QCD at finite T and μ

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1. Static thermodynamics

$\longrightarrow$ Euclidean, "understood" up to non-perturbative level, but only a limited class of observables

2. Real-time observables

$\longrightarrow$ Minkowskian, even leading-order perturbative computations very hard, but simple physical interpretations

3. Finite baryon density

$\longrightarrow$ adventurous, "condensed matter physics" of QCD, but largely model computations so far

1. Static thermodynamics

Let $\hat{H}$ be the Hamiltonian corresponding to $\mathcal{L}_{\text{QCD}}$. We would like to compute the partition function

$$\begin{aligned} \mathcal{Z} &= \text{Tr}\, e^{-\beta(\hat{H}-\mu\hat{Q})} = e^{-\beta\Omega(V,T,\mu)} \\ &= e^{\beta V p(T,\mu)} \,, \quad \beta \equiv \frac{1}{T} \,, \end{aligned}$$

where $\hat{Q}$ is the quark number, and $p(T,\mu)$ is the **pressure**.

We also consider **equal-time 2-point functions** like

$$\left\langle \hat{O}(\mathbf{x})\hat{O}(\mathbf{0}) \right\rangle \overset{|\mathbf{x}|\gg\beta}{\approx} A|\mathbf{x}|^{\alpha} e^{-m(T)|\mathbf{x}|} \,,$$

with $\langle \ldots \rangle \equiv \mathcal{Z}^{-1}\, \text{Tr}[e^{-\beta(\hat{H}-\mu\hat{Q})}(\ldots)]$, where $m(T) =$ "**screening mass**" $\equiv$ [correlation length]$^{-1}$.

Phenomenological motivation: Cosmology

In the Early Universe, $\frac{\mu}{T} \approx 10^{-10}$, so set $\mu = 0$, and denote $p(T) \equiv p(T,0) - p(0,0)$.

The cooling rate of the Universe is

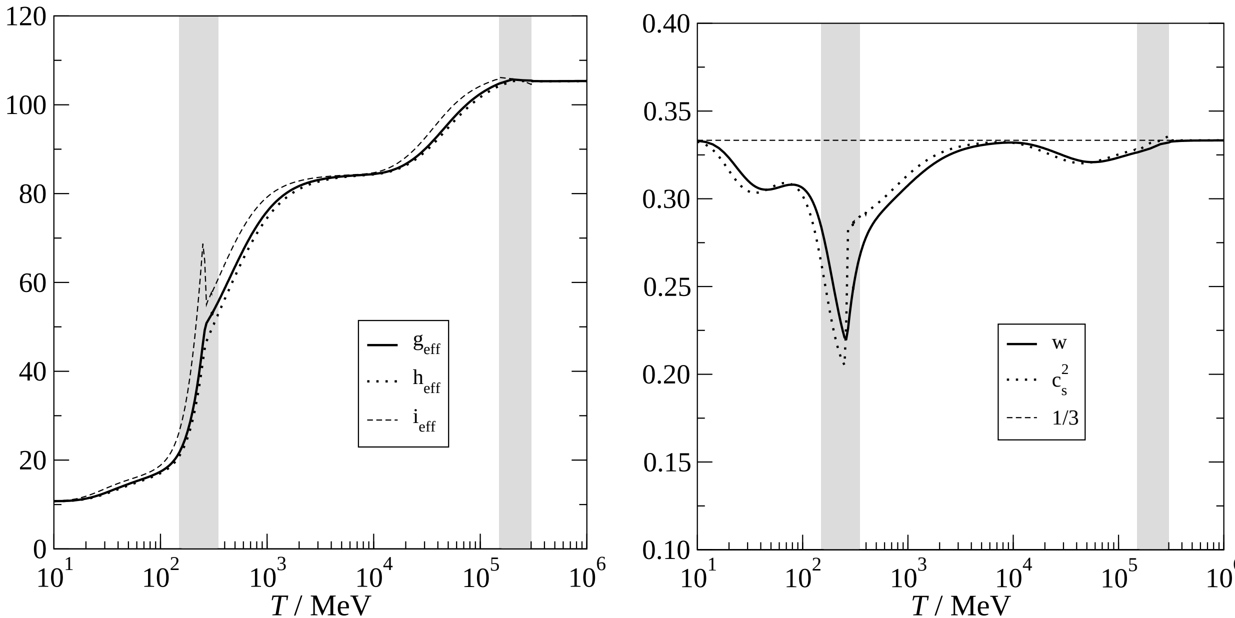
$$\frac{1}{T}\frac{\mathrm{d}T}{\mathrm{d}t} = -\frac{\sqrt{24\pi}}{m_{\mathsf{Pl}}}\frac{\sqrt{e(T)}s(T)}{c(T)} \,,$$

