QCD at finite T and μ

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1. Static thermodynamics

$\longrightarrow$ Euclidean, "understood" up to non-perturbative level, but only a limited class of observables

2. Real-time observables

$\longrightarrow$ Minkowskian, even leading-order perturbative computations very hard, but simple physical interpretations

3. Finite baryon density

$\longrightarrow$ adventurous, "condensed matter physics" of QCD, but largely model computations so far

2. Real-time observables

$\hat{O} \equiv$ spatial component of a conserved current ($\hat{T}^{\mu i}$, $\hat{J}^i$).

Heisenberg picture:

$$\hat{O}(t) = e^{i\hat{H}t}\hat{O}(0)e^{-i\hat{H}t} \, .$$

Spectral function:

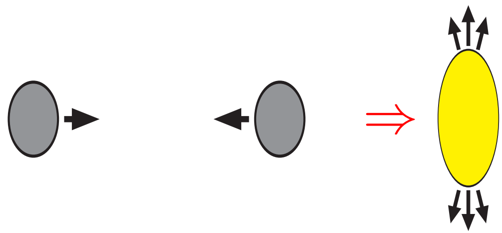
$$\rho(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t \, e^{i\omega t} \frac{1}{\mathcal{Z}} \, \mathrm{Tr} \left\{ e^{-\beta\hat{H}} \frac{1}{2} \left[\hat{O}(t), \hat{O}(0) \right] \right\} \, .$$

Transport coefficient: $\lim_{\omega\to 0} \frac{\rho(\omega)}{\omega}$.

This yields heat conductivity ($\hat{T}^{0i}$), electrical conductivity ($\hat{J}^i_{\text{em}}$), shear viscosity ($\hat{T}^{ji}$), bulk viscosity ($\hat{T}^{ii}$), flavour diffusion coefficient ($\hat{J}^i_{\text{f}}$), particle production rate ($\hat{J}^i_{\text{em}}$), . . .

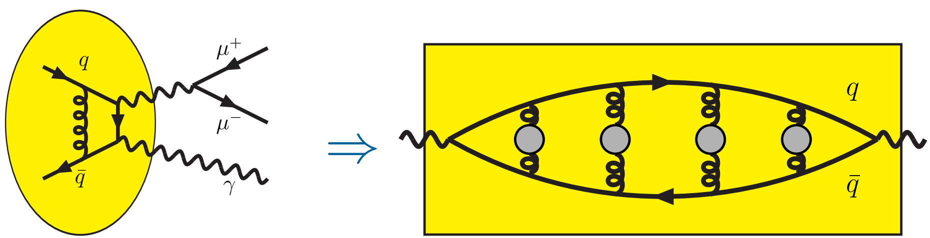
Phenomenological motivation: Heavy Ion Collisions

Hydrodynamics



Expansion due to $p_{\text{eff}}(T) = p_{\text{real}}(T) - \zeta\, \nabla \cdot \mathbf{v} + \mathcal{O}(\nabla^2)$, where $\zeta =$ "bulk viscosity" [in cosmology, $\nabla \approx 0$!].

Hard Probes



We rather consider a theoretical challenge today

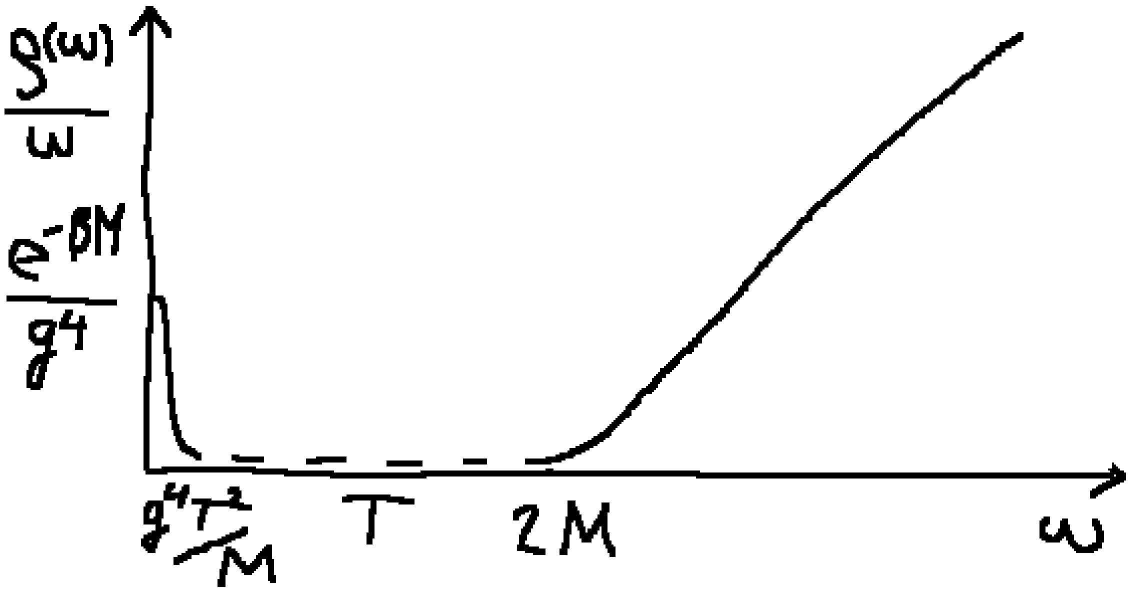
At zero temperature, there is no structure at small ω.

