QCD/QED NONRELATIVISTIC BOUND STATES

FOR THE PRACTITIONER IN THE WEAK COUPLING REGIME

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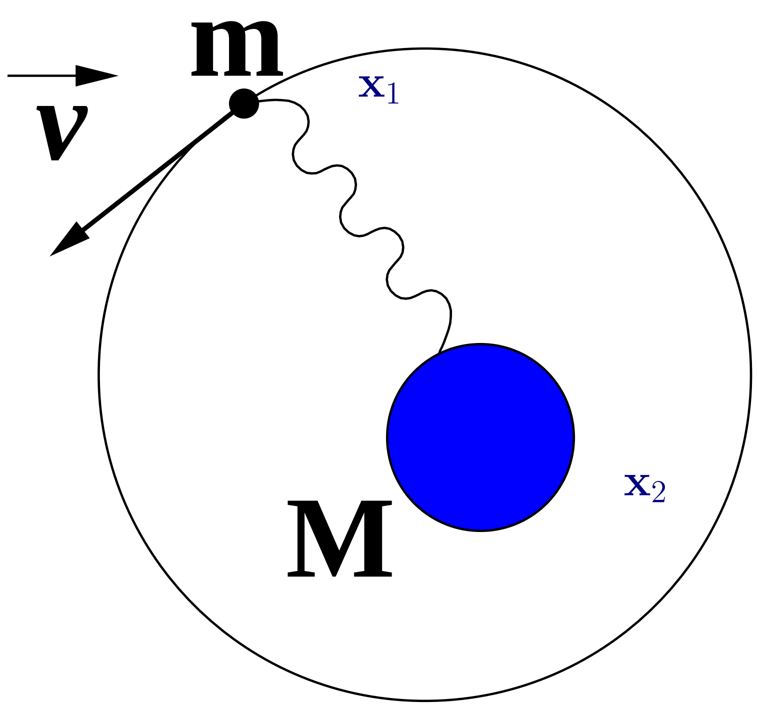

$$H = \frac{\mathbf{p}^2}{2m} + V(r) \qquad V(r) = -\frac{Z_1 Z_2 \alpha}{r}$$

$$v \sim \alpha \ll 1$$

$$\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2 \qquad \mathbf{X} = \frac{m}{m+M}\mathbf{x}_1 + \frac{M}{m+M}\mathbf{x}_2$$

Scales: m, $m\alpha$, $m\alpha^2$, ...

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1) NR bound state systems:

- **QED: positronium, Hydrogen-like/exotic atoms, atomic physics ...**
- **QCD: Heavy Quarkonium (Υ, J/ψ, B_c ...), ...**

2) Q-$\bar{Q}$ production near threshold (t-$\bar{t}$ at NLC).

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Tool: Effective Field Theories $\equiv$ **Factorization**

Why?: There is a hierarchy of different scales (hard, soft and ultrasoft).

$$\boxed{m >> mv >> mv^2\,, \quad (\Lambda_{QCD})}$$

EFTs are especially useful in these situations.

1) Perturbative calculations much easier and systematic.

2) Nonperturbative information is parameterized in a model independent way.

3) Power counting.

Index

1) Introduction

A) Kinematical regime: s plane

B) Warming up: Matching (NR)QFT with a Quantum mechanics description

2) Matching QCD to NRQCD

Relativistic Feynman diagrams $\leftarrow$

3) Matching NRQCD to pNRQCD (getting the potential)

Non-Relativistic (HQET-like) Feynman diagrams $\leftarrow$

4) Observable: Spectrum

A) Quantum mechanics perturbation theory $\leftarrow$

B) Ultrasoft loops (lamb shift) $\leftarrow$

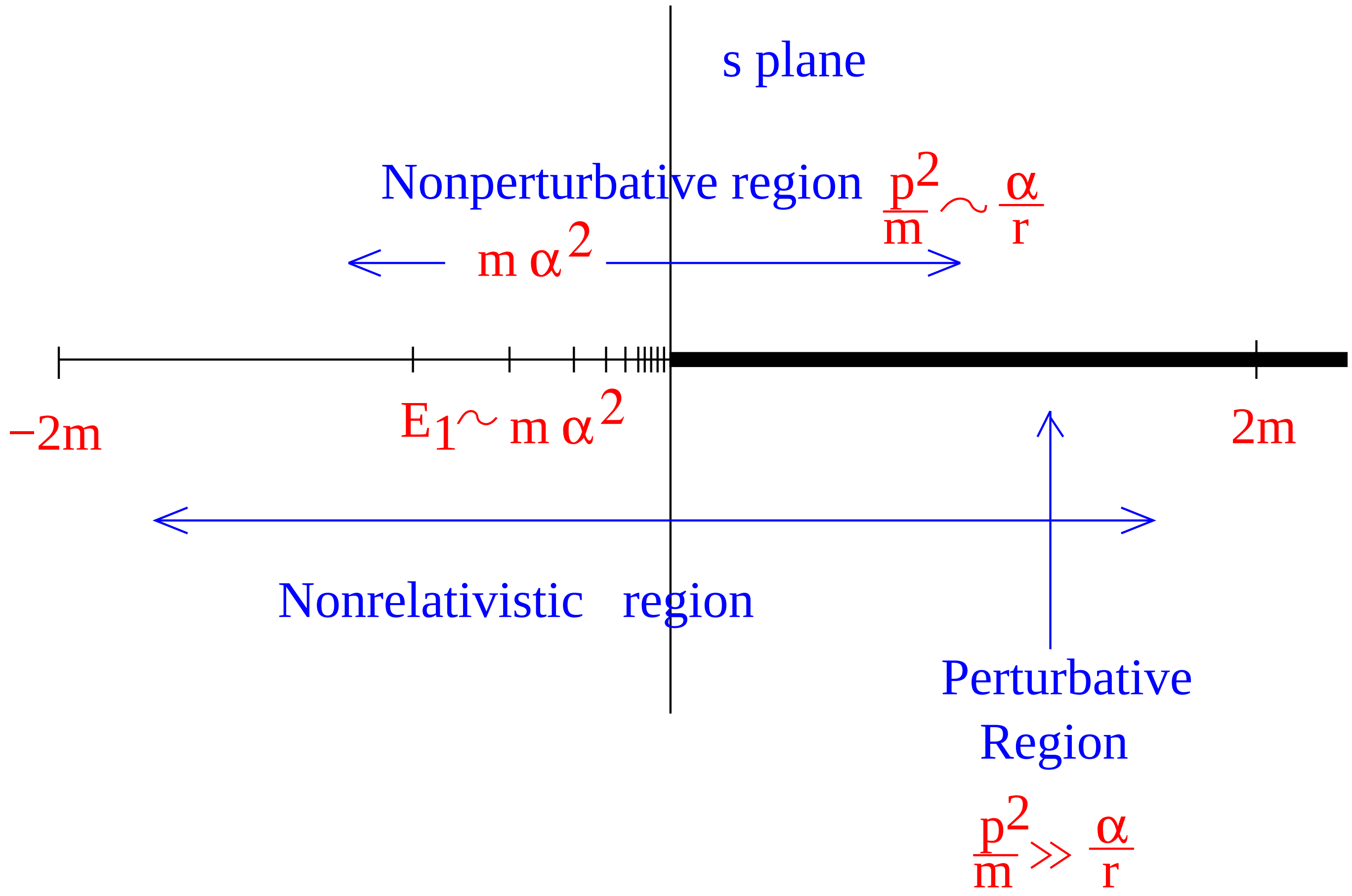
5) Heavy quarkonium spectrum

- **Non-perturbative effects**

6) Positronium spectrum

Exercise: Compute the $O(m\alpha^5 \ln \alpha)$ correction to the spectrum

1.a): kinematical situation



1st approximation: ∞ number of NR (bound states) free particles
Unusual situation in EFTs. What we will get is somewhat unusual from the EFT point of view.

