Non-relativistic EFTs at £nite T

- the static potential at one loop -

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Bibliography

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Lecture I

1. Quarkonium dissociation

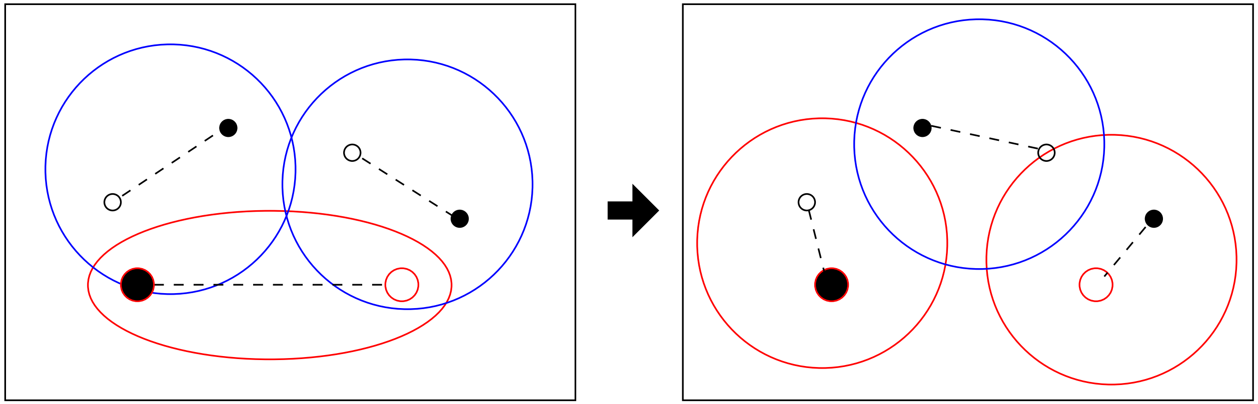
2. Static quarks in real time and £nite T

3. Momentum regions of the static potential

In medium quarkonium dissociation

- Thermal medium induces color screening.
- Color screening of the heavy quark potential, may induce quarkonium dissociation. Quarkonium dissociation may be a clear signature of quark-gluon plasma formation.
    - Matsui Satz PLB 178(86)416

In-medium string breaking through recoupling:



Color singlet $Q\bar{Q}$ free energy

The basic quantity for a canonical ensemble at temperature T is the partition function

$$Z = \sum_n \langle n | e^{-H/T} | n \rangle = e^{-F/T}$$

where F is the free energy.