where $s = p'(T)$, $e = Ts(T) - p(T)$, $c = e'(T)$.

Cosmological relics (dark matter, etc) are born when some reaction time $\tau(T)$ becomes longer than the time period $t_{\mathsf{now}} - t(T) \Rightarrow$ need to know $p(T)$!

A lot of the structure in $p(T)$ comes from QCD

$$g_{\text{eff}} \equiv \frac{e}{\left[\frac{\pi^2 T^4}{30}\right]} \,, \quad h_{\text{eff}} \equiv \frac{s}{\left[\frac{2\pi^2 T^3}{45}\right]} \,, \quad i_{\text{eff}} \equiv \frac{c}{\left[\frac{2\pi^2 T^3}{15}\right]} \,, \quad w \equiv \frac{p}{e} \,, \quad c_s^2 \equiv \frac{\mathrm{d}p}{\mathrm{d}e} = \frac{s}{c}$$



Laine, Schröder, hep-ph/0603048

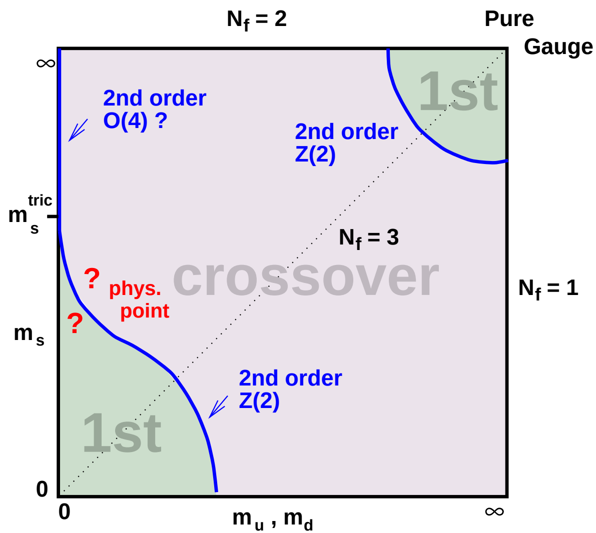
... however, let us "simplify" the task a bit, and rather pose a ...

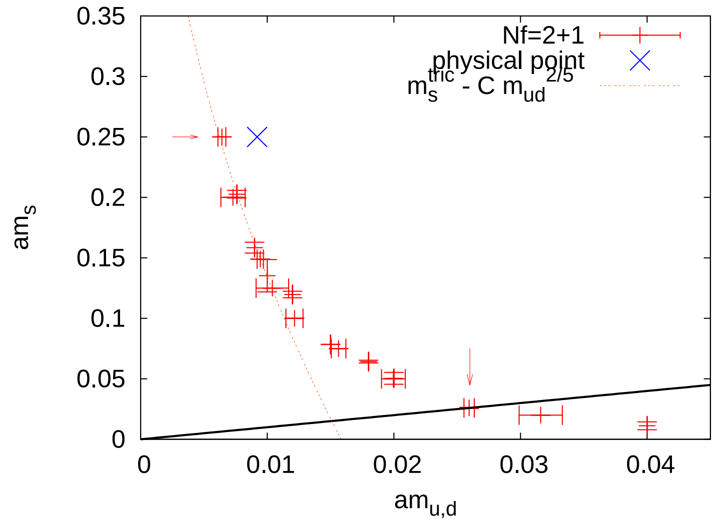
Theoretical challenge for today

Asymptotic freedom
$\Rightarrow$ effective coupling is small at $T \gg 1$ GeV
$\Rightarrow$ long-distance properties become more tractable
$\Rightarrow$ can we learn something about confinement?

