QCD at finite T and μ

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1. Static thermodynamics

$\longrightarrow$ Euclidean, "understood" up to non-perturbative level, but only a limited class of observables

2. Real-time observables

$\longrightarrow$ Minkowskian, even leading-order perturbative computations very hard, but simple physical interpretations

3. Finite baryon density

$\longrightarrow$ adventurous, "condensed matter physics" of QCD, but largely model computations so far

3. Finite baryon density

The QCD Lagrangian possesses the global $\mathrm{U}_V(1)$ symmetry

$$\psi_A \to e^{i\alpha}\psi_A \;, \quad \bar{\psi}_A \to e^{-i\alpha}\bar{\psi}_A \;, \quad A = 1, \ldots, N_\mathrm{c} \times N_\mathrm{f} \;.$$

The corresponding Noether current reads

$$J_\mu = \sum_A \frac{\partial \mathcal{L}_M}{\partial(\partial^\mu \psi_A)} \frac{\delta \psi_A}{\delta \alpha} = \sum_A \bar{\psi}_A \, i\gamma_\mu \, i\psi_A = -\sum_A \bar{\psi}_A \gamma_\mu \psi_A .$$

The charge $Q = \int_\mathbf{x} J_0$ is called the quark number ($\equiv N_\mathrm{c}\times$ baryon number). As an operator it commutes with the Hamiltonian, $[\hat{H}, \hat{Q}] = 0$. We then consider a system defined by the density matrix

$$\hat{\rho} \;=\; \frac{1}{\mathcal{Z}} e^{-\beta(\hat{H} - \mu \hat{Q})} \;.$$

Phenomenological motivation: Astrophysics

The relationship between the **mass and radius** of compact stars (neutron or "quark" stars) is described by the Tolman-Oppenheimer-Volkov equations:

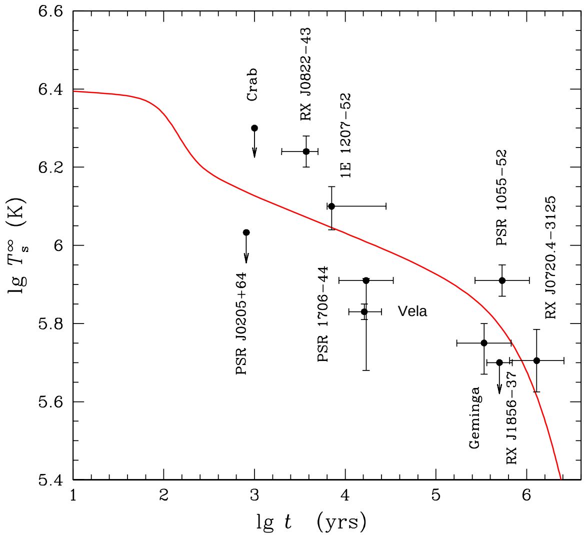
$$
\begin{aligned}
\frac{\mathrm{d}p(r)}{\mathrm{d}r} &= -\frac{[p(r)+e(r)][M(r)+4\pi r^3 p(r)]}{r^2 m_{\mathrm{Pl}}^2 - 2rM(r)} \,, \\
\frac{\mathrm{d}M(r)}{\mathrm{d}r} &= 4\pi r^2 e(r) \,.
\end{aligned}
$$

Here r-dependence comes through $p(r) \equiv p(T(r),\mu(r)) - p(0,0)$, and one can assume $T(r) \ll \mu(r)$.

In the core ($r \longrightarrow 0$) μ could perhaps reach values even up to ~ 0.7 GeV, so "deconfinement" could take place.

Current status: Alford et al, astro-ph/0606524

Many other properties of compact stars are also measured, such as their **cooling rate**, $\mathrm{d}T/\mathrm{d}t$.



Yakovlev, Pethick astro-ph/0402143

This might be explained by particle production (dominantly **neutrino emissivity**), in analogy with the Hard Probes of heavy ion collisions (see below).