For example, massive elementary scalar field:

$$G = \frac{i}{p_0^2 - E_{\mathbf{p}}^2} \Rightarrow \rho(\omega) = \frac{\pi}{2E_{\mathbf{p}}}\left[\delta(\omega - E_{\mathbf{p}}) - \delta(\omega + E_{\mathbf{p}})\right] \ .$$

Or a composite object, say $\hat{\bar{\psi}}\gamma^i\hat{\psi}$ (quark mass M):

So, the claim is that new structure emerges around $\omega = 0$ at $T > 0$:



What is the physics responsible for this?

To get a first idea, consider a conserved charge in Euclidean spacetime, with temporal extent $\beta = 1/T$.

Then the correlator is a constant:

$$\partial_\tau \left\langle \int \mathrm{d}^3\mathbf{x}\, \hat{J}_0(\tau,\mathbf{x}) \hat{J}_0(0,\mathbf{0}) \right\rangle = 0 \,.$$

In fact, for $\hat{J}_0 = \hat{\bar{\psi}}\gamma_0\hat{\psi}$, $T \ll M$, $g = 0$ (free limit),

$$\begin{aligned} \Delta^E_{00}(\tau) &\equiv \left\langle \int \mathrm{d}^3\mathbf{x}\, \hat{J}_0(\tau,\mathbf{x}) \hat{J}_0(0,\mathbf{0}) \right\rangle \\ &\approx -4N_\mathrm{c} \left(\frac{MT}{2\pi}\right)^{3/2} e^{-\beta M} \,. \end{aligned}$$

The Fourier transform: $\int_0^\beta \mathrm{d}\tau \, e^{i\omega_n\tau} \Delta^E_{00}(\tau) \propto \delta_{\omega_n,0}$.

The spectral function is a certain analytic continuation hereof; in fact (Exercise 4):

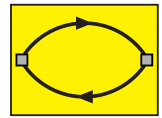
$$\frac{\rho_{00}(\omega)}{\omega} = \Delta^E_{00}(0)\beta\pi\delta(\omega) \; .$$

So, even for a conserved charge, there **is** a peak near the origin, it is just infinitely narrow!

With interactions turned on, the peak remains infinitely narrow, because the current is exactly conserved.

Consider then the correlator of the spatial components:

$$\Delta^E_{ii}(\tau) \quad \equiv \quad \left\langle \int \mathrm{d}^3\mathbf{x}\, \hat{J}_i(\tau,\mathbf{x})\hat{J}_i(0,\mathbf{0}) \right\rangle \ .$$

It turns out that in the free theory,  , this again contains a τ-independent "zero-mode":

$$\Delta^E_{ii}(\tau) = 4N_\mathrm{c}\frac{T}{M}\left(\frac{MT}{2\pi}\right)^{3/2} e^{-\beta M} + (\tau - \text{dep.})\ .$$

However, now interactions can smoothen $\delta(\omega)$ from $\rho(\omega)/\omega$. This yields the "transport peak" that we saw before.

Transport coefficient $\equiv$ intercept of $\rho(\omega)/\omega$ at origin.

In order to understand what interactions do, let us start with first quantized quantum mechanics.