$$\mathcal{L}_{(n)} = \psi_n^\dagger(\mathbf{X},t)\left(i\partial_0 + \frac{\boldsymbol{\nabla}_X^2}{2M} - E_n + i\epsilon\right)\psi_n(\mathbf{X},t)$$

Our case

$$\boxed{\mathcal{L} = \Sigma_n\, \psi_n^\dagger(\mathbf{X},t)(i\partial_0 + \frac{\boldsymbol{\nabla}_X^2}{2M} - E_n + i\epsilon)\psi_n(\mathbf{X},t)}$$

$\psi_n(\mathbf{X})$ **represents the quark-antiquark bound state**

Path integral formulation:

$$Z = \int \Pi D\psi_n^\dagger D\psi_n e^{i\int d^4X(\mathcal{L}+\psi_n^\dagger\eta_n+\eta_n^\dagger\psi_n)}$$

Connection with quantum mechanics (?):

particle-antiparticle wave function: $\Psi(\mathbf{X},\mathbf{x}) = \Psi_{\mathbf{x}}(\mathbf{X})$

$\mathbf{X}$ = **Center of mass coordinate**

$\mathbf{x}$ = **relative coordinate**

Ansatz: Promote $\Psi(\mathbf{X},\mathbf{x})$ **to a field**

$$Z = \int D\Psi(X,x)^\dagger D\Psi(X,x) e^{i\int d^4X d^3\mathbf{x}(\mathcal{L}+\Psi^\dagger J(X,x)+J^\dagger(X,x)\Psi)}$$

$$\mathcal{L} = \Psi^\dagger(i\partial_0 - \hat{h} + i\epsilon)\Psi$$

where

$$\hat{h} = \hat{h}_{\mathbf{X}} + \hat{h}_{\mathbf{x}}\,, \quad \hat{h}_{\mathbf{X}} = -\frac{\boldsymbol{\nabla}_X^2}{2M}\,, \quad \hat{h}_{\mathbf{x}} = -\frac{\boldsymbol{\nabla}_x^2}{2\mu_r} + V(\mathbf{x})$$

We have traded E_n **for** $\mathbf{V}(\mathbf{x})$**. The point is that we will able to relate** $\mathbf{V}(\mathbf{x})$ **with some Green functions in the underlying theory and in some kinematical regime to compute it perturbatively.**

Change of basis: $\Psi(\mathbf{X},\mathbf{x}) = \sum_n \phi_n(\mathbf{x})\psi_n(\mathbf{X})$**,**
where $\phi_n(\mathbf{x})$ **is a function and** $\psi_n(\mathbf{X})$ **a field and**

$$\hat{h}_{\mathbf{x}}\phi_n(\mathbf{x}) = E_n\phi_n(\mathbf{x})$$

$$Z = N\int \Pi D\psi_n^\dagger D\psi_n e^{i\int d^4X(\sum_n \psi_n^\dagger(i\partial_0+\frac{\boldsymbol{\nabla}_X^2}{2M}-E_n+i\epsilon)\psi_n+\int d^3x\sum_n(\phi_n^*\psi_n^\dagger J+J^\dagger\phi_n\psi_n))}$$

$$\int d^3x\phi_{n'}^*\hat{h}_{\mathbf{x}}\phi_n = \delta_{nn'}E_n$$

$$\int d^3x\phi_{n'}^*\phi_n = \delta_{nn'}$$

We have closed the connection between both formulations
Infinite number of states $\leftrightarrow$ Integro-diferential equation (Schrödinger equation)

NR Effective Field Theories

Our aim is to provide a systematic method to deal with NR bound state systems. We will introduce a hierarchy of EFTs when sequentially integrating out each scale (only one scale in each step, strong simplification).

$$
\begin{array}{ll}
\text{QCD} & \\
\Big\downarrow & \text{Integrating out the hard scale (m)} \\
\text{NRQCD} & \quad \text{Caswell, Lepage} \\
\Big\downarrow & \text{Integrating out the soft scale (mv)} \\
\text{pNRQCD} & \quad \text{Soto, Pineda}
\end{array}
$$

$$
\left.\begin{array}{l}
\left(i\partial_0 - \dfrac{\mathbf{p}^2}{2m} - V_0(r)\right)\Phi(\mathbf{r}) = 0 \\
+\text{corrections to the potential} \\
+\text{interaction with other low} \\
\quad \text{energy degrees of freedom}
\end{array}\right\} \text{potential NRQCD} \qquad E \sim mv^2
$$

In the perturbative case the starting point is $V_0 = -C_f \dfrac{\alpha}{r}$.

NRQCD: the scale m

- **Degrees of freedom**
- **Symmetries**
- **Cutoff**

NRQCD has an ultraviolet cutoff Λ such that $m \gg \Lambda$ and larger than any other dynamical scale in the problem. $\Psi = \psi + \chi$

$$\mathcal{L}_{NRQCD} = \bar{\Psi} i\gamma^0 D_0 \Psi + \bar{\Psi}\{\frac{\mathbf{D}^2}{2m} + c_F\, g\frac{\mathbf{\Sigma}\cdot\mathbf{B}}{2m} + c_D\, g\frac{\gamma^0\,(\mathbf{D}\cdot\mathbf{E} - \mathbf{E}\cdot\mathbf{D})}{8m^2}$$
$$+ic_S\, g\frac{\gamma^0\mathbf{\Sigma}\cdot(\mathbf{D}\times\mathbf{E} - \mathbf{E}\times\mathbf{D})}{8m^2} + \frac{\mathbf{D}^4}{8m^3}\}\Psi$$
$$-\frac{1}{4}c_1 F_{\mu\nu}F^{\mu\nu} + \frac{c_2}{m^2} g F_{\mu\nu} D^2 g F^{\mu\nu} + \frac{c_3}{m^2} g^3 f_{ABC} F^A_{\mu\nu} F^B_{\mu\alpha} F^C_{\nu\alpha}$$

$$\delta\mathcal{L}_{NRQCD} = \frac{d_{ss}}{m_1 m_2}\psi_1^\dagger\psi_1\chi_2^\dagger\chi_2 + \frac{d_{sv}}{m_1 m_2}\psi_1^\dagger\boldsymbol{\sigma}\psi_1\chi_2^\dagger\boldsymbol{\sigma}\chi_2$$
$$+\frac{d_{vs}}{m_1 m_2}\psi_1^\dagger \mathrm{T}^a\psi_1\chi_2^\dagger \mathrm{T}^a\chi_2 + \frac{d_{vv}}{m_1 m_2}\psi_1^\dagger \mathrm{T}^a\boldsymbol{\sigma}\psi_1\chi_2^\dagger \mathrm{T}^a\boldsymbol{\sigma}\chi_2\,.$$

Lepage, Caswell, Thacker

Matching QCD to NRQCD: the scale m

$c_i = 1+O(\alpha_s)$, $d_1 = 1+O(\alpha_s^2)$ **(relevant α_s at low energies)**, d_2 ,d_3, d_{ss}, ...$= O(\alpha_s)$.