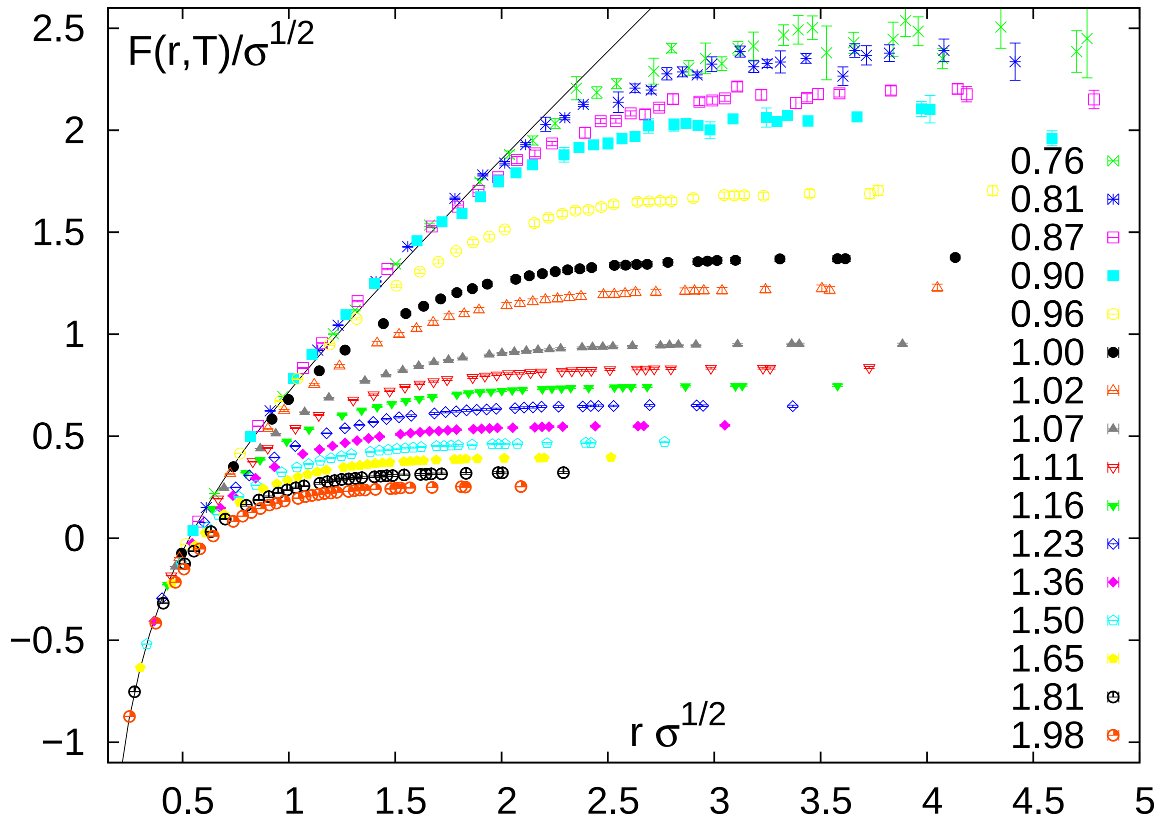
We may de£ne the color singlet $Q\bar{Q}$ free energy as the projection of F on the Fock space containing a static $Q\bar{Q}$ pair at distance r in a color singlet con£guration:

$$e^{-F(r,T)/T} = \sum_n \langle (Q\bar{Q})_1\, n | e^{-H/T} | (Q\bar{Q})_1\, n \rangle$$