To learn something about $\hat{J}^i$, let us give the quark a non-zero initial momentum, $\langle \hat{p}_i(t) \rangle_{\text{non-eq}} \neq 0$. In equilibrium, $\langle \hat{p}_i(t) \rangle_{\text{eq}} = 0$, so the system should relax:

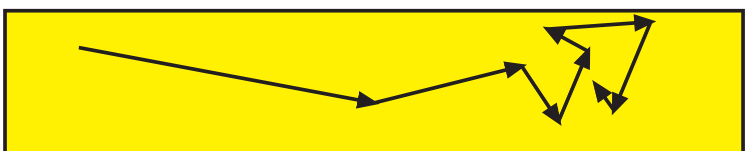
$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \hat{p}_i(t) \rangle_{\text{non-eq}} = -\eta_D \, \langle \hat{p}_i(t) \rangle_{\text{non-eq}} + \mathcal{O}(\langle \hat{p}_i(t) \rangle^2_{\text{non-eq}}) \; .$$

Here $\eta_D =$ "**drag coefficient**" $=$ "**relaxation rate**".

Once t is so large that $\langle \hat{p}_i(t) \rangle_{\text{non-eq}} \sim [\langle \hat{p}_i^2 \rangle_{\text{eq}}]^{1/2}$, Brownian motion sets in. The time scales we are interested in are slow, so let us assume that this can be described "classically" by Langevin-dynamics [though we may lose $\mathcal{O}(g^2)$ corrections]:

$$\begin{aligned}
\langle \hat{p}_i(t) \rangle_{\text{non-eq}} & \longrightarrow & p_i(t) \; , \\
\frac{\mathrm{d}}{\mathrm{d}t} p_i(t) & = & -\eta_D \, p_i(t) + \xi_i(t) \; , \\
\langle \xi_i(t) \xi_j(t') \rangle & = & \kappa \, \delta_{ij} \delta(t - t') \; , \qquad \langle \xi_i(t) \rangle = 0 \; .
\end{aligned}$$

Here $\kappa =$ "**momentum diffusion coefficient**".



Now, we can solve for the time evolution:

$$p_i(t) = p_i(0)e^{-\eta_D t} + \int_0^t \mathrm{d}t'\, e^{\eta_D(t'-t)}\xi_i(t')\ .$$

In particular, letting the system thermalize by waiting,

$$\begin{aligned}
&\lim_{t_0\to\infty} \langle p_i(t_0) p_i(t_0+t)\rangle \\
=\quad & \lim_{t_0\to\infty} \int_0^{t_0} \mathrm{d}t_1\, e^{\eta_D(t_1-t_0)} \int_0^{t_0} \mathrm{d}t_2\, e^{\eta_D(t_2-t_0-t)} \langle \xi_i(t_1)\xi_i(t_2)\rangle \\
=\quad & \frac{\kappa}{2\eta_D} e^{-\eta_D|t|} \quad \equiv \quad \langle p_i^2\rangle_{\text{eq}}\, e^{-\eta_D|t|}\ .
\end{aligned}$$

$\Bigg($ Equipartition tells that $\frac{\langle p_i^2\rangle_{\text{eq}}}{2M} = \frac{T}{2}[1+\mathcal{O}(g^2)]$, i.e. $\kappa = 2\eta_D T M$ "fluctuation-dissipation theorem" . $\Bigg)$

The quantum mechanical equivalent of the classical correlator $\lim_{t_0 \to \infty} \langle p_i(t_0) p_i(t_0 + t) \rangle$ must be

$$\Delta(t) \equiv \left\langle \frac{1}{2} \{ \hat{p}_i(t), \hat{p}_i(0) \} \right\rangle_{\text{eq}} ,$$

because operator ordering plays no role. We have thus learned that, **in equilibrium**,

$$\Delta(t) \simeq \Delta(0) e^{-\eta_D |t|} .$$

Consequently,

$$\tilde{\Delta}(\omega) \equiv \int_{-\infty}^{\infty} \mathrm{d}t \, e^{i\omega t} \, \Delta(t) \simeq \Delta(0) \frac{2\eta_D}{\omega^2 + \eta_D^2} .$$

Let us define a spectral function corresponding to the momentum operator:

$$C_\rho(t) \equiv \left\langle \frac{1}{2} \left[\hat{p}_i(t), \hat{p}_i(0)\right] \right\rangle_{\text{eq}} , \quad \rho(\omega) \equiv \int_{-\infty}^{\infty} \mathrm{d}t \, e^{i\omega t} \, C_\rho(t) \, .$$

