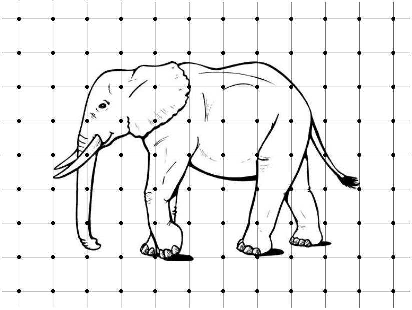
Introduction to lattice QCD (4)

Martin Lüscher, CERN Physics Department



School on Flavour Physics, Benasque, 13.–25. July 2008

How large is the lattice spacing?

Consider two-flavour QCD with quark mass matrix

$$M = \begin{pmatrix} m_0 & 0 \\ 0 & m_0 \end{pmatrix}, \qquad m_0: \text{ bare mass of the } u \text{ and } d \text{ quark}$$

The parameters in the lattice action are then

$$g_0,\, am_0 \quad \text{and} \quad \underset{\substack{\uparrow \\ \text{cancels out after substituting } \psi \to a^{-3/2}\psi,\ \overline{\psi} \to a^{-3/2}\overline{\psi}}}{a}$$

$\Rightarrow$ *the lattice spacing is a redundant parameter*

Now suppose the pion mass M_π is computed at some g_0, am_0

$$a^3 \sum_{\boldsymbol{x}} \langle (\bar{u}\gamma_5 d)(x)(\bar{d}\gamma_5 u)(0)\rangle \underset{x_0\to\infty}{\sim} \mathrm{e}^{-M_\pi x_0}$$

Since $x_0 = na$, $n = 1, 2, 3, \ldots$, one obtains aM_π not M_π

Similarly for the proton mass

$$\psi_p = \epsilon_{\alpha\beta\gamma}(d_\alpha^T \underset{\substack{\uparrow \\ \text{charge conjugation matrix}}}{C} \gamma_5 u_\beta)u_\gamma, \qquad a^3 \sum_{\boldsymbol{x}} \langle \psi_p(x)\overline{\psi}_p(0)\rangle \underset{x_0\to\infty}{\sim} \mathrm{e}^{-M_p x_0}$$

The computation thus yields

$$aM_\pi = \Phi_\pi(g_0, am_0)$$

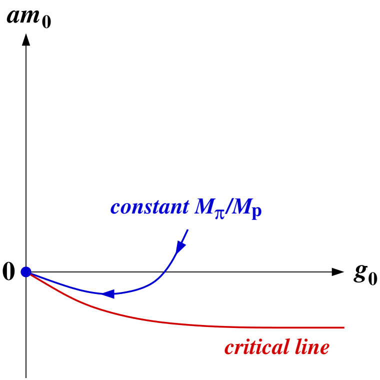
$$aM_p = \Phi_p(g_0, am_0)$$

Going to the continuum limit

$$M_\pi \ll a^{-1}, \qquad M_p \ll a^{-1}$$

amounts to taking $g_0, am_0 \to 0$

(QCD is “asymptotically free”)



Eventually we are only interested in the trajectory where

$M_\pi/M_p =$ physical value $\Rightarrow$ fixes am_0 as a function of g_0

Along this curve, the lattice spacing is then determined by

- setting $M_p = 938$ MeV
- and calculating

$$a = \frac{aM_p}{M_p} = 0.21 \times aM_p \text{ fm}$$

Note: other physical scales (F_π, M_ϱ, ...) can be used here

$\Rightarrow$ $a[\text{fm}]$ is slightly convention dependent!

