Lectures on Effective Field Theory

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Problem 3

The Lagrangian for a weakly-interacting Bose gas consisting of atoms with scattering length a and number density n is

$$\begin{aligned} \mathcal{L} &= \tfrac{1}{2}\left(\psi^* i\tfrac{\partial}{\partial t}\psi + \text{h.c.}\right) - \frac{1}{2m}\nabla\psi^*\cdot\nabla\psi - \mathcal{V}, \\ \mathcal{V} &= -\mu\psi^*\psi + \frac{2\pi a}{m}(\psi^*\psi)^2. \end{aligned}$$

A. Show that if $a > 0$, the potential energy density $\mathcal{V}$ is minimized by a nonzero value of ψ that corresponds to a number density given by

$$n = \psi^*\psi = \frac{m}{4\pi a}\mu.$$

B. Expand the field around its vacuum expectation value by expressing it in the form

$$\psi(\vec{r},t) = \sqrt{n} + \xi(\vec{r},t) + i\eta(\vec{r},t),$$

where ξ and η are real-valued fields. Show that the quadratic terms in the Lagrangian can be reduced to the form

$$\mathcal{L}_2 = i\left(\xi\dot{\eta} - \eta\dot{\xi}\right) - \frac{1}{2m}\left(\nabla\xi\cdot\nabla\xi + \nabla\eta\cdot\nabla\eta\right) - 2\mu\xi^2.$$

C. Find the dispersion relation $\omega(k)$ for the quasiparticles by expressing $\mathcal{L}_2$ in terms of a 2-component field $\binom{\xi}{\eta}$, finding the matrix equation of motion, and looking for solutions of the form

$$\begin{pmatrix} \xi(\vec{r},t) \\ \eta(\vec{r},t) \end{pmatrix} = \begin{pmatrix} \xi_0 \\ \eta_0 \end{pmatrix} \exp(i\vec{k}\cdot\vec{r} - i\omega t).$$

Show that the dispersion relation is

$$\omega(k) = \frac{k\sqrt{k^2 + k_B^2}}{2m},$$

where the Bogoliubov momentum is $k_B = \sqrt{16\pi a n}$.

D. The leading terms in the effective Lagrangian for the Goldstone mode have the form

$$\mathcal{L}_{\text{eff}} = \tfrac{1}{2}\dot{\phi}^2 - \tfrac{1}{2}v^2 \nabla\phi \cdot \nabla\phi.$$

Find the dispersion relation for the Goldstone mode. Determine the parameter v^2 by matching with the dispersion relation for the weakly-interacting Bose gas.

Problem 4

Consider a weakly-interacting Fermi gas consisting of atoms with two spin states and equal number densities $n_1 = n_2$. The equal number densities can arise from a common chemical potential $\mu = k_F^2/(2m)$, where k_F is the Fermi momentum. The Lagrangian for the fermion fields is

$$\begin{aligned}\mathcal{L} \; &= \; \sum_\sigma \left(\tfrac{1}{2}\left[\psi_\sigma^* i\tfrac{\partial}{\partial t}\psi_\sigma + \text{h.c.}\right] - \frac{1}{2m}\nabla\psi_\sigma^* \cdot \nabla\psi_\sigma + \mu\,\psi_\sigma^*\psi_\sigma \right) \\ &\quad -\frac{4\pi a}{m}\psi_1^*\psi_2^*\psi_2\psi_1,\end{aligned}$$

where a is the scattering length for a pair of fermions in the spin states 1 and 2. The Lagrangian is invariant under the $U(1)$ symmetry corresponding to multiplying both fields by a common phase. The propagator for either fermion field is

$$iD(\omega, k : k_F) = \frac{i\,\theta(k - k_F)}{\omega - k^2/2m + i\epsilon} + \frac{i\,\theta(k_F - k)}{\omega - k^2/2m - i\epsilon},$$

which reveals that the propagating degrees of freedom are atoms above the Fermi surface and holes below the Fermi surface.

A. Verify that the number density of fermions in the spin state 1 is equal to $n_1 = k_F^3/(6\pi^2)$ by evaluating the one-loop diagram for $\langle\psi_1^\dagger\psi_1\rangle$ and subtracting the number density of the vacuum, which is defined by $k_F = 0$:

$$n_1 = (-1)\int\frac{d\omega}{2\pi}\int\frac{d^3k}{(2\pi)^3}\left[iD(\omega,k;k_F) - iD(\omega,k;0)\right].$$

B. Use the thermodynamic relation $dP = (n_1 + n_2)d\mu$, to show that the pressure P (which is equal to the thermodynamic potential) is given by

$$P(\mu) = \frac{1}{15\pi^2 m}(2m\mu)^{5/2}.$$

C. If the scattering length a is negative, Cooper pairing between atoms in the spin states 1 and 2 near the Fermi surface breaks the $U(1)$ symmetry spontaneously and leads to a nonzero expectation value for the operator $\psi_1\psi_2$.

The field for the resulting Goldstone boson can be identified with the phase of this composite operator:

$$\psi_1\psi_2(\vec{r},t) = \langle\psi_1\psi_2\rangle \exp[2i\phi(\vec{r},t)].$$

Deduce the Galilean transformation of the Goldstone field from the Galilean transformations of the fermion fields:

$$\psi_\sigma(\vec{r},t) \longrightarrow \exp(im\vec{v}\cdot\vec{r} - i\tfrac{1}{2}mv^2 t)\psi_\sigma(\vec{r}-\vec{v}\,t,t).$$

D. Use the Galilean transformation of the Goldstone field to deduce that a time derivative of ϕ in the effective lagrangian can only appear in the combination

$$\dot{\phi} + \frac{1}{2m}\nabla\phi\cdot\nabla\phi.$$

E. The common chemical potential μ can be regarded as a constant background gauge field A_0 associated with a $U(1)$ gauge symmetry:

$$\psi_\sigma(\vec{r},t) \longrightarrow \exp[i\theta(\vec{r},t)]\psi(\vec{r},t).$$

Determine the $U(1)$ gauge transformation of the Goldstone field. Use it to deduce that $\dot{\phi}$ and μ can appear in the effective lagrangian only in the combination $\dot{\phi} - \mu$.

F. The above results imply that the leading terms in the effective lagrangian for the Goldstone bosons have the form

$$\mathcal{L}_{\text{eff}} = P(X), \qquad X = \mu - \dot{\phi} - \frac{1}{2m}\nabla\phi\cdot\nabla\phi,$$

where $P(\mu)$ is the pressure as a function of the chemical potential. Expand the expression for the pressure to second order around $X = \mu$ to deduce that the Goldstone bosons associated with Cooper pairing must have velocity $v = k_F/(\sqrt{3}m)$.