This can only be possible if there is no phase transition, so that our low-T world and the asymptotically free high-T world are analytically connected.

For physical quark masses, there is no order parameter and no spontaneously broken global symmetry ($\mu = 0$).





de Forcrand, Philipsen hep-lat/0607017

Aoki et al hep-lat/0611014

$\Rightarrow$ in principle we can use high T as a theoretical tool!

Concrete task

Assuming $T \gg 1$ GeV, so that $\alpha_s(T)/\pi \ll 1$, can we understand the fact that there is a mass gap, i.e. that the screening masses are positive, $m(T) > 0$, for any local operator $\hat{O}$?

$$\left\langle \hat{O}(\mathbf{x})\hat{O}(\mathbf{0}) \right\rangle \stackrel{|\mathbf{x}|\gg\beta}{\approx} A|\mathbf{x}|^{\alpha} e^{-m(T)|\mathbf{x}|} \; .$$

The basic formula of finite-temperature field theory:

$$\begin{aligned} \mathcal{Z}(V,T,\mu) &= \int_{\text{b.c.}} \mathcal{D}[A_\mu^a,\bar\psi,\psi]\exp\left(-S_E\right) \;, \\ S_E &\equiv \int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3\mathbf{x}\,\mathcal{L}_E \;, \\ \mathcal{L}_E &\equiv -\mathcal{L}_M(t\to -i\tau) \;, \end{aligned}$$

where b.c. are periodic (A_μ^a) or anti-periodic ($\bar\psi,\psi$) over τ, and integral is over all fields with these b.c.'s.

The Euclidean path integral works for any **equal-time** correlator, such as that needed for $m(T)$.

Given that by assumption $\alpha_s(T)/\pi \ll 1$, we analyse the system in simple-minded perturbation theory. [In principle it could also be treated on the lattice!]

Fourier decomposition:

$$\phi(\tau) = T \sum_n e^{i\omega_n \tau} \tilde{\phi}(\omega_n) \; , \quad \phi \in \{A^a_\mu, \psi, \bar{\psi}\}$$

where the possible values of ω_n are discretised:

$$\begin{aligned} A^a_\mu(\beta, \mathbf{x}) &= A^a_\mu(0, \mathbf{x}) \Rightarrow e^{i\omega_n \beta} = 1 \Rightarrow \omega_n^{\mathsf{b}} = 2\pi n T \; , \\ \psi(\beta, \mathbf{x}) &= -\psi(0, \mathbf{x}) \Rightarrow \omega_n^{\mathsf{f}} = 2\pi(n + \frac{1}{2})T \; . \end{aligned}$$

These are called "Matsubara frequencies".

So, any line with fermions is “massive”, and displays exponential decay with $m(T) \geq \pi T$.

In fact, fermions can be “integrated out”, without encountering infrared problems.

The same holds for **non-zero** bosonic modes.

The result: a “dimensionally reduced” effective field theory for hot QCD (or for the Standard Model).

P. Ginsparg, Nucl. Phys. B 170 (1980) 388;

T. Appelquist and R.D. Pisarski, Phys. Rev. D 23 (1981) 2305;

K. Kajantie et al, hep-ph/9508379;

E. Braaten and A. Nieto, hep-ph/9510408.

Cartoon of the general procedure

QCD $\equiv$ 4d YM + quarks; $\omega_n \sim \pi T$

$\Downarrow$ perturbation theory (1)

EQCD $\equiv$ 3d YM + A_0; $m_{\mathrm{E}} \sim gT$

$\Downarrow$ perturbation theory (2)

MQCD $\equiv$ 3d YM; $g_{\mathrm{M}}^2 \sim g^2 T$

$\Downarrow$ non-perturbative computation (3)

PHYSICS

Expansion parameter: $\epsilon_{(i)} \sim g^2 T / 4\pi |\mathbf{k}|_{(i)}$.

