QCD at finite T and μ (M. Laine, Benasque, July 18–21, 2008)

Lecture 1. Static thermodynamics

Exercise 1: "Where is physics hidden?"

Show that

$$T \sum_{n=-\infty}^{\infty} \int_{\mathbf{p}} \frac{1}{(2\pi T n)^2 + \mathbf{p}^2 + m^2} = \int_{\mathbf{p}} \frac{1}{E} \left[\frac{1}{2} + n_{\rm B}(E) \right] ,$$

where $E \equiv \sqrt{\mathbf{p}^2 + m^2}$ and $n_{\rm B}$ is the Bose-Einstein distribution function, $n_{\rm B}(E) \equiv 1/(e^{\beta E} - 1)$. [The term $\frac{1}{2E}$ corresponds to the vacuum result, $\int \frac{{\rm d}p_0}{2\pi} \frac{1}{p_0^2 + E^2}$, the rest to thermal corrections.]

Exercise 2: "Another effect of 3d confinement".

Consider the weak-coupling expansion for the QCD pressure, $p(T)$, in the effective theory framework. At which order would you expect a non-perturbative coefficient to first appear?

Lecture 2. Real-time observables

Exercise 3: "Another angle on diffusive motion".

Solve the 1-dimensional diffusion equation

$$\partial_t n(t,x) = D\, \partial_x^2 n(t,x) ,$$

with the initial condition $n(0,x) = \delta(x)$, and show that $\langle x^2(t) \rangle = 2Dt$. [Hint: write $\delta(x) = \lim_{\sigma\to 0} \frac{1}{\sigma\sqrt{\pi}} \exp(-\frac{x^2}{\sigma^2})$].

Exercise 4: "Extreme form of the transport peak".

Defining $\tilde\Delta_E(\omega_n) = \int_0^\beta {\rm d}\tau\, e^{i\omega_n \tau}\, \Delta_E(\tau)$, and making use of the spectral representation

$$\tilde\Delta_E(\omega_n) = \int_{-\infty}^{\infty} \frac{{\rm d}\omega}{\pi} \frac{\rho(\omega)}{\omega - i\omega_n} ,$$

argue that if $\Delta(\tau) = \Delta(0)\ \forall\ \tau$, then $\frac{\rho(\omega)}{\omega} = \pi\beta\Delta_E(0)\delta(\omega)$.

Lecture 3. Finite baryon density

Exercise 5: "Sum for particle number density".

In analogy with Exercise 1, it can be shown that $T \sum_{\omega_n^{\rm f}} \frac{1}{(\omega_n - i\mu)^2 + E^2} = \frac{1}{2E}[1 - n_{\rm F}(E-\mu) - n_{\rm F}(E+\mu)]$, where $n_{\rm F}(E) \equiv 1/(e^{\beta E} + 1)$. Starting from here (or elsewhere), show that

$$T \sum_{\omega_n^{\rm f}} \frac{i(\omega_n - i\mu)}{(\omega_n - i\mu)^2 + E^2} = -\frac{1}{2}\, [n_{\rm F}(E-\mu) - n_{\rm F}(E+\mu)] .$$

Exercise 6: "Triangle integral at finite μ".

Show that

$$\int \frac{{\rm d}^4 P}{(2\pi)^4} \frac{p_0 - i\mu}{[(p_0 - i\mu)^2 + \mathbf{p}^2]^2} = \frac{i\mu}{8\pi^2} .$$