Theoretical framework

We can treat the combination $\hat{H} - \mu\hat{Q}$ as an "effective" Hamiltonian, and directly write down the corresponding path integral, by simply adding (we leave out $\sum_A$)

$$-\mu\, Q = \mu \int_{\mathbf{x}} \bar{\psi}_A \gamma_0 \psi_A$$

to the **Euclidean** action. For perturbation theory, consider the quadratic part. In momentum space,

$$S_E = \sum\!\!\!\!\!\!\!\int_{P_\mathrm{f}} \bar{\tilde{\psi}}_A(P)[i\gamma_0\omega_n^{\mathrm{f}} + i\gamma_i p_i + \gamma_0\mu + m]\tilde{\psi}_A(P)\ .$$

Therefore, the chemical potential simply shifts $\omega_n^{\mathrm{f}} \to \omega_n^{\mathrm{f}} - i\mu$ in the Matsubara frequencies (below $\omega_n \equiv \omega_n^{\mathrm{f}}$).

To understand the corresponding physics, let us compute the conjugate variable, the quark number density:

$$
\begin{aligned}
n &\equiv \frac{\langle \hat{Q} \rangle}{V} = \frac{\partial p(T,\mu)}{\partial \mu} \\
&= \frac{\partial}{\partial \mu} \left\{ 4N_\mathrm{c} \times \frac{1}{2} \SumInt_{P_\mathsf{f}} \ln[(\omega_n - i\mu)^2 + \mathbf{p}^2 + m^2] \right\} \\
&= -4N_\mathrm{c} \SumInt_{P_\mathsf{f}} \frac{i(\omega_n - i\mu)}{(\omega_n - i\mu)^2 + E^2} \, , \quad E^2 \equiv \mathbf{p}^2 + m^2 \, .
\end{aligned}
$$

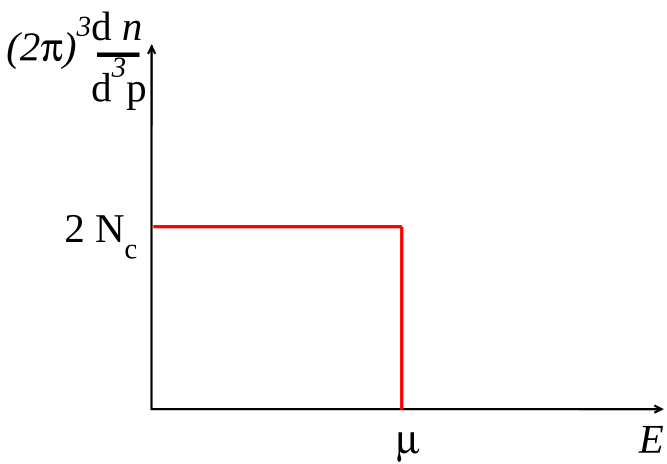
Using $T\sum_{\omega_n^\mathsf{f}} \frac{1}{(\omega_n - i\mu)^2 + E^2} = \frac{1}{2E}[1 - n_\mathsf{F}(E-\mu) - n_\mathsf{F}(E+\mu)]$ (cf. Exercise 1), where $n_\mathsf{F}(E) \equiv 1/(e^{\beta E}+1)$, it is not difficult to show that (Exercise 5)

$$
n = 2N_\mathrm{c} \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \left[n_\mathsf{F}(E-\mu) - n_\mathsf{F}(E+\mu) \right] \, .
$$

In the zero-temperature limit:

$$\lim_{T\to 0} n_{\mathsf{F}}(E-\mu) = \lim_{\beta\to\infty}\left\{\frac{1}{\exp[\beta(E-\mu)]+1}\right\}$$
$$= \theta(\mu - E)\ .$$

So we get a Fermi surface:



The only particles free to interact are those on the Fermi surface. In particular, there are 2$\longrightarrow$2 scatterings with momenta $(\mathbf{p}, -\mathbf{p}) \to (\mathbf{k}, -\mathbf{k})$, $E_{\mathbf{p}} = E_{\mathbf{k}} = \mu$.

