phi, \ddot{\phi}, \dots)$

$U(1) \Rightarrow$ no dependance on $\phi$

$\dot{\phi} + \frac{1}{2m}\nabla\phi\nabla\phi \to \dot{\phi} + \frac{1}{2m}\nabla\phi\nabla\phi$ invariant

galilean $\Rightarrow$ $\mathcal{L}$ depends on $\dot{\phi}$ only through $\dot{\phi} + \frac{1}{2m}\nabla\phi\cdot\nabla\phi$

Time reversal $\Rightarrow$ (# of $\phi$) + (# of $\frac{\partial}{\partial t}$) is even

④ Relative importance; Use free theory

$$\mathcal{L}_{free} = \frac{1}{2}Z\dot{\phi}^2 - \frac{1}{2}Zv^2\nabla\phi\cdot\nabla\phi$$

eq. of motion: $\ddot{\phi} - v^2\nabla^2\phi = 0$

$\Rightarrow \omega(k) = vk$ scaling dimension

Power counting:

| | |
|---|---|
| $\nabla$ | 1 |
| $\partial_t$ | 1 |
| $\phi$ | 1 |

Goldstone theorem: SSB of $U(1)$ $\Rightarrow$ mode with dispersion relation $\omega(k) \to vk$, goes to 0 as $k \to 0$ at least linearly in $k$

## (6) Simplify $\mathcal{L}_{eff}$

**A** Omit total derivatives

**B** Exploit field redefinitions $\phi \to \phi + \Phi(\ldots)$

Can omit terms that vanish if $\ddot{\phi} - v^2 \phi = 0$

Can omit $\ddot{\phi}$, $\dddot{\phi}$, ...

Can extend $U(1)$ phase symmetry
to $U(1)$ gauge "

$$\mathcal{L}_{fund.} = \frac{1}{2}\left[\psi^\dagger i D_0 \psi + h.c.\right] - \frac{1}{2m} \vec{D}\psi^\dagger \cdot \vec{D}\psi - \frac{2\pi a}{m}(\psi^\dagger\psi)^2$$

$$D_0 = \partial_t - i A_0(\vec{r}, t)$$
$$\vec{D} = \vec{\nabla} - i\vec{A}(\vec{r}, t)$$

$$\psi(\vec{r}, t) \to e^{i\theta(\vec{r}, t)}\psi(\vec{r}, t)$$
$$A_0(\vec{r}, t) \to A_0 + \dot{\theta}$$
$$\vec{A}(\vec{r}, t) \to \vec{A} + \vec{\nabla}\theta$$

Special case: $A_0(\vec{r}, t) = \mu$
$\vec{A}(\vec{r}, t) = 0$

$$\phi(\vec{r}, t) \to \phi(\vec{r}, t) + \theta(\vec{r}, t)$$
$$A_0(\vec{r}, t) \to A_0(\vec{r}, t) + \dot{\theta}(\vec{r}, t)$$

$\Rightarrow$ $\dot{\phi} - A_0 \to \dot{\phi} - A_0$

$\Rightarrow$ $\mathcal{L}_{eff}$ depends only on $\dot{\phi} - \mu$

$\mathcal{L}_{eff}$ depends on $X = \mu - \dot{\phi} - \frac{1}{2m} \nabla\phi \cdot \nabla\phi$

$$\mathcal{L}_{eff} = P(X) + Q(X)\, \nabla\nabla\phi \cdot \nabla\nabla\phi$$ suppressed by derivatives

$$= P\left(\mu - \dot{\phi} - \frac{1}{2m} \nabla\phi \cdot \nabla\phi\right) + \ldots$$

$$= P(\mu) + P'(\mu)\left(-\overset{\text{total deriv.}}{\cancel{\dot{\phi}}} + \frac{1}{2m} \nabla\phi\nabla\phi\right) + \ldots$$

$$= \underbrace{P(\mu)}_{\text{pressure}} + \underbrace{\frac{1}{2} P''(\mu)\, \dot{\phi}^2 - \frac{1}{2m} P'(\mu)\, \nabla\phi\nabla\phi}_{\frac{1}{2} Z \dot{\phi}^2 - \frac{1}{2} Z v^2 \nabla\phi\nabla\phi} + \ldots$$

$$P(\mu) = \frac{M}{8\pi a} \mu^2$$

velocity of Goldstone bosons

$$v^2 = \frac{P'(\mu)}{2m P''(\mu)} = \frac{4\pi a n}{m^2}$$

agrees with fundamental theory

$$\omega(k) = \frac{k\sqrt{k^2 + k_B^2}}{m}$$

the Goldstone B. are (physically) ripples of fluctuations in the density.