It can be shown that $\tilde{\Delta}(\omega) = [1 + 2n_{\text{B}}(\omega)]\,\rho(\omega)$:

More generally, *all* of the correlation functions defined above can be related to each other. In particular, all correlators can be expressed in terms of the spectral function, which in turn can be determined as a certain analytic continuation of the Euclidean correlator. In order to do this, we may first insert sets of energy eigenstates into the definitions of $\Pi^{>}_{\alpha\beta}$ and $\Pi^{<}_{\alpha\beta}$:

$$\begin{aligned}
\Pi^{>}_{\alpha\beta}(Q) &= \frac{1}{\mathcal{Z}} \int \mathrm{d}t\, \mathrm{d}^3\mathbf{x}\, e^{iQ\cdot x} \mathrm{Tr}\Big[e^{-\beta\hat{H}+i\hat{H}t} \underbrace{\mathbb{1}}_{\sum_m |m\rangle\langle m|} \hat{\phi}_\alpha(0,\mathbf{x}) e^{-i\hat{H}t} \underbrace{\mathbb{1}}_{\sum_n |n\rangle\langle n|} \hat{\phi}^\dagger_\beta(0,\mathbf{0}) \Big] \\
&= \frac{1}{\mathcal{Z}} \sum_{m,n} \int \mathrm{d}t\, \mathrm{d}^3\mathbf{x}\, e^{iQ\cdot x} e^{(-\beta+it)E_m} e^{-itE_n} \langle m|\hat{\phi}_\alpha(0,\mathbf{x})|n\rangle \, \langle n|\hat{\phi}^\dagger_\beta(0,\mathbf{0})|m\rangle \\
&= \frac{1}{\mathcal{Z}} \int_{\mathbf{x}} e^{-i\mathbf{q}\cdot\mathbf{x}} \sum_{m,n} e^{-\beta E_m}\, 2\pi\, \delta(q^0 + E_m - E_n) \langle m|\hat{\phi}_\alpha(0,\mathbf{x})|n\rangle \, \langle n|\hat{\phi}^\dagger_\beta(0,\mathbf{0})|m\rangle \, , \qquad (0.1) \\
\Pi^{<}_{\alpha\beta}(Q) &= \frac{1}{\mathcal{Z}} \int \mathrm{d}t\, \mathrm{d}^3\mathbf{x}\, e^{iQ\cdot x} \mathrm{Tr}\Big[e^{-\beta\hat{H}} \underbrace{\mathbb{1}}_{\sum_n |n\rangle\langle n|} \hat{\phi}^\dagger_\beta(0,\mathbf{0}) e^{i\hat{H}t} \underbrace{\mathbb{1}}_{\sum_m |m\rangle\langle m|} \hat{\phi}_\alpha(0,\mathbf{x}) e^{-i\hat{H}t} \Big] \\
&= \frac{1}{\mathcal{Z}} \sum_{m,n} \int \mathrm{d}t\, \mathrm{d}^3\mathbf{x}\, e^{iQ\cdot x} e^{(-\beta-it)E_n} e^{itE_m} \langle n|\hat{\phi}^\dagger_\beta(0,\mathbf{0})|m\rangle \, \langle m|\hat{\phi}_\alpha(0,\mathbf{x})|n\rangle \qquad (0.2) \\
&= \frac{1}{\mathcal{Z}} \int_{\mathbf{x}} e^{-i\mathbf{q}\cdot\mathbf{x}} \sum_{m,n} e^{-\beta E_n}\, 2\pi\, \underbrace{\delta(q^0 + E_m - E_n)}_{E_n = E_m + q^0} \langle m|\hat{\phi}_\alpha(0,\mathbf{x})|n\rangle \, \langle n|\hat{\phi}^\dagger_\beta(0,\mathbf{0})|m\rangle \\
&= e^{-\beta q^0}\, \Pi^{>}_{\alpha\beta}(Q) \, . \qquad (0.3)
\end{aligned}$$