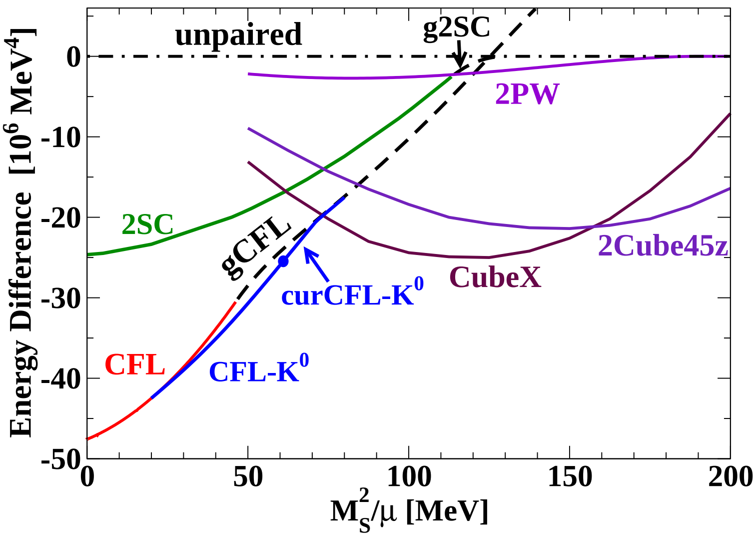
Now, if the two quarks with opposite momenta (Cooper pair) can be arranged into any spin/colour/flavour channel which is attractive, then the strength of the 2$\longrightarrow$2 scatterings on the Fermi surface can be argued to become strong (in a Renormalization Group for the scattering amplitude, a Landau pole develops).

This corresponds to the formation of a "colour superconducting" **condensate**, in analogy with BCS.

Proof for QCD in weak coupling: Son hep-ph/9812287

The problem

For realistic densities ($\mu \lesssim 0.7$ GeV), the coupling is large. So it is very difficult to determine which channel has the strongest attractive interaction, and what type of a condensate forms as a result.



Alford et al 0709.4635

In any case, interesting phenomena could arise.

The "spectrum" (in the sense of screening masses) is non-trivial: quarks have an energy gap across the Fermi momentum; gluons get screened by the Meissner effect (apart from confinement).

$\mathrm{U}_V(1)$ may break spontaneously $\Rightarrow$ superfluidity?

Condensate may break translational invariance $\Rightarrow$ crystal?

Consequently transport properties (viscosities, conductivities, etc) might become very exotic; these might affect compact star phenomenology.

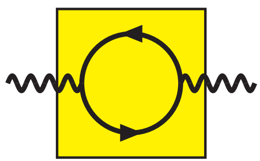
So, even more condensed matter and statistical physics knowledge needed than with real-time observables!

Theoretical challenge for today

How can a dense system lose energy through $\gamma \to \nu\bar{\nu}$?

As we saw in the first lecture, a heat bath can generate a (Debye-type) mass (m_{E}) for gauge fields.

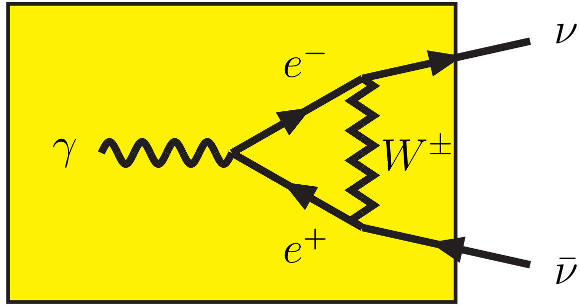
The same happens for photons in a dense system, through interactions with quarks on the Fermi surface.



$$m_\gamma^{\mathrm{eff}} \equiv \omega_{\mathrm{pl}} \sim e\mu$$

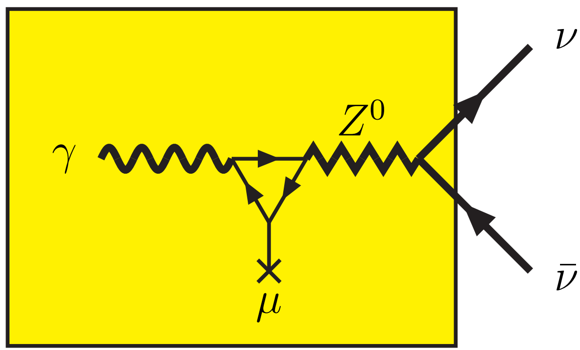
So, γ effectively becomes massive. On the other hand, neutrinos do not take part in thermal interactions; they remain massless, and the channel $\gamma \to \nu\bar{\nu}$ opens up.