Typically, $c_i \sim 1+\alpha_s\left(A\log\frac{m}{\mu}+B\right)$ $d_i \sim \alpha_s\left(1+\alpha_s\left(A\log\frac{m}{\mu}+B\right)\right)$

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Analytical expansion over the three-momentum and residual energy in the integrand before the integration is made in both the full and the effective theory.

QCD

$$\int d^4q f(q, m, |\mathbf{p}|, E) = \int d^4q f(q, m, 0, 0) + O\left(\frac{E}{m}, \frac{|\mathbf{p}|}{m}\right)$$

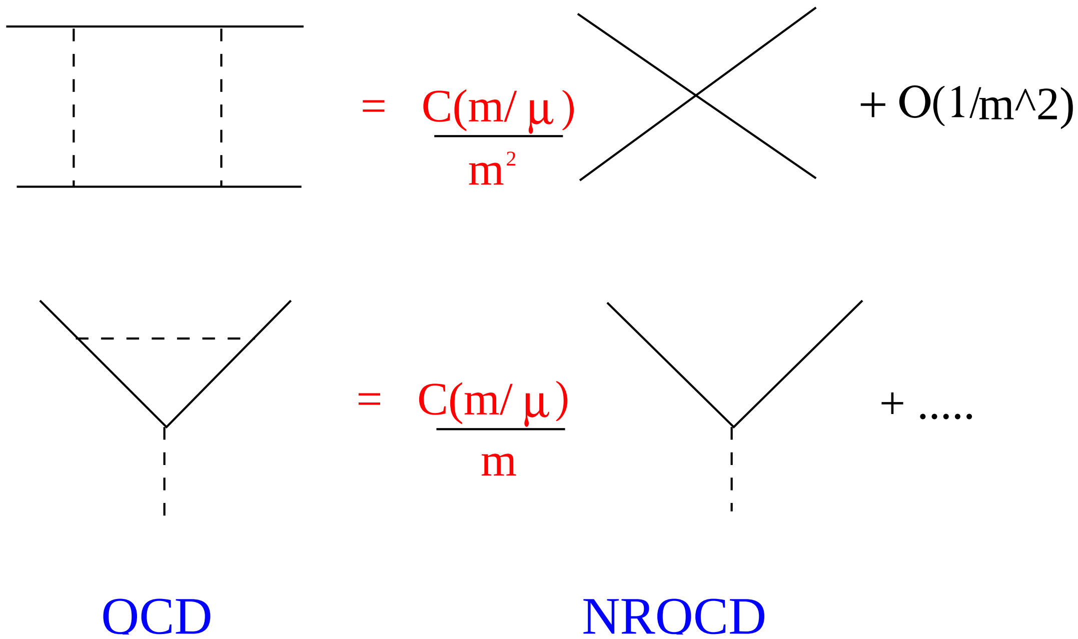
NRQCD

$$\int d^4q f(q, |\mathbf{p}|, E) = \int d^4q f(q, 0, 0) = 0\,!!$$

Dimensional regularization. The computation of loops in the effective theory just gives zero.

Final rules:

- **QCD tree level Feynman diagrams $\rightarrow$ non-relativistic reduction. They give the leading contribution to the NRQCD matching coefficients (this is equivalent to perform a Foldy-Wouthysen transformation for the bi-linear piece of the Lagrangian but not for four-fermion operators due to the annihilation terms).**
- **One matches loops in QCD with only one scale (the mass) to tree level diagrams in NRQCD.**
- **Matching to some given order in α and $1/m$, i.e. to $O(\alpha^n/m^s)$.**



Manohar; Soto, Pineda

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Next step:
Soto, Pineda→**How would we like the effective theory for Q–$\bar{Q}$ systems near threshold to be?** We do not want to describe all the degrees of freedom included in NRQCD, but rather only those with **US energy**. Moreover, we want to get a closer connection with a Schrödinger-like formulation for these systems (also, eventually, in the non-perturbative regime → **potential models**). We will call **potential NRQCD** this new effective theory.

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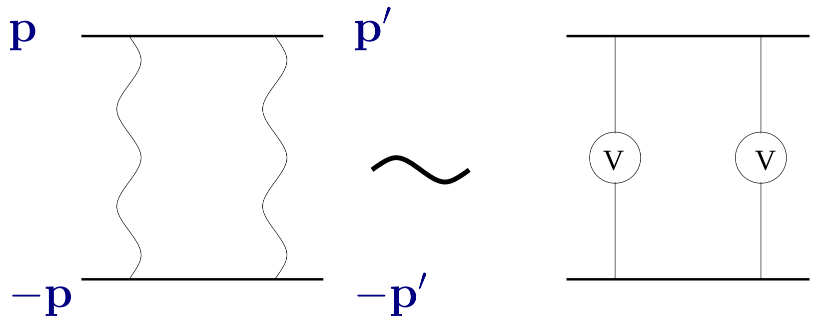
Beneke, Smirnov→ threshold expansion. Rigorous diagrammatic study of the (perturbative) momentum regions: hard, soft, potential, ultrasoft.
hard: particles with $E \sim |\mathbf{p}| \sim m$ → NRQCD
soft: particles with $E \sim |\mathbf{p}| \sim mv$ → pNRQCD
potential: particles with $E \sim mv^2$, $|\mathbf{p}| \sim mv$
ultrasoft: particles with $E \sim |\mathbf{p}| \sim mv^2$
Implementation of pNRQCD through the threshold expansion (integrating out potential gluons and all soft particles): Beneke, Smirnov; Kniehl, Penin; Kniehl, Penin, Smirnov, Steinhauser

Physical Picture



$$I \sim \int \frac{d^4 q}{(2\pi)^4} V(p,q) \frac{1}{E/2 + q^0 - \mathbf{q}^2/2m + i\epsilon} \frac{1}{E/2 - q^0 - \mathbf{q}^2/2m + i\epsilon} V(q,p')$$

$$V(p,q) \sim \frac{1}{(p-q)^2}$$

Counting (different possibilities):

