Introduction to lattice QCD (3)

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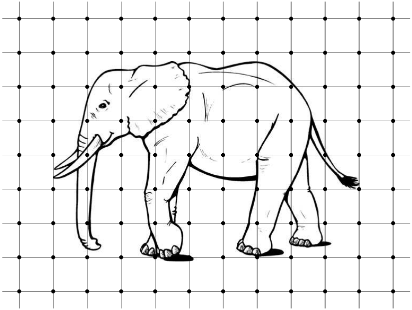

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Quantization of LQCD

Let $\phi_1(x), \ldots, \phi_n(x)$ be any gauge-invariant local fields

Their euclidean correlation function is then given by

$$\langle \phi_1(x_1) \ldots \phi_n(x_n) \rangle = \frac{1}{\mathcal{Z}} \int \mathrm{D}[U] \int \mathrm{D}[\psi]\mathrm{D}[\overline{\psi}]$$

$$\times\, \phi_1(x_1) \ldots \phi_n(x_n) \exp\{-S[U, \overline{\psi}, \psi]\}$$

$\uparrow$

lattice QCD action

$$\mathcal{Z} = \int \mathrm{D}[U] \int \mathrm{D}[\psi]\mathrm{D}[\overline{\psi}] \exp\{-S[U, \overline{\psi}, \psi]\}$$

Remarks

- *The functional integral can be taken as the definition of the quantum theory*

- *The measures* $\mathrm{D}[U]$ *and* $\mathrm{D}[\psi]\mathrm{D}[\overline{\psi}]$ *are ultra-local and largely symmetry-determined*

- *Most details do not matter as* $a \to 0$ *provided some basic principles are respected*

  locality, gauge symmetry, ...

Classical fermion fields

The quark fields integrated over are anticommuting classical fields, i.e. they take values in a Grassmann algebra

Generators

$$c_1, \dots, c_n, \bar{c}_1, \dots, \bar{c}_n, \qquad \{c_i, c_j\} = \{c_i, \bar{c}_j\} = \{\bar{c}_i, \bar{c}_j\} = 0$$

The elements of the algebra are linear combinations of the products

$$X_{b_1 \dots b_{2n}} = c_1^{b_1} \dots \bar{c}_n^{b_{2n}}, \qquad b_i \in \{0, 1\}$$

$\Rightarrow$ dimension $= 2^{2n}$

In particular, any function $f(\bar{c}, c)$ is given by

$$f(\bar{c}, c) = \sum_{b_1,\ldots,b_{2n}} f_{b_1\ldots b_{2n}} X_{b_1\ldots b_{2n}} = f_{00\ldots0} + \ldots + f_{11\ldots1} c_1 \ldots \bar{c}_n$$

and we may define

$$\int \mathrm{D}[c]\mathrm{D}[\bar{c}]\, f(\bar{c}, c) = f_{11\ldots1}$$

For example

$$\int \mathrm{D}[c]\mathrm{D}[\bar{c}] \exp\Big\{-\sum_{i,j} \bar{c}_i A_{ij} c_j\Big\} = \frac{(-1)^n}{n!} \int \mathrm{D}[c]\mathrm{D}[\bar{c}] \Big\{\sum_{i,j} \bar{c}_i A_{ij} c_j\Big\}^n$$

$$= \frac{(-1)^{\frac{1}{2}n(n-1)}}{n!} \sum_{i_1,\ldots,j_n} \epsilon_{i_1\ldots i_n} \epsilon_{j_1\ldots j_n} A_{i_1 j_1} \ldots A_{i_n j_n} = (-1)^{\frac{1}{2}n(n-1)} \det A$$

Similarly

$$\langle c_{k_1} \ldots \bar{c}_{l_m} \rangle_{\mathrm{F}} = \frac{1}{\mathcal{Z}_{\mathrm{F}}} \int \mathrm{D}[c]\mathrm{D}[\bar{c}]\, c_{k_1} \ldots \bar{c}_{l_m} \exp\Big\{-\sum_{i,j} \bar{c}_i A_{ij} c_j \Big\}$$

$$= \text{sum of Wick contractions}$$

where

$$\overbracket{c_k \bar{c}_l} = \langle c_k \bar{c}_l \rangle_{\mathrm{F}} = (A^{-1})_{kl}, \qquad \mathcal{Z}_{\mathrm{F}} = \int \mathrm{D}[c]\mathrm{D}[\bar{c}] \exp\Big\{-\sum_{i,j} \bar{c}_i A_{ij} c_j \Big\}$$