Process 1:



Adams, Ruderman, Woo Phys. Rev. 129(1963)1383
Braaten, Segel hep-ph/9302213

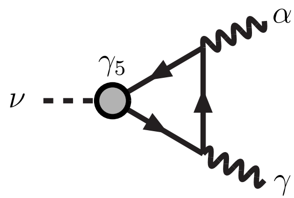
Process 2:



Harvey, Hill, Hill 0708.1281

For fun, consider the latter case now.

To start with, recall the anomalous Ward identity



$$\langle \partial_\nu [\bar{\psi}\gamma_\nu\gamma_5\psi]\rangle_{A_\nu^a} = 2m\langle\bar{\psi}\gamma_5\psi\rangle_{A_\nu^a} + \frac{g^2}{32\pi^2}\epsilon_{\alpha\beta\gamma\delta}F^a_{\alpha\beta}F^a_{\gamma\delta} \ .$$

Here the last term can be written as a total derivative,

$$\frac{g^2}{32\pi^2}\epsilon_{\alpha\beta\gamma\delta}F^a_{\alpha\beta}F^a_{\gamma\delta} = \partial_\mu K_\mu \ ,$$

$$K_\mu = \frac{g^2}{8\pi^2}\epsilon_{\mu\nu\lambda\rho}\left[A^a_\nu\partial_\lambda A^a_\rho + \frac{g}{3}f^{abc}A^a_\nu A^b_\lambda A^c_\rho\right] \ .$$

The component K_0 is called the Chern-Simons density.

On some level, this says that the "chiral charge", $\int d^3\mathbf{x}\,\bar{\psi}\gamma_0\gamma_5\psi$, and the Chern-Simons number, $\int d^3\mathbf{x}\,K_0$, are related to each other (for $m \to 0$).

At the same time, $\int d^3\mathbf{x}\,K_0$ is gauge invariant only in "small" gauge transformations (in "large" ones it changes by an integer), so the precise nature of the relation must be subtle.

In any case, if we give a chemical potential to the quarks (and leptons) of the Standard Model (thinking of $\mu\bar{\psi}\gamma_0\psi$ as a "vertex"), and choose the gauge fields on the external legs to be from SU$_L(2)$ or U$_Y(1)$, then γ_5 appears in some corners, and essentially the same graph could play a role.

To be more concrete, let us (naively) consider

$$\mathcal{L}_E = \bar{\psi}_L \left[\gamma_\nu D_\nu + \mu\gamma_0\right] \psi_L \ ,$$

with $\psi_L \equiv a_L\psi$, $a_L \equiv (1-\gamma_5)/2$. (We have set $m = 0$.)

Free propagator:

$$\langle \psi_L(P)\bar{\psi}_L(Q) \rangle = (2\pi)^4 \delta^{(4)}(P-Q)\ a_L \frac{-i\tilde{\not{P}}}{\tilde{P}^2} a_R \ ,$$

where $\tilde{P} \equiv (\omega_n^{\mathrm{f}} - i\mu, \mathbf{p})$, $a_R \equiv (1+\gamma_5)/2$.

Interaction:

$$\mathcal{L}_{E,I} = -ig\,\bar{\psi}_L \gamma_\nu A_\nu \psi_L \ .$$

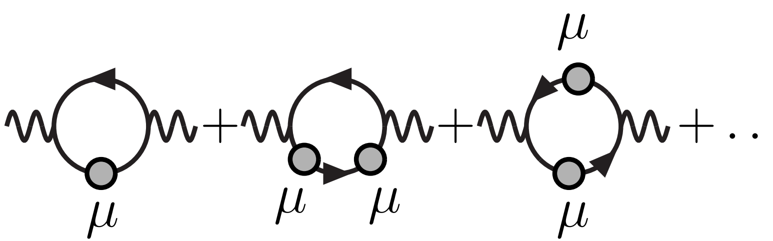
Compute now the **effective action** for A^a_μ (generating functional of 1PI Green's functions). Quadratic part:

$$S^{(2)}_{E,\text{eff}} = \frac{g^2 N_c}{4} \sum\!\!\!\!\!\!\!\int_Q A^a_\mu(Q) A^a_\nu(-Q) \sum\!\!\!\!\!\!\!\int_{\tilde{P}\text{f}} \frac{1}{\tilde{P}^2(\tilde{P}+Q)^2} \,\mathrm{Tr}\left[(\tilde{\not{P}} + \not{Q})\gamma_\mu \tilde{\not{P}} \gamma_\nu a_L\right]$$

$$\Rightarrow -\frac{g^2 N_c}{2} \sum\!\!\!\!\!\!\!\int_Q A^a_\mu(Q) A^a_\nu(-Q) \sum\!\!\!\!\!\!\!\int_{\tilde{P}\text{f}} \frac{1}{\tilde{P}^2(\tilde{P}+Q)^2} Q_\alpha \tilde{P}_\beta \epsilon_{\alpha\mu\beta\nu} \,,$$

where we inserted $\mathrm{Tr}[\gamma_5\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu] = 4\epsilon_{\alpha\mu\beta\nu}$.

This corresponds to the sum



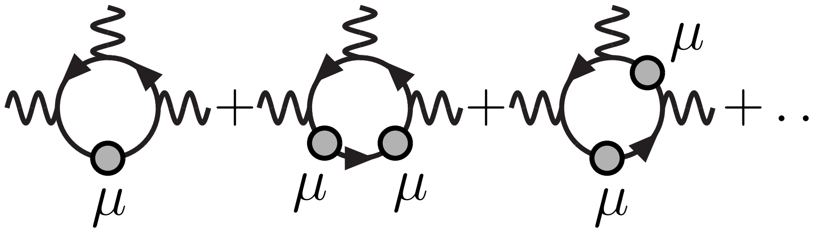
Let us expand to leading order in Q (low-energy expansion). The result can be non-zero only for the index choice $\beta = 0$. The sum-integral needed (Ex. 6):

$$
\begin{aligned}
\SumInt_{\tilde{P}\,\mathrm{f}} \frac{\tilde{P}_0}{(\tilde{P}^2)^2} &= T \sum_{\omega_n^{\mathrm{f}}} \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{\omega_n - i\mu}{[(\omega_n - i\mu)^2 + \mathbf{p}^2]^2} \\
&\overset{T \ll \mu}{=} \frac{i\mu}{8\pi^2} \, .
\end{aligned}
$$

So, in configuration space,

$$
S^{(2)}_{E,\mathrm{eff}} \approx \frac{\mu g^2 N_{\mathrm{c}}}{(4\pi)^2} \int_0^\beta \mathrm{d}\tau \int_{\mathbf{x}} \epsilon_{ijk} A_i^a \partial_j A_k^a \, .
$$

It turns out that the 3-point function leads to the same integral.



Combining the two; taking both $\mathsf{SU}_L(2)$ and $\mathsf{U}_Y(1)$ gauge fields as external legs; and letting the loop be either a quark ($\mu \longrightarrow \mu_Q$) or a lepton ($\mu \longrightarrow \mu_{L_i}$), yields

$$S^{(2)}_{E,\mathrm{eff}} \approx \sum_{i=1}^{3} (N_\mathrm{c}\mu_Q + \mu_{L_i}) \int_0^\beta \mathrm{d}\tau \int_\mathbf{x} \left[K_0^{\mathsf{SU}_L(2)} - K_0^{\mathsf{U}_Y(1)} \right] .$$

Redlich, Wijewardhana PRL 54(1985)970

Note that:

• Strictly speaking, such a term should not emerge: it is not gauge invariant (and not bounded from below).

• $\sum_{i=1}^{3}(N_c\mu_Q + \mu_{L_i})$ is the chemical potential for $B+L$, i.e. Baryon number + Lepton number.

• The reason for the problems: $B+L$ is anomalous in the Standard Model, i.e. **not** conserved $\Rightarrow$ in field theory no chemical potential should be assigned to it.

• It turns out, however, that the rate for $B+L$ violation is exponentially small at low temperatures. Therefore, there may exist **non-equilibrium** situations which can effectively be described by a conserved $B+L$.

