Introduction to lattice QCD (2)

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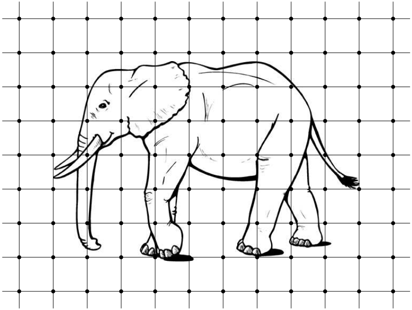

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SU(3) lattice gauge fields

Gauge transformations

$$\psi(x) \to \Lambda(x)\psi(x), \qquad \Lambda(x) \in \mathrm{SU}(3)$$

Covariant derivative (continuum theory)

$$D_\mu \psi = (\partial_\mu + A_\mu)\psi$$

$$A_\mu \to \Lambda A_\mu \Lambda^{-1} + \Lambda \partial_\mu \Lambda^{-1}$$

The gauge potential provides a connection between the colour spaces at infinitesimally separated points

On the lattice

$$\partial_\mu \psi(x) = \frac{1}{a}\left\{\psi(x+a\hat{\mu}) - \psi(x)\right\}$$

$$\to \frac{1}{a}\left\{\Lambda(x+a\hat{\mu})\psi(x+a\hat{\mu}) - \Lambda(x)\psi(x)\right\}$$

The colour connection is here provided by

$$U(x,\mu) \in \mathrm{SU}(3), \qquad U(x,\mu) \to \Lambda(x)U(x,\mu)\Lambda(x+a\hat{\mu})^{-1}$$

⇒ covariant difference operator

$$\nabla_\mu \psi(x) = \frac{1}{a}\left\{U(x,\mu)\psi(x+a\hat{\mu}) - \psi(x)\right\}$$

$$\nabla_\mu \psi(x) \to \Lambda(x)\nabla_\mu \psi(x)$$

Similarly

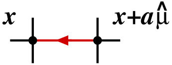
$$\nabla_\mu^* \psi(x) = \frac{1}{a} \left\{ \psi(x) - U(x - a\hat{\mu}, \mu)^{-1} \psi(x - a\hat{\mu}) \right\}$$

⇒ gauge-covariant Wilson–Dirac operator

$$D_{\rm w} = \sum_{\mu=0}^{3} \tfrac{1}{2} \left\{ \gamma_\mu (\nabla_\mu^* + \nabla_\mu) - a \nabla_\mu^* \nabla_\mu \right\}$$

An SU(3) lattice gauge field is an assignment of a matrix

$$U(x, \mu) \in \mathrm{SU}(3)$$

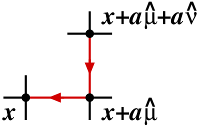


to every link $(x, x + a\hat{\mu})$ *on the lattice*

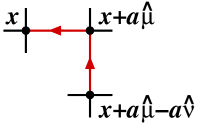
Wilson lines

May build gauge-covariant products

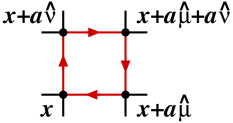
$$U(x,\mu)U(x+a\hat{\mu},\nu)$$



$$U(x,\mu)U(x+a\hat{\mu}-a\hat{\nu},\nu)^{-1}$$



$$U(x,\mu)U(x+a\hat{\mu},\nu)U(x+a\hat{\nu},\mu)^{-1}U(x,\nu)^{-1}$$

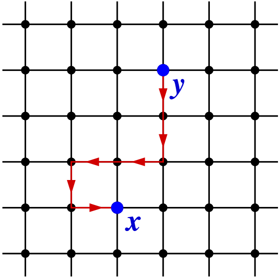


"plaquette loop"

More generally, for any lattice curve $\mathcal{C}$

$$U(x,y;\mathcal{C}) = \text{ordered product of } U\text{'s}$$

$$U(x,y;\mathcal{C}) \to \Lambda(x)U(x,y;\mathcal{C})\Lambda(y)^{-1}$$

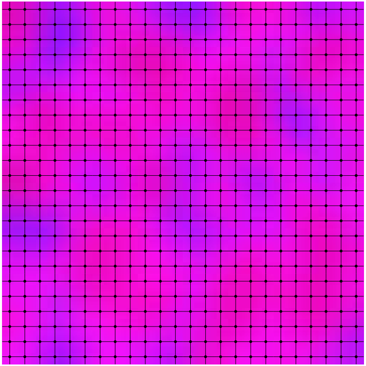


In particular, for any closed curve, the Wilson loop

$$W(\mathcal{C}) = \mathrm{tr}\{U(x,x;\mathcal{C})\}$$

is gauge-invariant and independent of x

Classical continuum limit



In which way can a continuum gauge field be approximated by a lattice gauge field?