∘ McLerran Svetitsky PRD 24(81)450

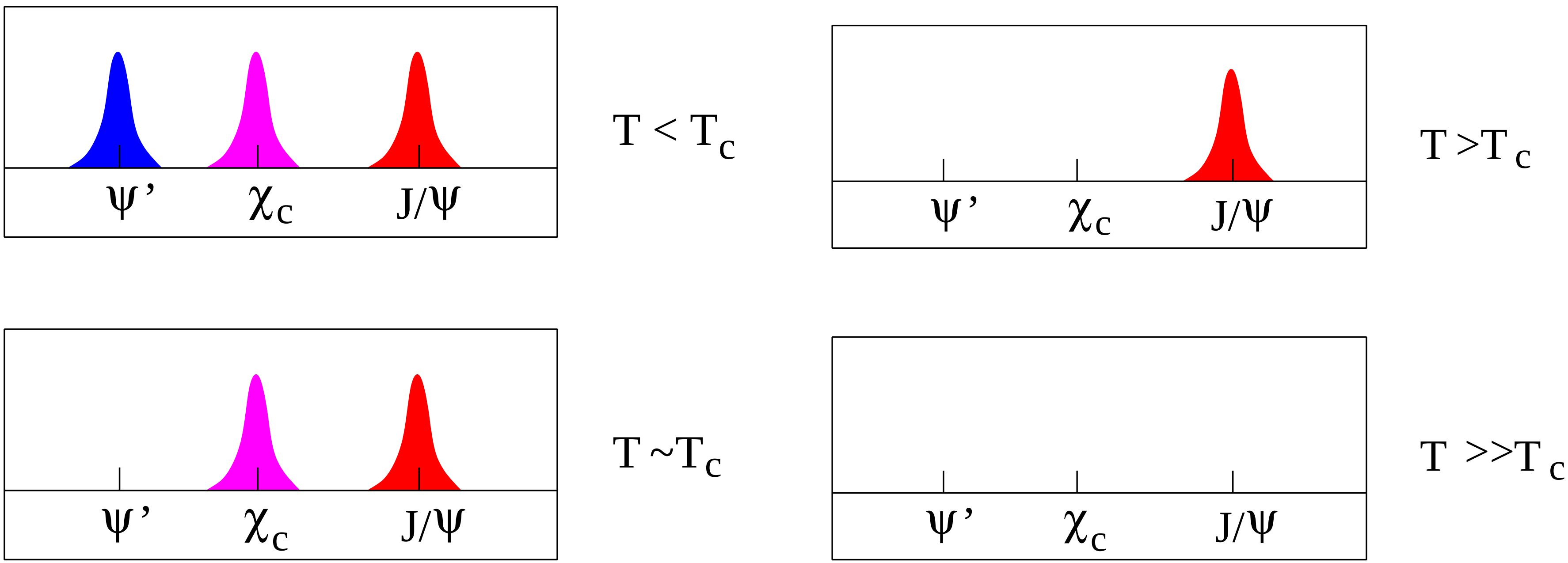
Color singlet $Q\bar{Q}$ free energy

The color singlet $Q\bar{Q}$ free energy $F(r,T)$ vs. r at different T.



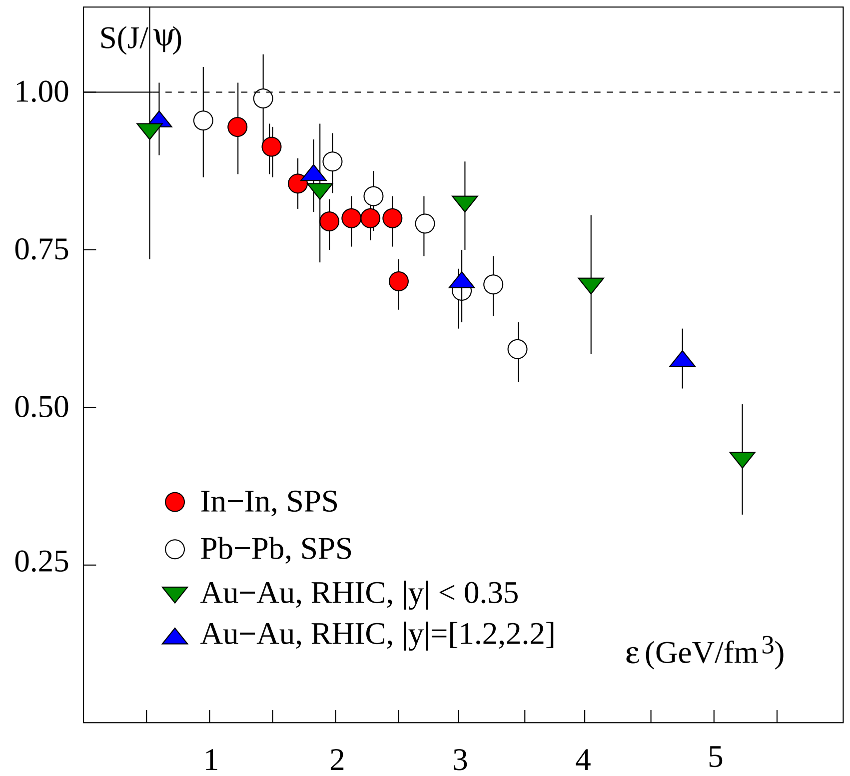
∘ Kaczmarek et al. PTP Sup 153(04)287; Kaczmarek Zantow hep-lat/0503017

Charmonium spectra at different temperature



From lattice studies: $T_c = 175 \pm 10$ MeV.

Charmonium dissociation at SPS and RHIC



The quarkonium potential at £nite T

In order to rigorously study quarkonium properties in a thermal bath at temperature T, the £rst quantity to determine is the quarkonium potential. The potential describes the real-time evolution of a $Q\bar{Q}$ pair through the Schrödinger equation

$$E\,\Phi = \left(\frac{p^2}{m} + V(r,T)\right)\,\Phi$$

In the full theory, it must come from

- a non-relativistic expansion in $1/m$, the leading term being the static potential;
- an expansion in the energy E.