A) $E \sim mv^2 \quad p^0, p'^0 \sim q^0 \sim |\mathbf{p}| \sim \mathbf{q} \sim \mathbf{p}' \sim mv$

Static propagator:

$$\frac{i}{q^0 + i\epsilon} \longrightarrow I \sim \delta V(\mathbf{p} - \mathbf{q})$$

B) $E \sim p^0, p'^0 \sim q^0 \sim mv^2 \quad |\mathbf{p}| \sim \mathbf{q} \sim \mathbf{p}' \sim mv$

Nonrelativistic propagator:

$$\frac{i}{q^0 - \mathbf{q}^2/(2m) + i\epsilon}$$

Leading contribution:

$$I \sim \int \frac{d^3\,\mathbf{q}}{(2\pi)^3} V(\mathbf{p},\mathbf{q}) \frac{1}{E - \mathbf{q}^2/m + i\epsilon} V(\mathbf{q},\mathbf{p}') \,.$$

But this is nothing but the usual NR Quantum Mechanics!!, I **can be written as**

$$I \sim \langle \mathbf{p}|\hat{V} \frac{1}{E - \hat{\mathbf{p}}^2/m + i\epsilon} \hat{V}|\mathbf{p}'\rangle$$

In fact, this can be done to any order considering ladder loops. Therefore, we obtain that the QFT amplitude can be written (within this approximation) as the Quantum Mechanical one

$$i\mathcal{A} = -i\langle \mathbf{p}| \left(\hat{V} + \hat{V} \frac{1}{E - \hat{\mathbf{p}}^2/m + i\epsilon} \hat{V} + \ldots \right) |\mathbf{p}'\rangle \,.$$

For the bound state $V \sim E \sim mv^2$**,** $p \sim mv$ **and one has to sum up the whole series.**

pNRQCD: the scale mv

The integration of the mv scale gives rise to potential terms. The Lagrangian is local in time but not in space.

Playing with the scales:

1) $mv \sim \Lambda_{QCD}$

2) $mv \gg \Lambda_{QCD} \gg mv^2$

3) $mv \gg mv^2 \sim \Lambda_{QCD}$

4) $mv \gg mv^2 \gg \Lambda_{QCD}$

Loosely speaking, when to trust the perturbative calculation and the size of NP corrections.

$\boxed{mv \gg \Lambda_{QCD}}$ **($\Upsilon(1S)$, t-$\bar{t}$, b-$\bar{b}$ sum rules)**

- **Degrees of freedom**
- **symmetries**
- **cutoff**

pNRQCD has two ultraviolet cut-offs, ν_{us} and ν_p. ν_{us} fulfils the relation $\mathbf{p}^2/m \ll \nu_{us} \ll |\mathbf{p}|$ and is the cut-off of the energy of the quarks, and of the energy and the momentum of the gluons. ν_p fulfils $|\mathbf{p}| \ll \nu_p \ll m$ and is the cut-off of the relative momentum of the quark–antiquark system, $\mathbf{p}$.

Power counting/scales

Scales: m**,** p**,** $1/r$**,** $\Lambda_{mp} = \{\Lambda_{QCD}, mv^2, ...\}$

Dimensionless quantities:

$$\boxed{\frac{p}{m}, \ \alpha_s, \ \frac{1}{m\,r}, \ \boxed{\Lambda_{mp} r} \ll 1}$$

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$$L_{pNRQCD} = L'_{NRQCD} + \int\int d^3x_1 d^3x_2 \psi(x_1)\chi_c(x_2)V(x_1 - x_2)\psi^\dagger(x_1)\chi_c^\dagger(x_2)$$

L'_{NRQCD}**, gluons multipole expanded (only ultrasoft gluons).**

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L'_{NRQCD}, **gluons multipole expanded (only ultrasoft gluons).**

$$V_s^{(0)} \equiv -C_F \frac{\alpha_{V_s}}{r}.$$

$$\frac{V_s^{(1)}}{m} \equiv -\frac{C_F C_A D_s^{(1)}}{2mr^2}.$$

$$\frac{V_s^{(2)}}{m^2} = -\frac{C_F D_{1,s}^{(2)}}{2m^2}\left\{\frac{1}{r}, \mathbf{p}^2\right\} + \frac{C_F D_{2,s}^{(2)}}{2m^2}\frac{1}{r^3}\mathbf{L}^2 + \frac{\pi C_F D_{d,s}^{(2)}}{m^2}\delta^{(3)}(\mathbf{r})$$
$$+ \frac{4\pi C_F D_{S^2,s}^{(2)}}{3m^2}\mathbf{S}^2\delta^{(3)}(\mathbf{r}) + \frac{3C_F D_{LS,s}^{(2)}}{2m^2}\frac{1}{r^3}\mathbf{L}\cdot\mathbf{S} + \frac{C_F D_{S_{12},s}^{(2)}}{4m^2}\frac{1}{r^3}S_{12}(\hat{\mathbf{r}}),$$

where $S_{12}(\hat{\mathbf{r}}) \equiv 3\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_1\,\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$ **and** $\mathbf{S} = \boldsymbol{\sigma}_1/2 + \boldsymbol{\sigma}_2/2$.

To go to the wave function description one has to project to the quark-antiquark sector.

$$\int d^3\mathbf{x}_1 d^3\mathbf{x}_2 \Psi(\mathbf{x}_1,\mathbf{x}_2)\psi(x_1)\chi_c(x_2)|0\rangle$$

$$H\int d^3\mathbf{x}_1 d^3\mathbf{x}_2 \Psi(\mathbf{x}_1,\mathbf{x}_2)\psi^\dagger(\mathbf{x}_1)\chi_c^\dagger(\mathbf{x}_2)|0\rangle = \int d^3\mathbf{x}_1 d^3\mathbf{x}_2 (\hat{h}\Psi(\mathbf{x}_1,\mathbf{x}_2))\psi^\dagger(\mathbf{x}_1)\chi_c^\dagger(\mathbf{x}_2)|0\rangle$$

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For QED

$$\begin{aligned} L_{pNRQED} &= \int d^3\mathbf{x}_1 d^3\mathbf{x}_2 \Psi^\dagger(\mathbf{x}_1,\mathbf{x}_2)\left(iD_0 + \frac{\mathbf{D}^2_{\mathbf{x}_1}}{2m_1} + \frac{\mathbf{D}^2_{\mathbf{x}_2}}{2m_2} - V(\mathbf{x},\mathbf{p})\right)\Psi(\mathbf{x}_1,\mathbf{x}_2) \\ &= \int d^3\mathbf{x}_1 d^3\mathbf{x}_2 \Psi^\dagger(\mathbf{x}_1,\mathbf{x}_2)\left(i\partial_0 + \frac{\boldsymbol{\nabla}^2_{\mathbf{x}}}{m} + \frac{\boldsymbol{\nabla}^2_{\mathbf{X}}}{4m}\right. \\ &\quad \left. - e\mathbf{x}\cdot\boldsymbol{\nabla} A_0(\mathbf{X}) - 2ie\frac{\mathbf{A}(\mathbf{X})\cdot\boldsymbol{\nabla}_{\mathbf{x}}}{m} - V(\mathbf{x},\mathbf{p})\right)\Psi(\mathbf{x}_1,\mathbf{x}_2) \end{aligned}$$