To be more specific, start at tree-level

$$\mathcal{L}_{\mathsf{QCD}} = \frac{1}{4}\sum_{a=1}^{N_\mathrm{c}^2-1} F^a_{\mu\nu}F^a_{\mu\nu} + \sum_{i=1}^{N_\mathrm{f}} \bar\psi_i[\gamma_\mu D_\mu + m_i]\psi_i \; .$$

Given that we can write

$$F^a_{0i} = \partial_0 A^a_i - \mathcal{D}^{ab}_i A^b_0 \; , \quad \mathcal{D}^{ab}_i \equiv \partial_i \delta^{ab} - g f^{abc} A^c_i \; ,$$

the static limit yields $F^a_{i0} = \mathcal{D}^{ab}_i A^b_0$, so that

$$\mathcal{L}^{(n=0)}_{\mathsf{QCD}} = \frac{1}{4} F^a_{ij} F^a_{ij} + \frac{1}{2}(\mathcal{D}^{ab}_i A^b_0)(\mathcal{D}^{ac}_i A^c_0) \; .$$

The Euclidean action in the path integral reads

$$S_E = \int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3\mathbf{x}\, \mathcal{L}_{\mathsf{QCD}}^{(n=0)} = \frac{1}{T} \int_V \mathrm{d}^3\mathbf{x}\, \mathcal{L}_{\mathsf{QCD}}^{(n=0)} \ .$$

If we rescale the fields as

$$A_i^a \to T^{1/2} A_i^a \ , \quad A_0^a \to T^{1/2} A_0^a \ ,$$

and the coupling as

$$g = T^{-1/2} g_{\mathsf{E}} \ , \quad [g_{\mathsf{E}}] = \mathsf{GeV}^{1/2} \ ,$$

then $1/T$ disappears from in front of the action.

What kind of operators are generally allowed?

Gauge transformation:

$$A'_\mu = U A_\mu U^{-1} + \frac{i}{g} U \partial_\mu U^{-1} \; .$$

Since we restrict to *static* fields, U should not depend on τ, to remain within the set. Thus, the effective theory should be invariant under

$$\begin{aligned} A'_i &= U A_i U^{-1} + \frac{i}{g} U \partial_i U^{-1} \; , \\ A'_0 &= U A_0 U^{-1} \; . \end{aligned}$$

So, the spatial components A_i remain gauge fields, while the temporal components A_0 turn into **scalar fields in the adjoint representation**.

General form of the effective theory

Respecting gauge as well as discrete symmetries,

$$\mathcal{L}_{\mathsf{EQCD}} = \frac{1}{4} F^a_{ij} F^a_{ij} + \frac{1}{2} (\mathcal{D}^{ab}_i A^b_0)(\mathcal{D}^{ac}_i A^c_0) + m^2_{\mathsf{E}} \, \mathrm{Tr}[A^2_0] + \lambda^{(1)}_{\mathsf{E}} (\mathrm{Tr}[A^2_0])^2 + \lambda^{(2)}_{\mathsf{E}} \, \mathrm{Tr}[A^4_0] + \ldots \ ,$$

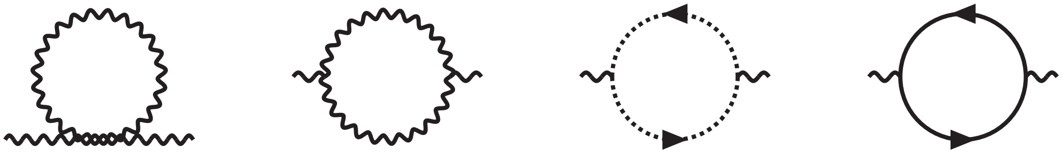
where we chose to write $A_0 \equiv \sum_{a=1}^{N_c^2-1} T^a A^a_0$, with T^a the Hermitean generators of SU(N_c).

(If $\mu \neq 0$, C is broken, and further operators can appear, like

$$\delta\mathcal{L}_{\mathsf{EQCD}} = i\gamma_{\mathsf{E}} \, \mathrm{Tr}[A^3_0] \ .$$
)

The parameters of $\mathcal{L}_{\mathsf{EQCD}}$ can be determined by **matching** suitable observables to the original theory.

To leading non-trivial order, need to consider



where the internal lines have non-zero Matsubara modes, while the external lines can be A_i^a or A_0^a.

In case of A_i^a: result must behave as $\mathbf{k}^2$, yielding a correction to the gauge coupling.