For example

$$\langle c_k \bar{c}_l c_i \bar{c}_j \rangle_{\mathrm{F}} = (A^{-1})_{kl}(A^{-1})_{ij} \underset{\uparrow}{-} (A^{-1})_{kj}(A^{-1})_{il}$$

Integration over the quark fields

The components

$$\psi(x)_{A\alpha q}, \quad \overline{\psi}(x)_{A\alpha q}$$

$$A = 1, \ldots, 4, \quad \alpha = 1, \ldots, 3, \quad q = 1, \ldots, N_{\rm f}$$

are taken to be the generators c_i and $\bar{c}_j$ of a Grassmann algebra

dimension $= 2^{2n}, \quad n = 12N_{\rm f} \times$ no of lattice points

$$\int {\rm D}[\psi]{\rm D}[\overline{\psi}] \ldots = \int {\rm D}[c]{\rm D}[\bar{c}] \ldots$$

$$\langle \psi(x_1) \ldots \overline{\psi}(y_m) \rangle_{\rm F} = \langle c_{k_1} \ldots \bar{c}_{l_m} \rangle_{\rm F}$$

In presence of an arbitrary gauge field U

$$\mathcal{Z}_\mathrm{F} = \int \mathrm{D}[\psi]\mathrm{D}[\overline{\psi}] \exp\Big\{-a^4 \sum_x \overline{\psi}(x)(D_\mathrm{w} + M)\psi(x)\Big\}$$

$$= \det(D_\mathrm{w} + M) = \prod_{q=1}^{N_\mathrm{f}} \det(D_\mathrm{w} + m_q) \qquad \text{(up to a power of } a\text{)}$$

Quark propagator & correlation functions

$$(D_\mathrm{w} + M)S(x, y; U) = a^{-4}\delta_{xy}$$

$$\langle\psi(x)\overline{\psi}(y)\rangle_\mathrm{F} = S(x, y; U)$$

$$\langle\psi(x_1)\overline{\psi}(y_1)\psi(x_2)\overline{\psi}(y_2)\rangle_\mathrm{F} = S(x_1, y_1; U)S(x_2, y_2; U) - \ldots$$

⇒ in the QCD functional integral, the quark fields may be integrated out

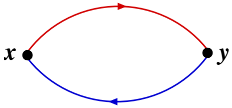
$$\langle \phi_1(x_1) \ldots \phi_n(x_n) \rangle =$$

$$\frac{1}{\mathcal{Z}} \int \mathrm{D}[U] \, \langle \phi_1(x_1) \ldots \phi_n(x_n) \rangle_{\mathrm{F}} \prod_{q=1}^{N_{\mathrm{f}}} \det(D_{\mathrm{w}} + m_q) \exp\{-S_{\mathrm{G}}[U]\}$$

In the case of the pion propagator

$$\langle (\bar{u}\gamma_5 d)(x)(\bar{d}\gamma_5 u)(y) \rangle_{\mathrm{F}} = -\mathrm{tr}\{\gamma_5 S(x,y;U)_{dd} \gamma_5 S(y,x;U)_{uu}\}$$

$=$



⇒ integral reduced to a purely bosonic integral

Integration over the gauge field

Require

$$\mathrm{D}[U] = \prod_{x,\mu} \mathrm{d}U(x,\mu) \qquad \text{(locality)}$$

$$\int_{\mathrm{SU}(3)} \mathrm{d}U\, f(U) = \int_{\mathrm{SU}(3)} \mathrm{d}U\, f(\Lambda U) \qquad \text{(gauge invariance)}$$