We will exploit these expansions either by direct computation over momentum regions (Lecture I) or by conctructing a suitable hierarchy of EFTs (Lecture II).

• Note that the free energy is a thermodynamical quantity, whichy may serve as a guide on what the quarkonium potential at £nite temperature may be, but it is NOT the potential we are looking for.

Static limit of QCD

In order to calculate the static $Q\bar{Q}$ potential, the fundamental theory is

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} + \sum_{i=1}^{n_f} \bar{q}_i \, i\not{D} q_i + \psi^\dagger i D_0 \psi + \chi^\dagger i D_0 \chi$$

ψ (χ) is the field that annihilates (creates) the static (anti)quark.

- This may be also seen as the LO EFT Lagrangian that follows from QCD by expanding in $1/m$ under the condition that $m \gg$ any other scale in the problem.
- The relevant scales for static $Q\bar{Q}$ pairs at £nite T are: $1/r$, E, ... T, m_D, ...
- Only longitudinal gluons couple to static quarks.

Propagators at £nite T

- $\langle O \rangle_T = \dfrac{\sum_n \langle n | e^{-H/T}\, O | n \rangle}{Z} = \dfrac{\mathrm{Tr}\, \{ e^{-H/T}\, O \}}{Z} = \dfrac{\int_{\text{p./antip.}} \mathcal{D}\phi\, O\, e^{-\int_0^{1/T} dt \int d^3x \mathcal{L}}}{Z}$

- $\phi \in \{A, \psi, \chi, q_i\}$ is a bosonic/fermionic field, we de£ne the following propagators

$$D^{>}(x) = \langle \phi(x)\phi(0) \rangle_T, \quad D^{<}(x) = \pm \langle \phi(0)\phi(x) \rangle_T$$
$$D(x) = \theta(x^0)\, D^{>}(x) \pm \theta(-x^0)\, D^{<}(x)$$

- From the equal-time canonical commutation relation it follows the sum rule (in momentum space)

$$\int \frac{dk^0}{2\pi} \left(D^{>}(k) - D^{<}(k) \right) = 1$$

- The spectral density ρ is de£ned as $\rho(x) = D^{>}(x) - D^{<}(x)$

Kubo–Martin–Schwinger formula

- From the cyclicity of the trace it follows that (for bosons/fermions)

$$
\begin{aligned}
D^{>}(x) &= \frac{\mathrm{Tr}\,\{e^{-H/T}\,\phi(x)\,e^{H/T}\,e^{-H/T}\,\phi(0)\}}{Z} = \frac{\mathrm{Tr}\,\{\phi(x^0+i/T,\vec{x})\,e^{-H/T}\,\phi(0)\}}{Z} \\
&= \pm\frac{\mathrm{Tr}\,\{e^{-H/T}\,\phi(0)\,\phi(x^0+i/T,\vec{x})\}}{Z} = \pm D^{<}(x^0+i/T,\vec{x})
\end{aligned}
$$

which in momentum space reads $D^{<}(k) = \pm e^{-k^0/T} D^{>}(k)$

- Since $D^{>}$ and $D^{<}$ are related, we may express them in term of ρ:

$$
\begin{aligned}
&D^{>}(k) = (1+n_{\mathrm{B}}(k^0))\rho(k), \quad D^{<}(k) = n_{\mathrm{B}}(k^0)\rho(k) && \text{for bosons} \\
&D^{>}(k) = (1-n_{\mathrm{F}}(k^0))\rho(k), \quad D^{<}(k) = -n_{\mathrm{F}}(k^0)\rho(k) && \text{for fermions}
\end{aligned}
$$

where

$$
n_{\mathrm{F}}(k^0) = \frac{1}{e^{k^0/T}+1}, \quad n_{\mathrm{B}}(k^0) = \frac{1}{e^{k^0/T}-1}
$$