Principal tools in LQCD

1. Strong-coupling expansion

Substitute

$$\psi \to a^{-2} m_0^{-1/2} \psi, \quad \overline{\psi} \to a^{-2} m_0^{-1/2} \overline{\psi}$$

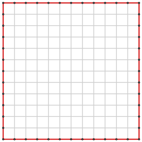
and let $m_0, g_0 \to \infty$

$$S = \sum_x \left\{ \overline{\psi}(x)\psi(x) + \frac{1}{m_0}\overline{\psi}(x) D_{\mathrm{w}} \psi(x) + \frac{1}{g_0^2} \sum_{\mu,\nu} P_{\mu\nu}(x) \right\}$$

$\Rightarrow$ the field variables at different points x decouple

$\Rightarrow$ theory is soluble in powers of $1/m_0$ and $1/g_0^2$

At $m_0 = \infty$, for example,

$$\Big\langle$$

$$\Big\rangle \sim \left(1/g_0^2\right)^{N_{\mathrm{plaq}}} = \exp\{-\sigma \times \text{area}\}$$

$\Rightarrow$ quark confinement

However, this limit is unphysical since

$$\sigma = \frac{1}{a^2}\ln(g_0^2) + \ldots$$

$$M_\pi = \mathrm{O}(1/a), \quad M_p = \mathrm{O}(1/a), \quad \text{etc.}$$

2. Numerical simulations

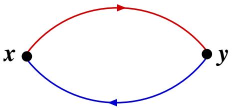
= Monte-Carlo integration of the (bosonized) functional integral

- Choose a finite lattice (64×32^3, for example)
- Generate a representative ensemble of gauge fields $\{U_1, U_2, \ldots, U_N\}$ using a Markov process
- $\Rightarrow \langle \mathcal{O} \rangle = \dfrac{1}{N} \displaystyle\sum_{k=1}^{N} \mathcal{O}[U_k] + \mathrm{O}(N^{-1/2})$

In the case of the pion propagator, for example,

$$\mathcal{O}[U] = \langle (\bar{u}\gamma_5 d)(x)(\bar{d}\gamma_5 u)(y)\rangle_{\mathrm{F}}$$

$$= -\mathrm{tr}\{\gamma_5 S(x,y;U)_{dd}\gamma_5 S(y,x;U)_{uu}\} =$$



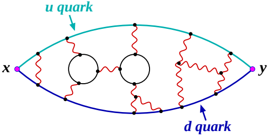
$\Rightarrow$ need to compute quark propagators for all $U = U_1, \ldots, U_N$

3. Weak-coupling expansion

For a, m_0 fixed and $g_0 \to 0$

$$\langle \phi(x_1) \dots \phi(x_n) \rangle \sim \sum_{k=0}^{\infty} g_0^{2k} C_k(x_1, \dots, x_n)$$

$$C_k(x_1, \dots, x_n) = \text{sum of Feynman diagrams}$$



Feynman rules derive from the lattice action

Lattice perturbation theory

For simplicity we omit the quarks. Then

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathrm{D}[U]\, \mathcal{O}\, \mathrm{e}^{-S_{\mathrm{G}}}$$

$$S_{\mathrm{G}} = \frac{1}{g_0^2} \sum_{\square} \mathrm{Re}\,\mathrm{tr}\{1 - U(\square)\} \geq 0$$

For $g_0 \to 0$ the minimal-action configurations dominate

$$S_{\mathrm{G}} = 0 \quad \Leftrightarrow \quad U = \text{pure gauge}$$

Perturbation theory = saddle-point expansion about these

In the vicinity of $U = 1$ we may set

$$U(x,\mu) = \exp\{g_0 a A_\mu(x)\}, \qquad A_\mu(x) = A_\mu^c(x) T^c$$

$$\mathrm{D}[U] \propto \prod_{x,\mu,c} dA_\mu^c(x) \left\{1 + \mathrm{O}(g_0^2)\right\}$$

Expansion of the action

$$S_{\mathrm{G}} = a^4 \sum_{x,\mu,\nu,c} \tfrac{1}{4}\left\{\partial_\mu A_\nu^c(x) - \partial_\nu A_\mu^c(x)\right\}^2 + \mathrm{O}(g_0)$$

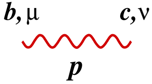
(lattice derivatives)