Lecture IV

low energy field theory

Atom field $\psi(\vec{r},t)$

$\mathcal{L}_{eff}$

Homogeneous Bose gas: pressure ...

quasi particle: $\omega = \frac{k\sqrt{k^2+k_B^2}}{2m}$

Low E eff field th. for Goldstone boson:

phase field: $\phi(\vec{r},t)$

$\mathcal{L}_{eff}$

velocity: $v = \sqrt{\frac{P'(\mu)}{mP''(\mu)}} = \frac{k_B}{2m}$

( Fermi gas (exercises): $\frac{k_F}{\sqrt{3}m} = v$ )

EFT method

(1)

(2)

(3)

(4) ...

Use $\mathcal{L}_{free}$

free theory is scale invariant $\vec{r} \to \lambda \vec{r}$ for any
$t \to \lambda^2 t$ $\lambda > 0$

Non trivial scale-invariant theory:

$$\mathcal{L}_* = \psi^\dagger \left( i\partial_t + \frac{1}{2m}\vec{\nabla}^2 \right)\psi + \mathcal{L}_{int}$$

$\Rightarrow$ UV cutoff $\Lambda$ is only scale, $\Lambda \to \infty$

if $\Lambda^{\gamma_n} O_n$ has finite matrix elements as $\Lambda \to \infty$
then $O_n$ has anomalous dimension $\gamma_n$
scaling dimension $d_n + \gamma_n$

$$\mathcal{L}_{eff} = \mathcal{L}_* + \sum_n C_n \left( \Lambda^{\gamma_n} O_n \right)$$

$$\Rightarrow C_n \sim \left( \frac{1}{\Lambda_{HI}} \right)^{d_n - 5 + \gamma_n}$$

"Strongly interacting" atoms
= atoms with large scattering length

S-wave contribution to T-matrix

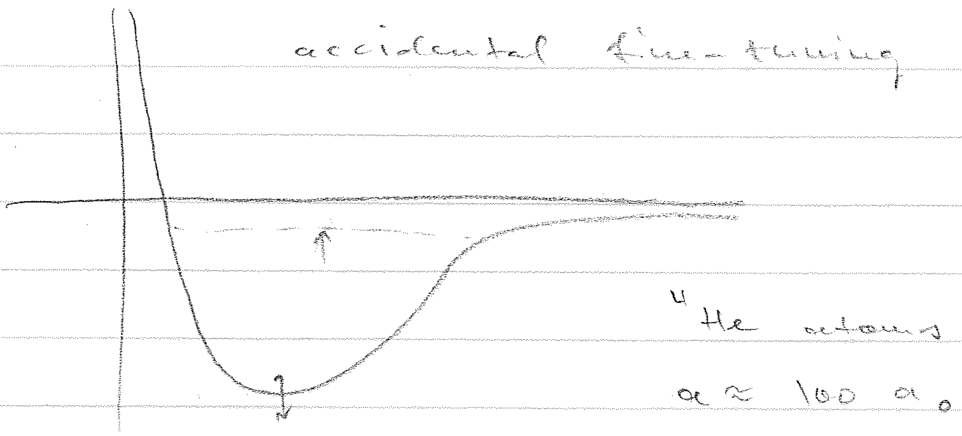
$$T(k) = \frac{8\pi/m}{k \cot\delta(k) - ik} \simeq \frac{8\pi/m}{-\frac{1}{a} - ik}$$

Effective range expansion:

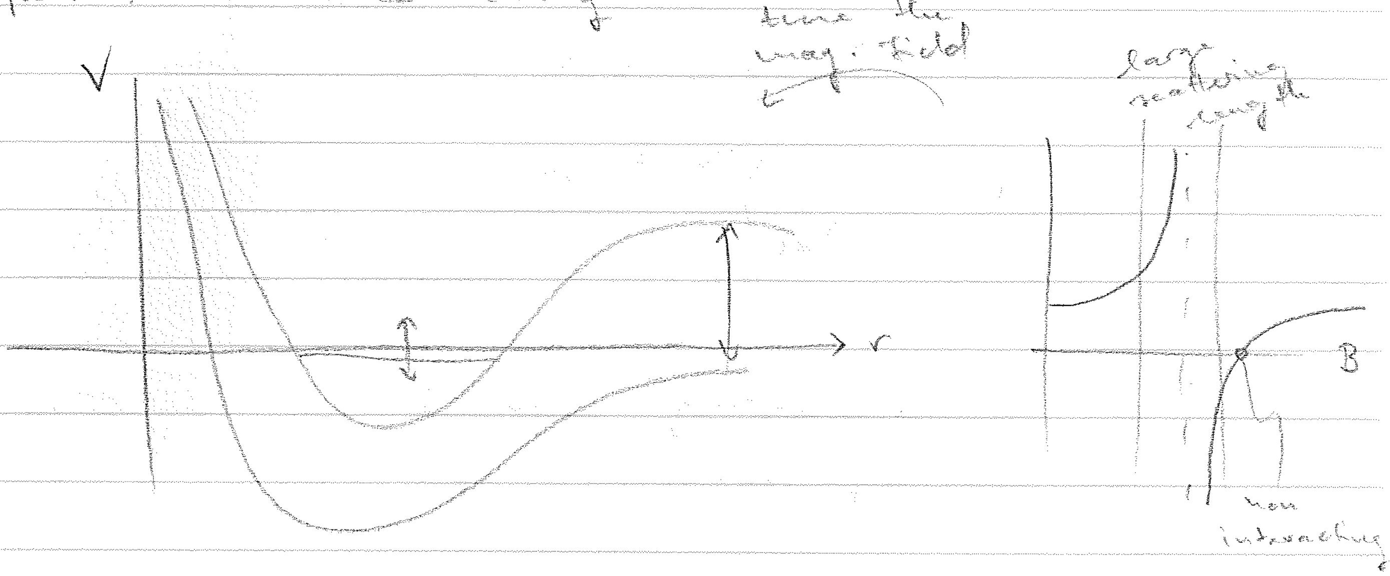
$$k\cot\delta(k) = -\frac{1}{a} + \boxed{\cancel{\frac{1}{2} r_s k^2}} + \ldots$$

Generically $|a|, |r_s| \sim \frac{1}{\Lambda_{HI}} \sim$ range

Large scattering length: $|a| \gg$ range
- requires fine-tuning
- " bound state near threshold




Experimental fine-tuning




The field theory:

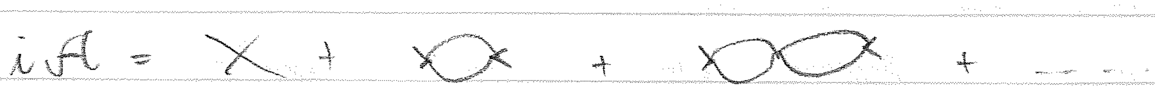
$$\mathcal{L} = \psi^* \left( i\partial_t + \frac{1}{2m}\vec{\nabla}^2 \right)\psi - \underbrace{\frac{g_0}{m}}_{\text{non-perturbative}} \psi^*\psi^*\psi\psi$$

Feynman rules:

$$\overline{\quad E, \vec{p} \quad} = \frac{i}{E - p^2/2m + i\epsilon}$$

$$\times = -i g_0/m$$

Amplitude for $2 \to 2$ scattering



$$= -ig_0 + \left(-i\frac{g_0}{m}\right)\left(i\mathcal{A}_{pair}\right)\left(-i\frac{g_0}{m}\right) + \left(-i\frac{g_0}{m}\right)^2\left(i\mathcal{A}_{pair}\right)^2 + \dots$$

$$= \frac{-ig_0/m}{1-\frac{g_0}{m}\mathcal{A}_{pair}} \qquad \mathcal{A}_{pair} = -\frac{m}{4\pi^2}\left(\Lambda - \frac{\pi}{2}\sqrt{-mE-i\epsilon}\right)$$

$$= \frac{8\pi/m}{\underbrace{-8\pi\left(\frac{1}{g_0} + \frac{1}{4\pi^2}\Lambda\right)}_{-1/a} + \sqrt{-mE-i\epsilon}} \qquad \begin{array}{l} E = k^2/m \\ \sqrt{-mE-i\epsilon} = -ik \end{array}$$