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New fields: Singlet S, Octet (O) and US gluons.
Gauge transformation:

$$S(\mathbf{x},\mathbf{X},t) \to S(\mathbf{x},\mathbf{X},t)\,, \qquad O(\mathbf{x},\mathbf{X},t) \to g(\mathbf{X},t)O(\mathbf{x},\mathbf{X},t)g^{-1}(\mathbf{X},t)\,.$$

Field Redefinitions.

$$\Psi(\mathbf{x}_1,\mathbf{x}_2) = \phi(\mathbf{x}_1,\mathbf{x}_2)S(\mathbf{x},\mathbf{X}) + \phi(\mathbf{x}_1,\mathbf{X})O(\mathbf{x},\mathbf{X})\phi(\mathbf{X},\mathbf{x}_2)$$

$$\phi(\mathbf{y},\mathbf{x},t) \equiv \mathrm{P}\exp\left\{ig\int_0^1 ds\,(\mathbf{y}-\mathbf{x})\cdot\mathbf{A}(\mathbf{x}-s(\mathbf{x}-\mathbf{y}),t)\right\}$$

pNRQCD Lagrangian at $O(r)$

$$\begin{aligned}
\mathcal{L}_{pNRQCD} &= \mathrm{Tr}\Big\{S^\dagger\left(i\partial_0 - V_s^{(0)}(\mathbf{x})\right)S + O^\dagger\left(iD_0 - V_o^{(0)}(\mathbf{x})\right)O\Big\} \\
&+ gV_A(\mathbf{x})\mathrm{Tr}\left\{O^\dagger\mathbf{x}\cdot\mathbf{E}\,S + S^\dagger\mathbf{x}\cdot\mathbf{E}\,O\right\} + g\frac{V_B(\mathbf{x})}{2}\mathrm{Tr}\left\{O^\dagger\mathbf{x}\cdot\mathbf{E}\,O + O^\dagger O\mathbf{x}\cdot\mathbf{E}\right\} \\
&- \mathrm{Tr}\Big\{S^\dagger\left(\frac{\mathbf{p}^2}{m} + \sum_n \frac{V_s^{(n)}(\mathbf{x})}{m^n}\right)S - O^\dagger\left(\frac{\mathbf{p}^2}{m} + \sum_n \frac{V_o^{(n)}(\mathbf{x})}{m^n}\right)O\Big\},
\end{aligned}$$

Interpolating fields:

$$Q_2^\dagger(\mathbf{x}_2,t)\phi(\mathbf{x}_2,\mathbf{x}_1;t)Q_1(\mathbf{x}_1,t) = Z_s^{1/2}(\mathbf{x})S(\mathbf{X},\mathbf{x},t)$$

$$Q_2^\dagger(x_2)\phi(\mathbf{x}_2,\mathbf{X};t)T^a\phi(\mathbf{X},\mathbf{x}_1;t)Q_1(x_1) = Z_o^{1/2}(\mathbf{x})O^a(\mathbf{X},\mathbf{x},t)$$

Matching NRQCD to pNRQCD

Same idea than in NRQCD. Expansion in the scales that are left in the effective theory. We integrate out the scale k (transfer momentum between the quark and antiquark).
Analytical expansion of 1/m (and therefore p) and E before the integration is made in both the full and the effective theory. Effectively HQET-like rules (HQET quark propagator).
NRQCD

$$\int d^4q f(q,k,|\mathbf{p}|,E) = \int d^4q f(q,k,0,0) + O\left(\frac{E}{k},\frac{|\mathbf{p}|}{k}\right) \qquad \text{potentials}$$

pNRQCD

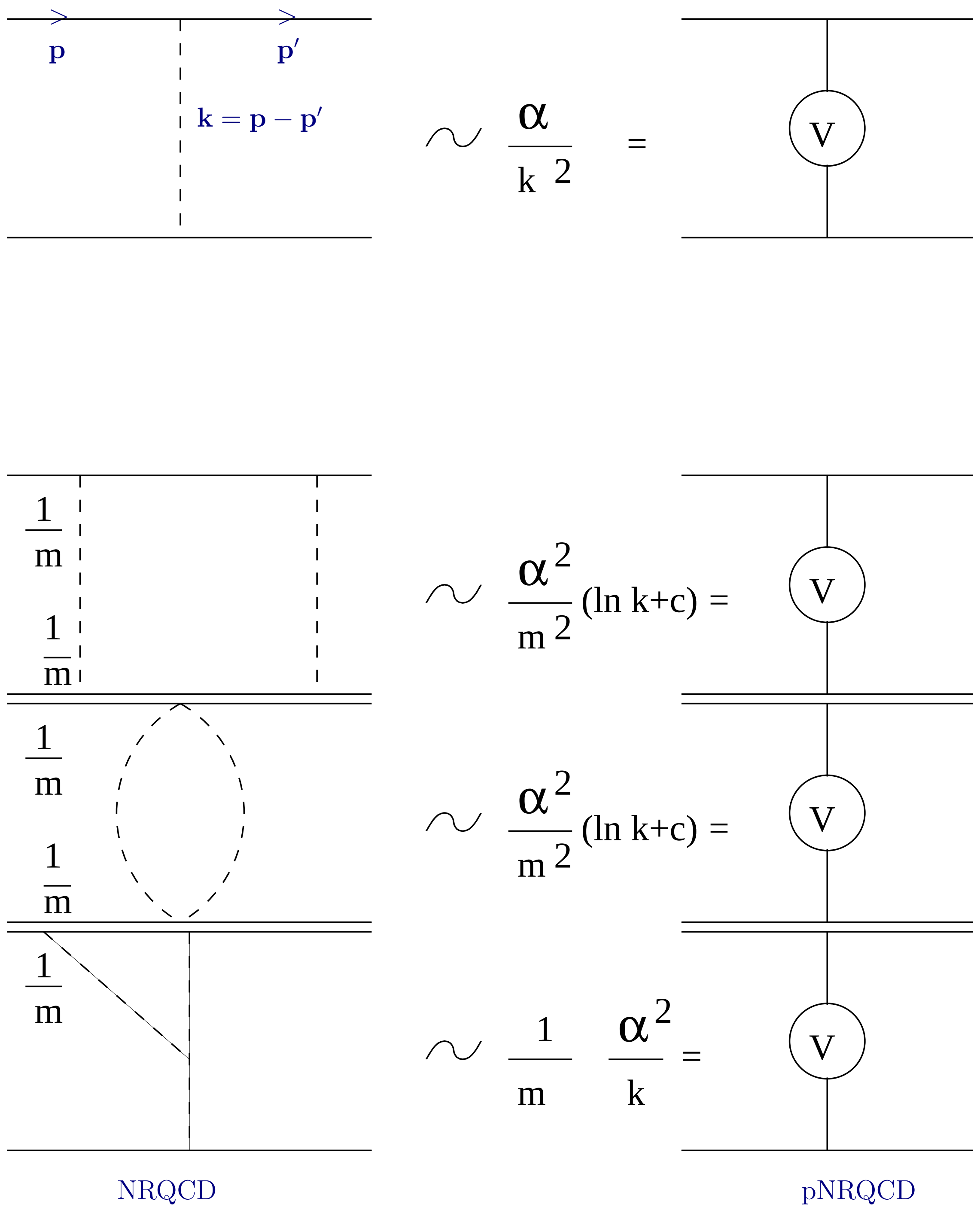
$$\int d^4q f(q,|\mathbf{p}|,E) = \int d^4q f(q,0,0) = 0\,!!$$

Dimensional regularization. The computation in the effective theory just gives zero.