In case of A_0^a: result can remain non-zero as $\mathbf{k} \to \mathbf{0}$, yielding a non-zero mass parameter m_{E}^2.

The typical sum-integral appearing:

$$T \sum_{\omega_n} \int \frac{\mathrm{d}^d \mathbf{k}}{(2\pi)^d} \frac{1}{\omega_n^2 + \mathbf{k}^2}$$

$$= \quad 2T \sum_{n=1}^{\infty} \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(1-\frac{d}{2})}{\Gamma(1)} \frac{1}{(2\pi n T)^{2-d}}$$

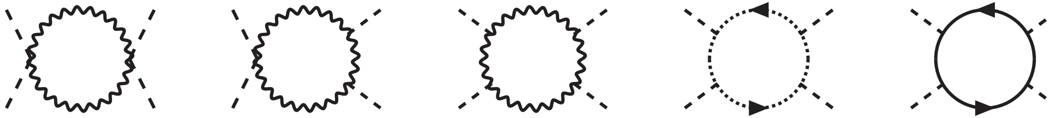
$$= \quad 2T \frac{1}{(4\pi)^{d/2}(2\pi T)^{2-d}} \Gamma(1-\frac{d}{2}) \zeta(2-d)$$

$$\stackrel{d=3-2\epsilon}{=} \quad \mu^{-2\epsilon} \frac{T^2}{12} \Big\{ 1 \quad \Big|\Big| \quad \zeta(-1) = -\frac{1}{12}!$$

$$+\epsilon \left[2 \ln \left(\frac{\bar{\mu} e^{\gamma_{\mathrm{E}}}}{4\pi T} \right) + 2 - 2\gamma_{\mathrm{E}} + 2 \frac{\zeta'(-1)}{\zeta(-1)} \right]$$

$$+\mathcal{O}(\epsilon^2) \Big\} , \quad \bar{\mu}^2 \equiv 4\pi \mu^2 e^{-\gamma_{\mathrm{E}}} .$$

Graphs for quartic couplings:



A typical integral:

$$T\sum_{\omega_n}\int\frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}\frac{1}{(\omega_n^2+\mathbf{k}^2)^2}$$

$$=\quad\frac{\mu^{-2\epsilon}}{(4\pi)^2}\left[\frac{1}{\epsilon}+2\ln\left(\frac{\bar{\mu}e^{\gamma_{\mathsf{E}}}}{4\pi T}\right)+\mathcal{O}(\epsilon)\right]\ .$$

So the "thermal scale" is $\bar{\mu}_T \simeq 4\pi e^{-\gamma_{\mathsf{E}}}T \approx 7.0555T$, and the effective coupling runs faster than expected.

Collecting together, we have the effective theory

$$\mathcal{L}_{\mathsf{EQCD}} = \frac{1}{4}F^a_{ij}F^a_{ij} + \frac{1}{2}(\mathcal{D}^{ab}_i A^b_0)(\mathcal{D}^{ac}_i A^c_0) + m^2_{\mathsf{E}}\,\mathrm{Tr}[A^2_0] + \lambda^{(1)}_{\mathsf{E}}(\mathrm{Tr}[A^2_0])^2 + \lambda^{(2)}_{\mathsf{E}}\,\mathrm{Tr}[A^4_0] + \ldots \;,$$

where at 1-loop

$$m^2_{\mathsf{E}} = g^2T^2\Big(\frac{N_\mathrm{c}}{3} + \frac{N_\mathrm{f}}{6}\Big) \;,$$

$$\lambda^{(1)}_{\mathsf{E}} = \frac{g^4T}{4\pi^2} \;, \quad \lambda^{(2)}_{\mathsf{E}} = \frac{g^4T}{12\pi^2}(N_\mathrm{c} - N_\mathrm{f}) \;,$$

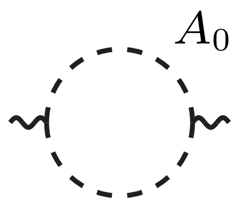
$$g^2_{\mathsf{E}} = T\Big\{g^2(\bar\mu) + \frac{g^4(\bar\mu)}{(4\pi)^2}\Big[-\beta_0\ln\Big(\frac{\bar\mu e^{\gamma_{\mathsf{E}}}}{4\pi T}\Big) + \frac{N_\mathrm{c} - 8N_\mathrm{f}\ln 2}{3}\Big]\Big\} \;.$$

But now A_0 is massive $\Rightarrow$ it can be integrated out.