Retarded and advanced propagators at £nite T

We may express the full propagator also in terms of retarded and advanced propagators, which for the bosonic/fermionic case read (in momentum space)

$$D^{\mathrm{R}}(k) = \int d^4x\, e^{ik\cdot x}\, \theta(x_0)\langle \phi(x)\phi(0) \mp \phi(0)\phi(x)\rangle_T$$

$$D^{\mathrm{A}}(k) = -\int d^4x\, e^{ik\cdot x}\, \theta(-x_0)\langle \phi(x)\phi(0) \mp \phi(0)\phi(x)\rangle_T$$

then we have

$$\begin{aligned}
\rho(k) &= D^{\mathrm{R}}(k) - D^{\mathrm{A}}(k), \\
D(k) &= D^{\mathrm{R}}(k) + D^{<}(k) = D^{\mathrm{A}}(k) + D^{>}(k) \\
&= \frac{D^{\mathrm{R}}(k) + D^{\mathrm{A}}(k)}{2} + \left(\frac{1}{2} + n_{\mathrm{B,F}}(k^0)\right)\rho(k).
\end{aligned}$$

Free static quark propagator

The free static quark propagators (dropping Lorentz indeces) satisfy the e.o.m.

$$k^0 S^{>\,(0)}(k) = m S^{>\,(0)}(k), \qquad k^0 S^{<\,(0)}(k) = m S^{<\,(0)}(k)$$

the Kubo–Martin–Schwinger relation

$$S^{<\,(0)}(k) = -e^{-k^0/T} S^{>\,(0)}(k)$$

and the sum rule

$$\int \frac{dk^0}{2\pi} \left(S^{>\,(0)}(k) - S^{<\,(0)}(k) \right) = 1$$

whose solution is

$$S^{>\,(0)}(k) = (1 - n_{\rm F}(k^0))\, 2\pi\delta(k^0 - m), \qquad S^{<\,(0)}(k) = -n_{\rm F}(k^0)\, 2\pi\delta(k^0 - m)$$

$$S^{(0)}(k) = \frac{i}{k^0 - m + i\epsilon} - n_{\rm F}(k^0)\, 2\pi\delta(k^0 - m)$$

$$\rho^{(0)}(k) = S^{>\,(0)}(k) - S^{<\,(0)}(k) = 2\pi\delta(k^0 - m)$$

Free static quark propagator

In the static limit $m \to \infty$, the propagators simplify because $n_{\mathrm{F}}(m) \to 0$; moreover we may get rid of the explicit mass dependence by means of the field rede£nition $\psi \to \psi e^{-imt}$

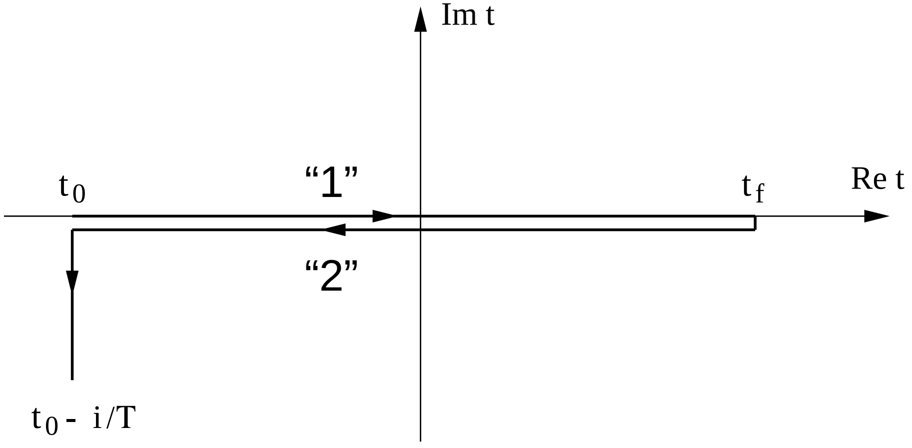
$$S^{>\,(0)}(k) = 2\pi\delta(k^0), \qquad S^{<\,(0)}(k) = 0$$
$$S^{(0)}(k) = \frac{i}{k^0 + i\epsilon}$$
$$\rho^{(0)}(k) = 2\pi\delta(k^0)$$