Final rules:

- **NRQCD tree level Feynman diagrams $\longrightarrow$ non-relativistic reduction. They give the leading contribution to the potentials.**
- **One matches loops in NRQCD with only one scale (k) to tree level diagrams in pNRQCD (potentials).**
- **Matching to some given order in α and $1/m$, i.e. $O(\alpha^n/m^s)$.**

It can also be understood within the threshold expansion: integrating out potential gluons and soft particles.



p
p′
k = p − p′
~ α/k 2 =
V
1/m
1/m
~ α 2/m 2 (ln k+c) =
V
1/m
1/m
~ α 2/m 2 (ln k+c) =
V
1/m
~ 1/m α 2/k =
V
NRQCD
pNRQCD

$$\alpha_{V_s} = \alpha_{\rm s}(r)\left\{1 + (a_1 + 2\gamma_E\beta_0)\frac{\alpha_{\rm s}(r)}{4\pi} + \left[\gamma_E\left(4a_1\beta_0 + 2\beta_1\right) + \left(\frac{\pi^2}{3} + 4\gamma_E^2\right)\beta_0^2 + a_2\right]\frac{\alpha_{\rm s}^2(r)}{16\,\pi^2} \right.$$
$$\left. + \frac{C_A^3}{12}\frac{\alpha_{\rm s}^3}{\pi}\ln\mu r\right\}, \qquad (a_2,\ \text{Schroeder},\ \text{Peter})$$
$$D_s^{(1)} = \alpha_{\rm s}^2(r)\left\{1 + \frac{2}{3}(4C_F + 2C_A)\frac{\alpha_s}{\pi}\ln\mu r\right\},$$
$$D_{1,s}^{(2)} = \alpha_{\rm s}(r)\left\{1 + \frac{4}{3}C_A\frac{\alpha_s}{\pi}\ln\mu r\right\},$$
$$D_{2,s}^{(2)} = \alpha_{\rm s}(r),$$
$$D_{d,s}^{(2)} = \alpha_{\rm s}(r)\left\{1 + \frac{\alpha_s}{\pi}\left(\frac{2C_F}{3} + \frac{17C_A}{3}\right)\ln mr + \frac{16}{3}\frac{\alpha_s}{\pi}\left(\frac{C_A}{2} - C_F\right)\ln\mu r\right\},$$
$$D_{S^2,s}^{(2)} = \alpha_s(r)\left\{1 - \frac{7C_A}{4}\frac{\alpha_s}{\pi}\ln mr\right\},$$
$$D_{LS,s}^{(2)} = \alpha_s(r)\left\{1 - \frac{2C_A}{3}\frac{\alpha_s}{\pi}\ln mr\right\},$$
$$D_{S_{12},s}^{(2)} = \alpha_s(r)\left\{1 - C_A\frac{\alpha_s}{\pi}\ln mr\right\}.$$

Previous work: Gupta, Radford, Repko; Titard, Ynduráin

Logs: Brambilla, Soto, Vairo, Pineda; Kniehl, Penin; Hoang, Manohar Stewart

Finite terms $1/m$ and $1/m^2$ potentials: Kniehl, Penin, Smirnov, Steinhauser

Renormalization group improved expressions: Soto, Pineda; Pineda

Another possibility: to perform directly the matching to singlet and octet fields, one naturally ends up with Wilson loops. **See Vairo's lectures. Expansion in $1/M$.** HQET can be used in the matching (static sources).

A) **Wilson loop formalism. Suitable if we can not work perturbatively. Example, the computation of the following Green function in both theories**

$$\langle 0|Q_2^\dagger(x_2)\phi(x_2,x_1)Q_1(x_1)Q_1^\dagger(y_1)\phi(y_1,y_2)Q_2(y_2)|0\rangle,$$

NRQCD

$$\delta^3(\mathbf{x}_1-\mathbf{y}_1)\delta^3(\mathbf{x}_2-\mathbf{y}_2)\langle W_\Box\rangle,$$

pNRQCD

$$Z_s(\mathbf{r})\delta^3(\mathbf{x}_1-\mathbf{y}_1)\delta^3(\mathbf{x}_2-\mathbf{y}_2)e^{-iTV_s^{(0)}(\mathbf{r})}$$

One obtains:

$$V^{(0)}(\mathbf{r})=\lim_{T\to\infty}\frac{i}{T}\log\langle W_\Box\rangle=-C_f\frac{\alpha_s}{r}+O(\alpha_s^2)\quad \text{Wilson, Susskind}$$

$$V^{(1,0)}=-\lim_{T\to\infty}\int_0^T dt\,\frac{t}{2}\,\langle\langle g\mathbf{E}_1(t)\cdot g\mathbf{E}_1(0)\rangle\rangle_c=-C_fC_A\frac{\alpha_s^2}{4r^2}+O(\alpha_s^3)\ \text{Brambilla, Soto, Vairo, P.}$$

and $O(1/m^2)$ ...

B) **Even another method to get the potentials in terms of Wilson loops. Comparison between states and matrix elements in NRQCD and pNRQCD starting from the static solution. Closer in philosophy, and to some extent in the procedure, to standard quantum mechanics perturbation theory.**

$\boxed{\Lambda_{QCD} \lesssim mv^2}$ **($\Upsilon(1S)$, b-$\bar{b}$ sum rules, t-$\bar{t}$)**

At leading order the singlet Hamiltonian reads ($v \sim C_f \alpha_s / n$)

$$h_s^{(0)} = -\frac{\Delta}{m} - \frac{C_f \alpha_s(\mu)}{r} \rightarrow M(n,l) = 2m - \frac{m C_f^2 {\alpha_s}^2}{4n^2}$$

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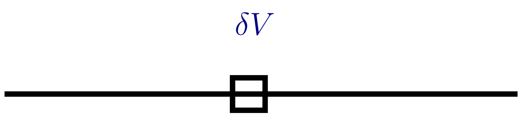
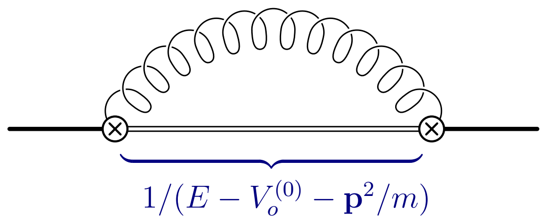
Corrections to the Green Function

$$G_s(E) = P_s \frac{1}{h_s^{(0)} - H_I - E} P_s = G_s^{(0)} + \delta G_s \qquad G_s^{(0)}(E) = \frac{1}{h_s^{(0)} - E}$$

Corrections:. Expansion in $1/M$, α and the multipole expansion.