The subsequent effective theory:

$$\mathcal{L}_{\mathsf{MQCD}} = \frac{1}{4} F^a_{ij} F^a_{ij} + \ldots.$$

The parameters are again determined by matching:



At 1-loop:

$$g^2_{\mathsf{M}} = g^2_{\mathsf{E}} \left[1 - \frac{1}{48} \frac{g^2_{\mathsf{E}} N_c}{\pi m_{\mathsf{E}}} \right] .$$

Infrared problem of thermal field theory

Linde PLB 96 (1980) 289

Gross, Pisarski, Yaffe RMP 53 (1981) 43

The remaining theory has only one parameter, $g_{\mathsf{M}}^2 \approx g^2 T$, which is dimensionful. There is no other scale.

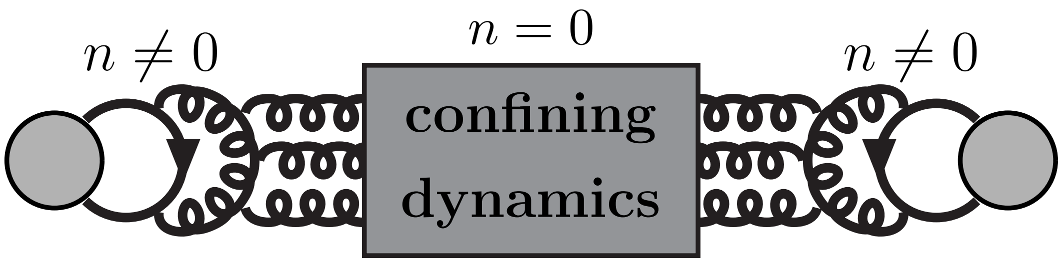
The theory is also confining, with the confinement scale proportional to g_{M}^2.

Numerically, for $N_{\rm c} = 3$,

$$
\begin{aligned}
\sqrt{\sigma} &\simeq 0.553(2) g_{\mathsf{M}}^2 \,, \\
m_{0^{++}} &\simeq 2.39(3) g_{\mathsf{M}}^2 \,.
\end{aligned}
$$

Teper hep-lat/9804008

So, at long distances, any correlation function decays exponentially, with a screening mass given by that of the lightest MQCD glueball with the correct quantum numbers:



$$m(T) = \#_{\text{non-pert}}\, g_{\mathrm{M}}^2 + \mathcal{O}(g^3 T) \; .$$

Summary

Arguing that a mass gap exists is easier at finite temperatures, since one only needs to do this for pure Yang-Mills theory in three dimensions. But it is still a non-perturbative problem.

A weak-coupling expansion ($g \ll 1$) can still be constructed, but it comes with in-general non-perturbative coefficients. In other words:

$$\text{weak-coupling expansion} \neq \text{loop expansion.}$$

Exercise 1: "Where is physics hidden?"

Show that

$$T \sum_{n=-\infty}^{\infty} \int_{\mathbf{p}} \frac{1}{(2\pi T n)^2 + \mathbf{p}^2 + m^2} = \int_{\mathbf{p}} \frac{1}{E} \left[\frac{1}{2} + n_{\text{B}}(E) \right] ,$$

where $E \equiv \sqrt{\mathbf{p}^2 + m^2}$ and n_{B} is the Bose-Einstein distribution function, $n_{\text{B}}(E) \equiv 1/[e^{\beta E} - 1]$. [The term $\frac{1}{2E}$ corresponds to the vacuum result, $\int \frac{\mathrm{d}p_0}{2\pi} \frac{1}{p_0^2 + E^2}$, the rest to thermal corrections.]

Exercise 2: "Another effect of 3d confinement".

Consider the weak-coupling expansion for the QCD pressure, $p(T)$, in the effective theory framework. At which order would you expect a non-perturbative coefficient to first appear?