The retarded and advanced propagators read

$$S^{\mathrm{R}}(k) = -i\int \frac{d\omega}{2\pi}\,\frac{\rho^{(0)}(\omega,\vec{k})}{\omega - k^0 - i\epsilon}, \qquad S^{\mathrm{A}}(k) = -i\int \frac{d\omega}{2\pi}\,\frac{\rho^{(0)}(\omega,\vec{k})}{\omega - k^0 + i\epsilon}$$

Real time

In order to have a perturbative expansion that involves real-time propagators, and hence it is very close to the zero temperature case, it is convenient to modify the contour of the time integration in the partition function in order to allow for real time:



Green functions may be obtained from the modified partition function by the usual functional differentiation with respect to sources $j(x)$:

$$G(x_1, ..., x_n) = \frac{1}{Z} \frac{\delta^n Z(j)}{i\delta j(x_1)...i\delta j(x_n)}\bigg|_{j=0}$$

Real time

The price to pay is that the degrees of freedom double (“1” and “2”)

$$
\begin{aligned}
D_{11}(x) &= D(x) \\
D_{22}(x) &= D(x)^* \\
D_{12}(x) &= D^<(x) \\
D_{21}(x) &= D^>(x)
\end{aligned}
$$

This is an alternative approach to the equivalent calculations in imaginary time formalism + analytical continuation in real time, the possible advantages being that

- the framework becomes very close to the one for $T = 0$;
- in the static quark sector, the second degrees of freedom, labeled “2”, decouple from the physical degrees of freedom, labeled “1”.

Real-time static quark propagator

- Free static quark propagator:

$$\mathbf{S}_Q^{(0)}(p) = \begin{pmatrix} \dfrac{i}{p^0 + i\epsilon} & 0 \\ 2\pi\delta(p^0) & \dfrac{-i}{p^0 - i\epsilon} \end{pmatrix}$$

Since $[\mathbf{S}_Q^{(0)}(p)]_{12} = 0$, the static quark fields labeled “2” never enter in any physical amplitude, i.e. any amplitude that has the physical fields, labeled “1”, as initial and £nal states.