First attempt

$$U(x,\mu) = \exp\{aA_\mu(x)\} = 1 + aA_\mu(x) + \mathrm{O}(a^2)$$

$$\Rightarrow \quad \nabla_\mu \psi(x) = \frac{1}{a}\{(1 + aA_\mu(x))\psi(x + a\hat{\mu}) - \psi(x)\} + \mathrm{O}(a)$$

$$= (\partial_\mu + A_\mu(x))\psi(x) + \mathrm{O}(a)$$

However, this choice of U is not gauge-covariant

A more educated choice is

$$U(x,\mu) = \mathcal{T}\exp\left\{a\int_0^1 \mathrm{d}t\, A_\mu(x+(1-t)a\hat{\mu})\right\}$$

In this case

- ⋆ $U(x,\mu) = 1 + aA_\mu(x) + \mathrm{O}(a^2)$ as before
- ⋆ the mapping $A \to U$ is gauge-covariant
- ⋆ $U(x,y;\mathcal{C})$ = continuum Wilson line, for all lattice curves $\mathcal{C}$

$\Leftrightarrow$ lattice gauge field = set of Wilson-line elements

Gauge-invariant local lattice fields

Examples of quark bilinear fields

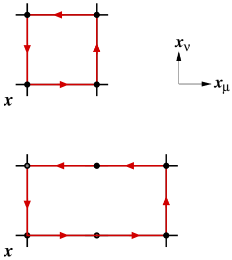
$$\overline{\psi}\psi, \quad \overline{\psi}\gamma_5\tau^a\psi, \quad \overline{\psi}Q\gamma_\mu\psi,$$

$$\overline{\psi}\gamma_\mu\nabla_\nu\psi, \quad \overline{\psi}\nabla_\mu\nabla_\nu\psi, \quad \ldots$$

"Plaquette" and "rectangle" fields

$$P_{\mu\nu}(x) = \mathrm{Re}\,\mathrm{tr}\{1 - U(x,x;\square)\},$$

$$R_{\mu\nu}(x) = \mathrm{Re}\,\mathrm{tr}\{1 - U(x,x;\sqsubset\!\sqsupset)\}$$



In the classical continuum limit

$$\mathcal{O}(x) \underset{a\to 0}{\sim} \sum_{n\geq 0} a^n \mathcal{O}_n(x)$$

$\mathcal{O}_n(x)$: gauge-invariant polynomial in $\psi(x), \overline{\psi}(x), A_\mu(x)$
and their derivatives of dimension n

For example

$$P_{\mu\nu}(x) = -\tfrac{1}{2}a^4 \mathrm{tr}\{F_{\mu\nu}(x)F_{\mu\nu}(x)\}$$

$$-\tfrac{1}{2}a^5 \mathrm{tr}\{F_{\mu\nu}(x)(D_\mu + D_\nu)F_{\mu\nu}(x)\} + \ldots$$

$$R_{\mu\nu}(x) = -2a^4 \mathrm{tr}\{F_{\mu\nu}(x)F_{\mu\nu}(x)\} + \ldots$$

Remarks

- ⋆ Lattice fields can be classified by their leading behaviour in the classical continuum limit
- ⋆ Any gauge-invariant, local continuum field can be represented on the lattice
- ⋆ However, the representation is not unique

Lattice QCD action

Wilson action (1974)

$$S = S_{\mathrm{G}} + S_{\mathrm{F}}$$

$$S_{\mathrm{G}} = \frac{1}{g_0^2} \sum_x \sum_{\mu,\nu} P_{\mu\nu}(x),$$

$P_{\mu\nu}$: plaquette field

$$S_{\mathrm{F}} = a^4 \sum_x \overline{\psi}(x)(D_{\mathrm{w}} + M)\psi(x),$$

D_{w}: Wilson–Dirac operator

M: quark mass matrix

However, there are alternative lattice actions, e.g.