This is the Fourier-space version of the KMS relation. Consequently

$$\rho_{\alpha\beta}(Q) = \frac{1}{2} [\Pi^{>}_{\alpha\beta}(Q) - \Pi^{<}_{\alpha\beta}(Q)] = \frac{1}{2} (e^{\beta q^0} - 1) \Pi^{<}_{\alpha\beta}(Q) \qquad (0.4)$$

and, conversely,

$$\begin{aligned}
\Pi^{<}_{\alpha\beta}(Q) &= 2n_{\text{B}}(q^0) \rho_{\alpha\beta}(Q) \, , \qquad (0.5) \\
\Pi^{>}_{\alpha\beta}(Q) &= 2 \frac{e^{\beta q^0}}{e^{\beta q^0} - 1} \rho_{\alpha\beta}(Q) = 2[1 + n_{\text{B}}(q^0)] \rho_{\alpha\beta}(Q) \, , \qquad (0.6)
\end{aligned}$$

where $n_{\text{B}}(x) \equiv 1/[\exp(\beta x) - 1]$. Moreover,

$$\Delta_{\alpha\beta}(Q) = \frac{1}{2} [\Pi^{>}_{\alpha\beta}(Q) + \Pi^{<}_{\alpha\beta}(Q)] = [1 + 2n_{\text{B}}(q^0)] \rho_{\alpha\beta}(Q) \, . \qquad (0.7)$$

Note that $1 + 2n_{\text{B}}(-q^0) = -[1 + 2n_{\text{B}}(q^0)]$, so that if ρ is odd in $Q \to -Q$, then Δ is even.

Thus, for $\omega \ll T$ where $n_{\mathrm{B}}(\omega) = 1/(e^{\omega/T} - 1) \approx T/\omega$, we find that

$$
\begin{aligned}
\rho(\omega) &\approx \frac{\omega}{2T}\tilde{\Delta}(\omega) = \Delta(0)\frac{\beta\omega\eta_D}{\omega^2 + \eta_D^2} \\
\Leftrightarrow \frac{\rho(\omega)}{\omega} &\approx \Delta(0)\beta\,\mathrm{Im}\,\frac{1}{\omega - i\eta_D}\,.
\end{aligned}
$$

For $\eta_D \to 0^+$, $\frac{1}{\omega - i0^+} = P(\frac{1}{\omega}) + i\pi\delta(\omega)$, and we get $\rho(\omega)/\omega = \Delta(0)\beta\pi\delta(\omega)$, like in a free theory!

Hence, it is the multiple collisions within the plasma, leading to momentum diffusion/dissipation, that are responsible for a finite transport peak.

Further concepts

Apart from η_D, κ, there are two other quantities which essentially contain the same information. If we define

$$x_i(t) - x_i(0) \equiv \int_0^t \mathrm{d}t' \frac{p_i(t')}{M} \, ,$$

then the previous result for $\langle p_i(t_1) p_i(t_2) \rangle_{\text{eq}}$ leads to

$$\left\langle [x_i(t) - x_i(0)]^2 \right\rangle_{\text{eq}} \overset{t \gg \eta_D^{-1}}{\approx} 2Dt \, .$$

Here the (flavour) **diffusion coefficient** D is given by

$$D = \frac{\langle p_i^2 \rangle_{\text{eq}}}{\eta_D M^2} \overset{\text{equipartition}}{\simeq} \frac{T}{\eta_D M} \overset{\text{fluct.-dissip.}}{\simeq} \frac{2T^2}{\kappa} \qquad \text{(Einstein)} \, .$$

Yet another related concept is the **energy loss** of a heavy quark:

$$\frac{\mathrm{d}E}{\mathrm{d}x} = \frac{\mathrm{d}t}{\mathrm{d}x}\frac{\mathrm{d}p_i}{\mathrm{d}t}\frac{\mathrm{d}E}{\mathrm{d}p_i} \simeq \frac{E}{p}(-\eta_D p_i)\frac{p_i}{E} = -\eta_D p \, .$$

So, at least to leading order, all four quantities $(\eta_D, \kappa, D, \mathrm{d}E/\mathrm{d}x)$ are related to each other — a dear child has many names![1]

[1] In the case of other transport coefficients, like viscosities, there is a unique definition, but practical computations and the related physics are harder.

Intermediate summary

In the limit where Langevin dynamics is applicable, we have defined four different quantities characterizing the dissipative motion of heavy quarks within a plasma, all of them related to each other.

If they are finite, then a genuine transport peak exists.

