• *Problem 1*

Prove that

$$\sum_{a=1}^{N^2-1} t^a_{ij} t^a_{lk} = \frac{1}{2}\left(\delta_{ik}\delta_{jl} - \frac{1}{N}\delta_{ij}\delta_{kl}\right) . \tag{1}$$

where t^a are the SU(N) generators in the fundamental representation, normalized as follows:

$$\mathrm{Tr}\left(t^a t^b\right) = T_R \delta^{ab} \equiv \frac{1}{2}\delta^{ab} . \tag{2}$$

• *Problem 2*

The parameter Λ_{QCD} is defined as follows:

$$\log \frac{Q^2}{\Lambda^2_{QCD}} = -\int_{\alpha_S(Q^2)}^{\infty} da \frac{1}{\beta(a)} , \tag{3}$$

with

$$\beta(\alpha_S) = -\beta_0 \alpha_S^2 \left(1 + b'\alpha_S + b''\alpha_S^2 + \mathcal{O}(\alpha_S^3)\right) . \tag{4}$$

i) Express $\alpha_S(Q^2)$ as a function of Λ_{QCD} at one loop (i.e. neglecting the contributions of b' and of b'') and at two loops (i.e. neglecting the contribution of b'').

ii) If A and B denote two renormalization schemes such that

$$\alpha_S^B = \alpha_S^A \left\{1 + c_1 \alpha_S^A + c_2 \left(\alpha_S^A\right)^2 + \mathcal{O}\left(\left(\alpha_S^A\right)^3\right)\right\} \tag{5}$$

prove that the parameters Λ_{QCD} in the two schemes are given by

$$\Lambda^B_{QCD} = \Lambda^A_{QCD} \exp\left(\frac{c_1}{2\beta_0}\right) \tag{6}$$

and that β_0 and b' are scheme independent.

• *Problem 3*

Consider the processes

$$u\bar{u} \longrightarrow \gamma\gamma , \tag{7}$$

$$u\bar{u} \longrightarrow gg . \tag{8}$$

i) Draw the corresponding Feynman diagrams (in QED and QCD respectively), and using Feynman rules write down the amplitudes for all diagrams.

ii) Prove, at the level of amplitudes (i.e., without squaring them), that the non-transverse polarization degrees of freedom of the photons and of the gluons do not give contributions to the cross sections. Compare the cases of QED and QCD and discuss the differences. [Hint: contract with the photon/gluon four-momenta; what happens if the polarization of the gluon whose four-momentum is not contracted is not transverse? Why is the situation different wrt QED?].

iii) Compute the matrix element squared for the QCD process, eq. (8). In doing so, express the colour factors in terms of N_c (i.e., do not use $N_c = 3$ in the computation). In the $N_c \to \infty$ limit, discuss the pattern of interference among Feynman diagrams.

• *Problem 4*

Using the definition of the plus prescription, prove that the regularized lowest order Altarelli-Parisi qq kernel obeys:

$$P_{qq}^{(0)}(z) \equiv C_F \left(\frac{1+z^2}{1-z}\right)_+ = C_F \frac{1+z^2}{(1-z)_+} + \frac{3C_F}{2}\delta(1-z)\,. \tag{9}$$

Furthermore, using the definition

$$\gamma_{qq}^{(0)}(j) = \int_0^1 dz z^{j-1} P_{qq}^{(0)}(z) \tag{10}$$

prove that

$$\gamma_{qq}^{(0)}(j) = C_F \left\{ -\frac{1}{2} + \frac{1}{j(j+1)} - 2\sum_{i=2}^{j} \frac{1}{i} \right\}\,. \tag{11}$$

• *Problem 5*

Consider the branching

$$q(E) \to q(E_q, \theta_q) + g(E_g, \theta_g) \tag{12}$$

with θ_q and θ_g the polar angles wrt the direction of flight of the quark that splits. Introducing z such that

$$E_q = zE\,, \qquad E_g = (1-z)E \tag{13}$$

prove that the leading behaviour (for $\theta_q, \theta_g \to 0$) of the matrix element squared relevant to the branching with the gluon polarized in the plane determined by the two quarks is

$$|\mathcal{A}_{in}|^2 \to \frac{(1+z)^2}{1-z} \frac{1}{Q^2}, \tag{14}$$

with Q^2 the invariant mass squared of the qg pair emerging from the branching. Prove also that, if the gluon is polarized out of the plane

$$|\mathcal{A}_{out}|^2 \to (1-z) \frac{1}{Q^2}, \tag{15}$$

and therefore that, putting together eqs. (14) and (15), one recovers the usual Altarelli-Parisi qq (unpolarized) kernel.

• *Problem 6*
Consider a system S of mass M_s produced in the collisions of two incoming partons:

$$a + b \longrightarrow S. \tag{16}$$

Keeping in mind the definitions of rapidity and pseudorapidity:

$$y = \frac{1}{2} \log \frac{k^0 + k^3}{k^0 - k^3}, \qquad \eta = -\log \tan \frac{\theta}{2}, \tag{17}$$

with θ the polar angle of the four-momentum k wrt to the axis parallel to k^3.

i) Express (in an arbitrary frame) the pseudorapidity η_S of S as a function of its rapidity y_S and its transverse momentum $p_T^{(S)}$.

ii) Denoting by E the total collider c.m. energy, and the four-momenta of partons a and b by

$$p_a = x_1 \frac{E}{2}(1,0,0,1), \qquad p_b = x_2 \frac{E}{2}(1,0,0,-1) \tag{18}$$

in the incoming hadron c.m. frame, express x_1 and x_2 as a function of the variables of the system S.