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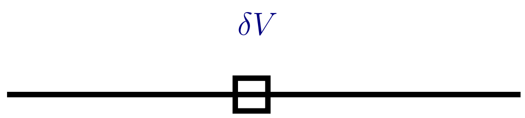
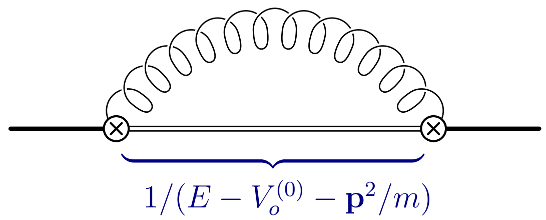




From ultrasoft gluons:

$$H_I^{us} = -\frac{g}{2} \xi^a \mathbf{x} \cdot \mathbf{E}^a(0) \,,$$

$$\begin{aligned} \delta G_s &\sim \frac{1}{h_s^{(0)} - E} \int \frac{d^3\mathbf{k}}{(2\pi)^{D-1}} \mathbf{r} \frac{k}{k + h_o^{(0)} - E} \mathbf{r} \frac{1}{h_s^{(0)} - E} \\ &\sim \frac{1}{h_s^{(0)} - E} \mathbf{r} (h_o^{(0)} - E)^3 \left\{ \frac{1}{\epsilon} + \gamma + \ln \frac{(h_o^{(0)} - E)^2}{\mu_{us}^2} + C \right\} \mathbf{r} \frac{1}{h_s^{(0)} - E} \end{aligned}$$

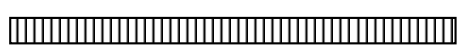
From the potential:

$$h_s = c_k\frac{\mathbf{p}^2}{m} - C_f\frac{\alpha_{V_s}}{r} - c_4\frac{\mathbf{p}^4}{4m^3} - \frac{C_f C_A D_s^{(1)}}{2mr^2} - \frac{C_f D_{1,s}^{(2)}}{2m^2}\left\{\frac{1}{r},\mathbf{p}^2\right\} + \frac{C_f D_{2,s}^{(2)}}{2m^2}\frac{1}{r^3}\mathbf{L}^2$$
$$+\frac{\pi C_f D_{d,s}^{(2)}}{m^2}\delta^{(3)}(\mathbf{r}) + \frac{4\pi C_f D_{S^2,s}^{(2)}}{3m^2}\mathbf{S}^2\delta^{(3)}(\mathbf{r}) + \frac{3C_f D_{LS,s}^{(2)}}{2m^2}\frac{1}{r^3}\mathbf{L}\cdot\mathbf{S} + \frac{C_f D_{S_{12},s}^{(2)}}{4m^2}\frac{1}{r^3}S_{12}(\hat{\mathbf{r}}),$$

where $C_f = (N_c^2-1)/(2N_c)$ and $c_k = c_4 = 1$ (we only use c_4 for tracking of the contribution due to this term). The propagator of the singlet is (formally)

$$\frac{1}{E-h_s}.$$

At leading order (within an strict expansion in α_s) the propagator of the singlet reads



$$= G_c(E) = \frac{1}{E-h_s^{(0)}} = \frac{1}{E-\mathbf{p}^2/m - C_f\alpha_s/r}.$$

$$\delta G_s \sim \frac{1}{h_s^{(0)}-E}\delta V\frac{1}{h_s^{(0)}-E}$$

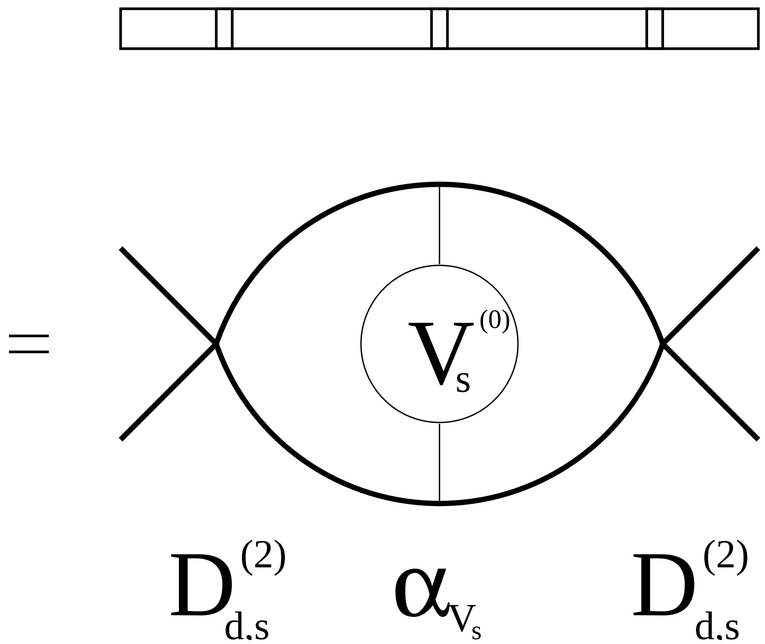
If we were interested in computing the spectrum at $O(m\alpha_s^6)$, one should consider the iteration of subleading potentials (δh_s) in the propagator as follows:

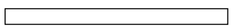
$$\delta G_s \sim \frac{1}{h_s^{(0)}-E}\delta h_s\frac{1}{h_s^{(0)}-E} + \frac{1}{h_s^{(0)}-E}\delta h_s\frac{1}{h_s^{(0)}-E}\delta h_s\frac{1}{h_s^{(0)}-E} + \cdots$$

Example:

$$\delta(r)\frac{1}{h_s^{(0)}-E}\frac{C_f\alpha_s}{r}\frac{1}{h_s^{(0)}-E}\delta(r)$$

In general, these contributions will produce logarithmic divergences due to potential loops. These divergences can be absorbed in the matching coefficients, $D_{d,s}^{(2)}$ and $D_{S^2,s}^{(2)}$, of the local potentials (proportional to the $\delta^{(3)}(\mathbf{r})$) providing with the renormalization group equations of these matching coefficients in terms of ν_p. Let us explain how it works in detail. Since the singular behavior of the potential loops appears for $\mathbf{p}^2/m \gg \alpha_s/r$, a perturbative expansion in α_s is licit in $G_c(E)$, which can be approximated by



 $= G_c^{(0)}(E) = \dfrac{1}{E-\mathbf{p}^2/m}\,.$

$$\langle \mathbf{r}=0|\frac{1}{E-\mathbf{p}^2/m}C_f\frac{\alpha_{V_s}}{r}\frac{1}{E-\mathbf{p}^2/m}|\mathbf{r}=0\rangle$$
$$\sim \int \frac{\mathrm{d}^d p'}{(2\pi)^d}\int \frac{\mathrm{d}^d p}{(2\pi)^d}\frac{m}{\mathbf{p}'^2-mE}C_f\frac{4\pi\alpha_{V_s}}{\mathbf{q}^2}\frac{m}{\mathbf{p}^2-mE} \sim -C_f\frac{m^2\alpha_{V_s}}{16\pi}\frac{1}{\epsilon},$$

where $D=4+2\epsilon$ and $\mathbf{q}=\mathbf{p}-\mathbf{p}'$. This divergence is absorbed in $D_{d,s}^{(2)}$.