Remaining challenges

Which of them (if any) can be given a meaningful (non-perturbative) definition within QCD?

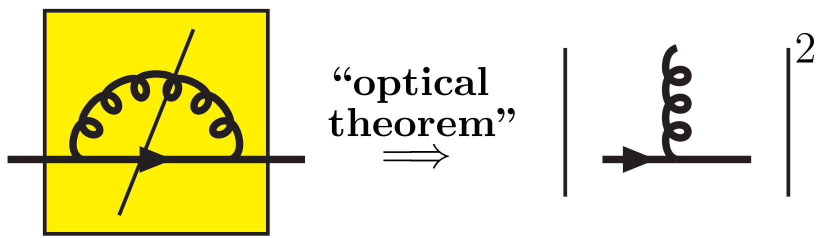
How do the answers look like? $g^p T^q M^r$?

Approach 1/3

In principle the information should be extractable from the imaginary part of the on-shell **self-energy** of the heavy quark. Historically, this was first used in order to determine $\mathrm{d}E/\mathrm{d}x$.

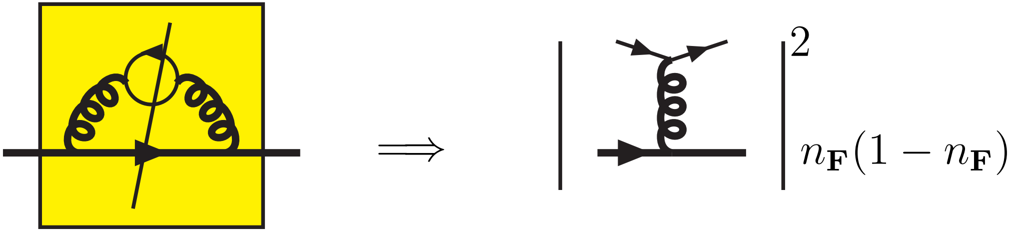
Braaten, Thoma PRD 44(1991)1298 & 2625

Let us use this to estimate the parametric magnitude of the effect.



$\Rightarrow$ not allowed kinematically.

At the next order:



$\Rightarrow$ The effect is $\mathcal{O}(g^4)$!

This method cannot, however, be easily promoted to the non-perturbative level.

Approach 2/3

Looking at the Langevin-equation,

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = -\eta_D \, p_i + \xi_i \; , \quad \langle \xi_i(t)\xi_i(t') \rangle = \kappa \, \delta(t-t') \; ,$$

we note that κ can be obtained as

$$\kappa = \int_{-\infty}^{\infty} \mathrm{d}t \, \langle \xi_i(t)\xi_i(0) \rangle \; .$$

Moreover, we may identify ξ_i as the **force** acting on the heavy quark: $\xi_i \sim g E_i$, where E_i is the colour-electric field. Hence

$$\kappa \equiv g^2 \int_{-\infty}^{\infty} \mathrm{d}t \, \frac{1}{N_\mathrm{c}} \, \mathrm{Tr} \left\langle W^\dagger(t,0) E_i(t,\mathbf{x}) W(t,0) E_i(0,\mathbf{x}) \right\rangle \; .$$

Casalderrey-Solana, Teaney hep-ph/0605199

To estimate κ, write $E_i \sim \partial_i A_0$, and note that the t-integral corresponds to the $\omega \to 0$ limit of a Fourier transform. So,

$$\kappa \quad \sim \quad g^2 \lim_{\omega \to 0} \frac{2T}{\omega} \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \mathbf{p}^2 \rho_{00}(\omega, \mathbf{p}) \ ,$$

where ρ_{00} is the spectral function related to A_0. After (Hard Thermal Loop) resummation,

$$\rho_{00}(\omega, \mathbf{p}) \overset{\omega \ll p \ll T}{\approx} \frac{\pi m_\mathsf{E}^2 \omega}{2p(p^2 + m_\mathsf{E}^2)^2} \ , \quad \text{and}$$

$$\kappa \quad \sim \quad g^2 T \frac{m_\mathsf{E}^2}{2\pi} \int_0^T \frac{\mathrm{d}p\, p^3}{(p^2 + m_\mathsf{E}^2)^2} \sim g^4 T^3 \ln \frac{T}{m_\mathsf{E}} \ .$$

Proper treatment of UV &
determination of NLO correction: Caron-Huot, Moore 0708.4232