$\Rightarrow$ gluon propagator $= \dfrac{\delta_{\mu\nu}}{\hat{p}^2} +$ gauge terms

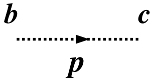
As in the continuum, we need to fix the gauge

$$S_{\mathrm{G}} \rightarrow S_{\mathrm{G}} + a^4 \sum_{x,c} \frac{\lambda_0}{2} \left\{ \sum_{\mu} \partial_\mu^* A_\mu^c(x) \right\}^2 + S_{\mathrm{FP}}$$

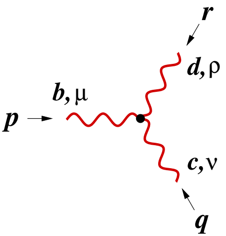
Feynman rules



$$= \quad \frac{\delta^{bc}}{\hat{p}^2} \left\{ \delta_{\mu\nu} - (1 - \lambda_0^{-1}) \frac{\hat{p}_\mu \hat{p}_\nu}{\hat{p}^2} \right\}$$



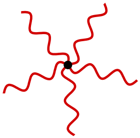
$$= \quad \frac{\delta^{bc}}{\hat{p}^2}$$

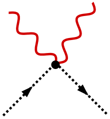


$$= \quad -ig_0 f^{bcd} \left\{ \delta_{\mu\nu} (\widehat{p-q})_\rho \cos(\tfrac{1}{2} a r_\mu) + \text{cyclic} \right\}$$

$$\vdots$$

On the lattice there are further vertices of order $a, a^2, \ldots$





$\cdots$

Remarks

- ⋆ Feynman integrands are rational functions in the sines and cosines of the momenta
- ⋆ Integrals are finite since $|p_\mu| \leq \pi/a$
- ⋆ Many more diagrams than in the continuum

Renormalization & continuum limit

At fixed external momenta, any l-loop diagram $\mathcal{J}$ can be expanded

$$\mathcal{J} \underset{a \to 0}{\sim} a^{-\omega} \sum_{n=0}^{\infty} a^n \sum_{k=0}^{l} c_{n,k} (\ln a)^k$$

ω : superficial degree of divergence

There is a power-counting theorem (Reisz 1988) and a rigorous proof of renormalizability to all orders

Renormalization

$$g_0 = Z_1 Z_3^{-3/2} g, \qquad \lambda_0 = Z_3^{-1} \lambda$$

$$G_0(p_1, \ldots, p_n) = Z_3^{n/2} G(p_1, \ldots, p_n) \qquad \text{(gluon n-point function)}$$

where

$$Z_k = 1 + \sum_{l=1}^{\infty} Z_{k,l}\, g^{2l}$$

$Z_{k,l} =$ polynomial in $\ln(a\mu)$ of degree l, $\qquad \mu :$ normalization mass

"Minimal subtraction" scheme: $Z_{k,l}$ has no constant term

Then, with properly chosen $Z_{k,l}$, the continuum limit

$$\lim_{a \to 0} G(p_1, \ldots, p_n)$$

can be taken order by order in g

Remarks

- Up to finite renormalizations, the n-point functions $\lim_{a\to 0} G(p_1, \ldots, p_n)$ are <u>universal</u>
- Confirms that LQCD is just a regularization of QCD
- The leading lattice corrections are of O(a)

Renormalization group

For μ, g fixed

$$g_0 = Z_1 Z_3^{-3/2} g = \text{function of } a$$

$$a\frac{\partial g_0}{\partial a} = b_0 g_0^3 + b_1 g_0^5 + \ldots, \qquad b_0 = \frac{1}{(4\pi)^2}\left\{11 - \tfrac{2}{3}N_\mathrm{f}\right\}$$

$\Rightarrow$ the lattice coupling vanishes in the continuum limit

$$g_0^2 \underset{a\to 0}{\sim} -\frac{1}{b_0 \ln(a\mu)} + \ldots$$

$\Rightarrow$ *studying the continuum limit in perturbation theory is meaningful!*