$\Rightarrow$ the measure is uniquely determined up to a normalization factor

Implied properties

$$\int \mathrm{d}U\, f(U) = \int \mathrm{d}U\, f(U\Lambda) = \int \mathrm{d}U\, f(U^{-1}) = \int \mathrm{d}U\, f(U^*)$$

We may, for example, parametrize $U \in \mathrm{SU}(3)$ through

$$U = (\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3)$$

$$\|\boldsymbol{u}_1\|^2 = \|\boldsymbol{u}_2\|^2 = 1, \quad (\boldsymbol{u}_1, \boldsymbol{u}_2) = 0, \quad \boldsymbol{u}_3 = \boldsymbol{u}_1^* \times \boldsymbol{u}_2^*$$

The one-link integral is then given by

$$\int_{\mathrm{SU}(3)} \mathrm{d}U\, f(U) =$$

$$\int_{\mathbb{C}^3 \times \mathbb{C}^3} \mathrm{d}^6 u_1 \mathrm{d}^6 u_2\, \delta(1 - \|\boldsymbol{u}_1\|^2)\delta(1 - \|\boldsymbol{u}_2\|^2)\delta((\boldsymbol{u}_1, \boldsymbol{u}_2))\, f(U)$$

This completes the definition of the (Wilson) lattice theory

Elementary properties

1. Regularity

In finite volume

- *the space of all gauge fields is compact*
- *after the fermions are integrated out, one is normally left with a continuous integrand*
- *the partition function $\mathcal{Z}$ is positive*

$\Rightarrow$ the correlation functions $\langle \phi_1(x_1) \ldots \phi_n(x_n) \rangle$ are entirely well-defined

$\Rightarrow$ lattice QCD provides a non-perturbative regularization of QCD

2. Gauge invariance

For any observable $\mathcal{O}[U,\overline{\psi},\psi]$ and (classical) gauge function $\Lambda(x)$

$$\langle\mathcal{O}\rangle = \langle\mathcal{O}^{\Lambda}\rangle$$

$$\mathcal{O}^{\Lambda}[U,\overline{\psi},\psi] = \mathcal{O}[U^{\Lambda},\overline{\psi}^{\Lambda},\psi^{\Lambda}]$$

$$U^{\Lambda}(x,\mu) = \Lambda(x)U(x,\mu)\Lambda(x+a\hat{\mu})^{-1},\ \ldots$$

Example

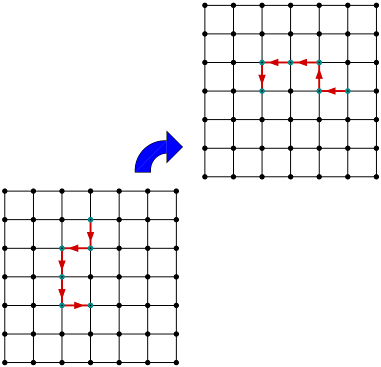
$$\langle\psi(x)\overline{\psi}(y)\rangle = \Lambda(x)\langle\psi(x)\overline{\psi}(y)\rangle\Lambda(y)^{-1}$$

$$= 0 \quad \text{if} \quad x \neq y$$

3. Space-time symmetries

Correlation functions are invariant under

- ⋆ *translations by lattice vectors*
- ⋆ *space-time rotations* $x_\mu \to \Lambda_{\mu\nu} x_\nu$ *where* $\Lambda \in \mathrm{SO}(4,\mathbb{Z})$
- ⋆ *parity, time reversal and charge conjugation*



4. Flavour symmetries

The vector $\mathrm{U}(N_\mathrm{f})$ symmetry is realized as in the continuum

However, the axial symmetries are broken

$$A_\mu = \bar{u}\gamma_\mu\gamma_5 d, \quad P = \bar{u}\gamma_5 d$$

$$\tfrac{1}{2}(\partial_\mu^* + \partial_\mu)\langle A_\mu(x)P(y)\rangle = (m_u + m_d)\langle P(x)P(y)\rangle$$

$$+ \text{contact terms} + \mathrm{O}(a)$$

Does not invalidate the lattice theory as a regularization of QCD, but can be an inconvenience in practice

5. Unitarity

... can be shown to be rigorously guaranteed!