Let us now inspect the quadratic part of $S^{(2)}_{E,\text{eff}}$. We rewrite it in terms of the physical fields ($\tan\theta_w \equiv \frac{g_1}{g_2}$)

$$\begin{aligned} Z_i &= \cos\theta_w\, A_i^3 + \sin\theta_w\, B_i \ , \\ Q_i &= -\sin\theta_w\, A_i^3 + \cos\theta_w\, B_i \ . \end{aligned}$$

The structure is:

$$\begin{aligned} g_2^2(A_i^3)^2 - g_1^2 B_i^2 &= (g_2 A_i^3 + g_1 B_i)(g_2 A_i^3 - g_1 B_i) \\ &= (g_1^2 + g_2^2)\left[-Z_i Q_i \sin 2\theta_w + Z_i^2 \cos 2\theta_w\right] \ . \end{aligned}$$

$$\Rightarrow \begin{cases} \bullet\ \nexists\ K_0^{\text{QED}} \text{ because QED is vectorlike.} \\ \bullet\ \exists \text{ coupling between } Z_i \text{ and } Q_i. \\ \bullet\ \exists \text{ new ``mass term'' (actually tachyonic) for } Z_i. \end{cases}$$

Remark

In 0708.1281, Harvey et al reach the same conclusion via a totally different route: they consider a gauged WZW-action, with external SU$_L$(2) and U$_Y$(1) gauge fields as well as a flavour-singlet vector meson ω_μ.

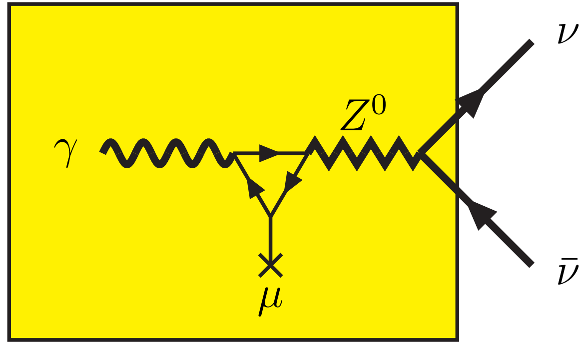
The field ω_μ couples to a baryon density, and integrating it out, they find a $Z_i Q_i$-coupling analogous to the one above, but with

$$\mu_B \equiv N_\mathrm{c} \mu_Q \longrightarrow \frac{g_\omega^2}{m_\omega^2} n_B \sim (0.4 \ldots 4.0)\ \text{GeV}\ ,$$

with a phenomenological coupling $g_\omega = 10 \ldots 30$.

Physics consequence

From the amplitude



one can now estimate the "emissivity":

$$\frac{\mathrm{d}E}{\mathrm{d}^3\mathbf{x}\,\mathrm{d}t} \sim -\frac{\alpha_{\mathsf{em}}}{16\pi^6}\mu_B^2\, G_F^2\, \omega_p^{9/2} T^{5/2} e^{-\omega_p/T}\;,$$

where the "plasmon frequency" ω_p plays the role of a dynamical photon mass ($T \lesssim \omega_p \sim 1$ MeV?).

Summary

Despite asymptotic freedom, QCD at finite baryon density remains an extremely complicated and difficult system to solve (Fermi surface phenomena play a role).

Indeed, some of its properties are not properly understood even at leading order in the gauge coupling.

Yet, there appears to be exciting physics taking place, some of which could be relevant for astrophysics.

Exercise 5: "Sum for particle number density".

In analogy with Exercise 1, it can be shown that $T\sum_{\omega_n^{\mathsf{f}}} \frac{1}{(\omega_n - i\mu)^2 + E^2} = \frac{1}{2E}[1 - n_{\mathsf{F}}(E-\mu) - n_{\mathsf{F}}(E+\mu)]$. Starting from here (or elsewhere), show that

$$T\sum_{\omega_n^{\mathsf{f}}} \frac{i(\omega_n - i\mu)}{(\omega_n - i\mu)^2 + E^2} = -\frac{1}{2}\left[n_{\mathsf{F}}(E-\mu) - n_{\mathsf{F}}(E+\mu)\right] \, .$$

Exercise 6: "Triangle integral at finite μ".

Show that

$$\int \frac{\mathrm{d}^4 P}{(2\pi)^4} \frac{p_0 - i\mu}{[(p_0 - i\mu)^2 + \mathbf{p}^2]^2} = \frac{i\mu}{8\pi^2} \, .$$