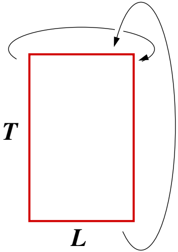
$$S_{\mathrm{G}} = \frac{1}{g_0^2} \sum_x \sum_{\mu,\nu} \left\{ c_0 P_{\mu\nu}(x) + c_1 R_{\mu\nu}(x) \right\}, \qquad c_0 + 4c_1 = 1$$

Symanzik 1980, Weisz 1983, M.L. & Weisz 1985, Iwasaki 1985, ...

- The differences are of order a^p in the classical continuum limit

- Additional terms may be tuned so as to accelerate the convergence to the continuum limit

  $\Rightarrow$ *"Symanzik improvement programme"*



QCD in finite volume

Consider $T \times L^3$ box with periodic b.c.

$$A_\mu(x)_{x_k=L} = A_\mu(x)_{x_k=0} \qquad (k=1,2,3)$$

$$A_\mu(x)_{x_0=T} = \Omega(\boldsymbol{x}) A_\mu(x)_{x_0=0} \Omega(\boldsymbol{x})^{-1} + \Omega(\boldsymbol{x}) \partial_\mu \Omega(\boldsymbol{x})^{-1}$$

where $\Omega(\boldsymbol{x})$ is a periodic gauge function

The mapping

$$\Omega: \ \mathbb{T}^3 \mapsto \mathrm{SU}(3)$$

may have a non-zero winding number $Q \Rightarrow$ cannot be "gauged away"

Q is a gauge-invariant property of the gauge field

$$Q = -\frac{1}{16\pi^2} \int_0^T \mathrm{d}x_0 \int_0^L \mathrm{d}^3\boldsymbol{x}\, \epsilon_{\mu\nu\rho\sigma} \mathrm{tr}\{F_{\mu\nu}(x) F_{\rho\sigma}(x)\}$$

$$= 0, \pm 1, \pm 2, \ldots$$

"Topological charge" or "instanton number"

$\Rightarrow$ *in finite volume, the field space divides into disconnected sectors*

Lattice gauge fields in finite volume

The independent field variables are

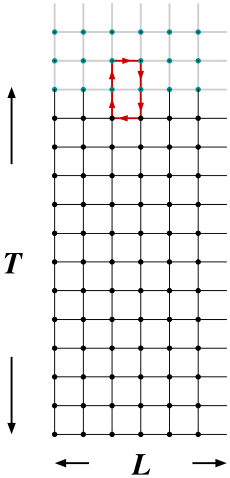
$$U(x,\mu), \quad 0 \le x_0 \le T-a$$
$$0 \le x_k \le L-a$$

Elsewhere the field is determined through

$$U(x+L\hat{\boldsymbol{k}},\mu) = U(x,\mu)$$

$$U(x+T\hat{0},\mu) = \Omega(\boldsymbol{x})U(x,\mu)\Omega(\boldsymbol{x}+a\hat{\mu})^{-1}$$

⇒ classical continuum limit works out as before



On the lattice, the b.c. are not a property of the field space

Moreover, by applying the gauge transformation

$$U(x,\mu) \to \Lambda(x)U(x,\mu)\Lambda(x+a\hat{\mu})^{-1}$$

$$\Lambda(x) = \Omega(\boldsymbol{x})^{-\lfloor x_0/T \rfloor}$$

we may "gauge away" $\Omega(\boldsymbol{x})$

⇒ *no need to consider b.c. "up to gauge transformations"*

⇒ *We may impose ordinary periodic b.c. from the beginning and nevertheless have all topological sectors included in the theory*

Somewhat puzzling may be the fact that the space of lattice fields

$$\mathcal{F} \cong \mathrm{SU}(3)^{4N}, \quad N = \# \text{ of lattice points}$$

is connected!

action $\sim 1/a^4$ →