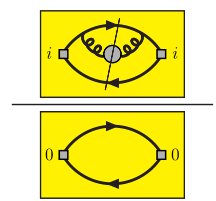
Approach 3/3

To obtain a definition which is guaranteed to be finite at any order, we need to make use of the current, $\hat{J}^\mu = \hat{\bar{\psi}}\gamma^\mu\hat{\psi}$. Physically, $\hat{J}^\mu \sim nu^\mu \sim M^{-1}np^\mu$, so in fact we can apply the argument previously used for $\hat{p}_i$:

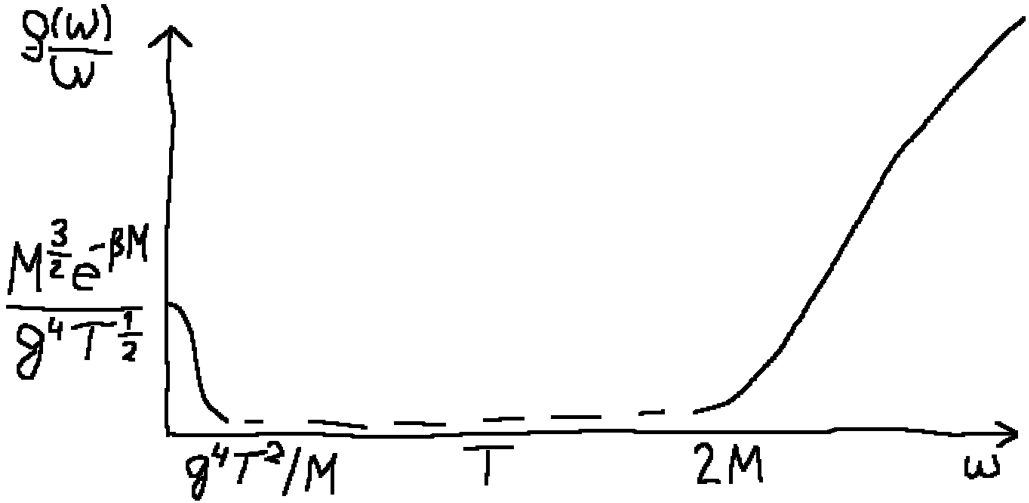
$$\frac{\rho_{ii}(\omega)}{\omega} \overset{\omega \ll T}{\approx} \Delta_{ii}(0)\frac{\beta\eta_D}{\omega^2 + \eta_D^2} \; .$$

Thus, η_D can be extracted from the "line shape" around origin. Alternatively, looking at the intercept, and adjusting the normalizations to work correctly at leading order, we can define

$$D \equiv \frac{\lim_{\omega\to 0} \frac{\rho^{\mu}{}_{\mu}(\omega)}{\omega}}{\int_0^\beta \mathrm{d}\tau \, \Delta_{00}(\tau)} \; .$$



Summary



Note: $\kappa \sim g^4 \ln(\frac{1}{g})T^3$, so $\eta_D \simeq \kappa/2TM \sim g^4 \ln(\frac{1}{g})\frac{T^2}{M}$.

$\Rightarrow$ Simple textbook statistical physics is hidden in a very far and difficult corner of quantum field theory!

Exercise 3: "Another angle on diffusive motion".

Solve the 1-d diffusion equation for the number density,

$$\partial_t n(t,x) = D\,\partial_x^2 n(t,x) \ ,$$

with the initial condition $n(0,x) = \delta(x)$, and show that $\langle x^2(t)\rangle = 2Dt$. [Hint: write $\delta(x) = \lim_{\sigma\to 0}\frac{1}{\sigma\sqrt{\pi}}\exp(-\frac{x^2}{\sigma^2})$].

Exercise 4: "Extreme form of the transport peak".

Defining $\tilde{\Delta}_E(\omega_n) = \int_0^\beta \mathrm{d}\tau\, e^{i\omega_n\tau}\,\Delta_E(\tau)$, and making use of the spectral representation

$$\tilde{\Delta}_E(\omega_n) = \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{\pi}\frac{\rho(\omega)}{\omega - i\omega_n} \ ,$$

argue that if $\Delta(\tau) = \Delta(0)\ \forall\ \tau$, then $\frac{\rho(\omega)}{\omega} = \pi\beta\Delta_E(0)\delta(\omega)$.