$g_0$ is very small, but the loops give very high contribution since they go with $\Lambda$. That's why we have to sum up all diagrams.

$$\mathcal{A}(E) = \frac{8\pi/m}{-\frac{1}{a} + \sqrt{-mE-i\epsilon}}$$

Bound states = poles in $E$

$a > 0$ : pole at $E = -\frac{1}{ma^2}$ — wave function $\psi(r) = \frac{e^{-r/a}}{r}$

$a < 0$ : no bound state

Scale invariant limits

$$T(k) = \frac{8\pi/m}{-1/a - ik}$$

A. $a = 0$ "free theory"

$T(k) = 0$

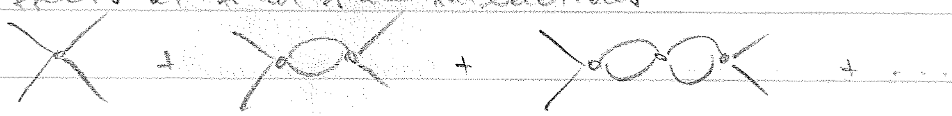
$$g_0 = -\frac{4\pi^2}{\Lambda}$$

B. $a = \pm\infty$

$$T(k) = \frac{8\pi/m}{-ik}$$

Matrix element of $\psi^\dagger\psi^\dagger\psi\psi$ = 

effects of final state interactions



$$= 1 + 1\left(i\mathcal{A}_{pair}\right)\left(-i\frac{g_0}{m}\right) + \ldots =$$

$$= \frac{\mathcal{A}(E)}{-ig_0 m}$$

effects of all interactions: $= \left(\frac{\mathcal{A}(E)}{-ig_0 m}\right)^2$

$$= -\frac{m^2}{g_0^2}\mathcal{A}^2(E) \sim \Lambda^2\mathcal{A}^2(E)$$

$\Rightarrow \Lambda^{-2}\psi^\dagger\psi^\dagger\psi\psi$ has finite matrix elements as $\Lambda\to\infty$

$\psi^\dagger\psi^\dagger\psi\psi$ has anomalous dim $-2$

scaling dim $6-2=4$

Same for other operators.

$\psi^\dagger\psi^\dagger(i\partial_t)^l\psi\psi$ has anomalous dim $= -2$

scaling dim $= 6+2l-2$

$= 6, 8, \ldots$

Operators with $\overleftrightarrow{\nabla}$ $\sim$ 7, 9, 10 ...

3-body operators $\psi^\dagger\psi^\dagger\psi^\dagger\psi\psi\psi$ sc. dim $5+\sqrt{\lambda}$

where $\sqrt{\lambda}\cos\left(\sqrt{\lambda}\frac{\pi}{2}\right) = \frac{8}{\sqrt{3}}\sin\left(\sqrt{\lambda}\frac{\pi}{6}\right)$ $\lambda = -s_0^2$

$\sqrt{\lambda} = \pm i s_0$

$s_0 = 1.00624$

$$\left(\frac{\Lambda}{\Lambda_*}\right)^{+is_0} + \left(\frac{\Lambda}{\Lambda_*}\right)^{-is_0}$$

others: 7.86, 7.82, 9.08

$$= e^{is_0\log(\Lambda/\Lambda_*)} + e^{-is_0\log(\Lambda/\Lambda_*)} = 2\cos\left(s_0\log(\Lambda/\Lambda_*)\right)$$

Log-periodic: invariant under $\Lambda \to e^{\pi/s_0}\Lambda$

$= 22.7\,\Lambda$

$\Rightarrow$ discrete scale invariance:

$$\vec{r} \to (22.7)^n\,\vec{r}$$