Heavy Quarkonium mass

The Mass ($E_n = -mC_F^2\alpha_s^2/(4n^2)$)

$$M(n,l) = 2m + E_n$$
$$+O(m\alpha_s^3) \quad \text{Billoire}$$
$$+O(m\alpha_s^4) \quad \text{Yndurain, Pineda}$$
$$+O(m\alpha_s^5 \log) \quad \text{Brambilla, Soto, Vairo, Pineda}$$
$$+O(m\alpha_s^5)\ \text{(almost)} \quad \text{Kniehl, Penin, Smirnov, Steinhauser}$$
$$+O(mv^4 \times F(mv^2/\Lambda_{QCD})) \quad \text{Voloshin}$$

$$\delta E_{nl}^{NP} = \langle 0|\langle n,l|H_I \frac{1}{E_n - h_o - H_g} H_I|n,l\rangle|0\rangle$$
$$= \frac{T_F}{3N_c}\int_0^\infty dt \langle n,l|\mathbf{r} e^{-t(h_o - E_n)}\mathbf{r}|n,l\rangle\langle g\mathbf{E}^a(t)\phi(t,0)_{ab}^{\text{adj}} g\mathbf{E}^b(0)\rangle(\nu_{us})$$
$$\sim \langle \alpha G^2\rangle \left(\frac{1}{mv}\right)^2 \frac{1}{mv^2}\left(1 + \left(\frac{\Lambda_{QCD}}{mv^2}\right)^2 + \ldots\right).$$

Leading term: Voloshin-Leutwyler; Subleading term: Pineda

Positronium

$$L_{pNRQED} = \int d^3\mathbf{x} d^3\mathbf{X} S^\dagger(\mathbf{x},\mathbf{X},t)$$
$$\Big\{i\partial_0 - \frac{\mathbf{p}^2}{m} + \frac{\alpha}{|\mathbf{x}|} + \frac{\mathbf{p}^4}{4m^3} - \frac{\delta^{(3)}(\mathbf{x})}{m^2}\left(\pi\alpha\left(c_D - 2{c_F}^2\right) + d_s + 3d_v - 16\pi\alpha d_2\right)$$
$$+\frac{\alpha}{2m^2}\frac{1}{|\mathbf{x}|}\left(\mathbf{p}^2 + \frac{1}{\mathbf{x}^2}\mathbf{x}\cdot(\mathbf{x}\cdot\mathbf{p})\mathbf{p}\right) - \frac{\delta^{(3)}(\mathbf{x})}{m^2}\mathbf{S}^2\left(\pi\alpha\frac{4}{3}{c_F}^2 - 2d_v\right)$$
$$-\frac{\alpha}{4m^2}\frac{1}{|\mathbf{x}|^3}\mathbf{L}\cdot\mathbf{S}\left(2c_S + 4c_F\right) - \frac{\alpha {c_F}^2}{4m^2}\frac{1}{|\mathbf{x}|^3}S_{12}(\mathbf{x}) - \delta V(\mathbf{x}) + \mathbf{x}\cdot e\mathbf{E}(\mathbf{X},t)\Big\}S(\mathbf{x},\mathbf{X},t)\,.$$

$$\delta V = \frac{\delta^{(3)}(\mathbf{x})}{m^2}\left(\frac{\alpha^2}{3} - \frac{7\alpha^2}{3}\log\mu^2\right) - \frac{7\alpha^2}{6\pi m^2}\mathrm{reg}\frac{1}{|\mathbf{x}|^3}$$

The positronium spectrum can be obtained with $O(m\alpha^5)$ accuracy with this Lagrangian. The cancellation of the scale dependence coming from the different scales involved in the problem is nicely seen.

The calculations can be performed using dimensional regularization.

$$
\begin{aligned}
E_{n,l,j} = {} & 2m - m\frac{\alpha^2}{4n^2} + \frac{m\alpha^4}{8}\{-\frac{4}{n^3(2l+1)} + \frac{11}{8n^4} \\
& -\frac{2\alpha}{3\pi}\frac{\delta_{l0}}{n^3}\,(9\log\alpha + 7\log n + 8\log R(n,l) - 14\log 2 \\
& \qquad\qquad -\frac{49}{15} - 7\left(\sum_{k=1}^{n}\frac{1}{k} + \frac{n-1}{2n}\right)\Big) \\
& -\frac{16\alpha}{3\pi}\frac{1-\delta_{l0}}{n^3}\left(\log R(n,l) + \frac{7}{16}\frac{1}{l(l+1)(2l+1)}\right) \\
& +\frac{14}{3}\frac{\delta_{l0}\delta_{s1}}{n^3}\left\{1 + \frac{3\alpha}{7\pi}\left(-\frac{32}{9} - 2\log 2\right)\right\} \\
& +\frac{(1-\delta_{l0})\delta_{s1}}{l(2l+1)(l+1)n^3}\,C_{j,l}\}\,,
\end{aligned}
$$

where $\log R(n,l) = \log\frac{2\langle E_{n,l}\rangle}{m\alpha^2}$ **is called the Bethe logarithm and**

$$
C_{j,l} = \begin{cases}
-\dfrac{l+1}{2\,l-1}\left(2(3l-1) + \dfrac{\alpha}{\pi}(4\,l-1)\right) & ,\ j = l-1\ , \\
-2 - \dfrac{\alpha}{\pi} & ,\ j = l\ , \\
\dfrac{l}{2\,l+3}\left(2(3l+4) + \dfrac{\alpha}{\pi}(4\,l+5)\right) & ,\ j = l+1\ .
\end{cases}
$$

Conclusions

The use of Effective Field Theories has lead us to a better understanding of the dynamics of NR systems.

A rigorous connection between Quantum Field Theories and a NR Quantum-mechanical formulation of the NR systems now exists for both the perturbative and nonperturbative case.

It is now understood how to perform calculations in dimensional regularization.

Power counting is now better understood.

Plenty of Observables: Decay widhts ($\Gamma(n^3S_1 \to e^+e^-)$, $\Gamma(n^1S_0 \to \gamma\gamma)$), Bottomonium sum rules. Determination of m_b. t–$\bar{t}$ production near threshold. Determination of m_t. QED, ...