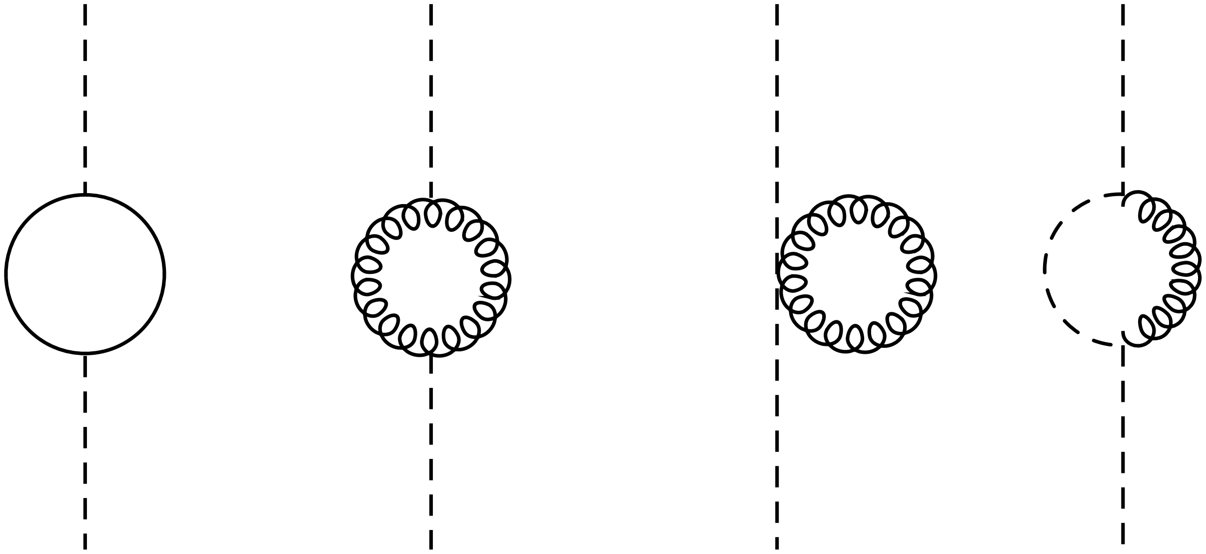
Real-time gluon propagator

- Free gluon propagator in Coulomb gauge:

$$
\begin{aligned}
\mathbf{D}_{00}^{(0)}(\vec{k}) &= \frac{i}{\vec{k}^2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
\mathbf{D}_{ij}^{(0)}(k) &= \left(\delta_{ij} - \frac{k^i k^j}{\vec{k}^2}\right)\left\{\begin{pmatrix} \dfrac{i}{k^2+i\epsilon} & \theta(-k^0)\,2\pi\delta(k^2) \\ \theta(k^0)\,2\pi\delta(k^2) & -\dfrac{i}{k^2-i\epsilon} \end{pmatrix}\right. \\
&\qquad \left. +2\pi\delta(k^2)\,n_{\mathrm{B}}(|k^0|)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right\}
\end{aligned}
$$

In Coulomb gauge, only transverse gluons carry a thermal part.

Gluon self energy



- We only need Π_{00};
- We only need $[\Pi_{00}]_{11}$ (for short Π_{00} from now on);
- We only need $\Pi_{00}(k)$ in some speci£c momentum regions.

$\Pi_{00}(k)$ for $k^0 \ll T \sim |\vec{k}|$

$$\mathrm{Re}\,\left[\Pi^{\mathrm{R}}_{00}(k)\right]_{\mathrm{thermal}} = \mathrm{Re}\,\left[\Pi^{\mathrm{A}}_{00}(k)\right]_{\mathrm{thermal}} =$$

$$\frac{g^2\,T_F\,n_f}{\pi^2}\int_0^{+\infty} dq^0\,q^0\,n_{\mathrm{F}}(q^0)\left[2+\left(\frac{|\vec{k}|}{2q^0}-2\frac{q^0}{|\vec{k}|}\right)\ln\left|\frac{|\vec{k}|-2q^0}{|\vec{k}|+2q^0}\right|\right]$$

$$+\frac{g^2\,N_c}{\pi^2}\int_0^{+\infty} dq^0\,q^0\,n_{\mathrm{B}}(q^0)\left[1-\frac{\vec{k}^2}{2(q^0)^2}+\left(-\frac{q^0}{|\vec{k}|}+\frac{|\vec{k}|}{2q^0}-\frac{|\vec{k}|^3}{8(q^0)^3}\right)\ln\left|\frac{|\vec{k}|-2q^0}{|\vec{k}|+2q^0}\right|\right]$$

$$\mathrm{Im}\,\left[\Pi^{\mathrm{R}}_{00}(k)\right]_{\mathrm{thermal}} = -\mathrm{Im}\,\left[\Pi^{\mathrm{A}}_{00}(k)\right]_{\mathrm{thermal}} =$$

$$\frac{2\,g^2\,T_F\,n_f}{\pi}\,\frac{k^0}{|\vec{k}|}\int_{|\vec{k}|/2}^{+\infty} dq^0\,q^0\,n_{\mathrm{F}}(q^0)$$

$$+\frac{g^2\,N_c}{\pi}\,\frac{k^0}{|\vec{k}|}\left[\frac{\vec{k}^2}{8}\,n_{\mathrm{B}}(|\vec{k}|/2)+\int_{|\vec{k}|/2}^{+\infty} dq^0\,q^0\,n_{\mathrm{B}}(q^0)\left(1-\frac{\vec{k}^4}{8(q^0)^4}\right)\right]$$

$\Pi_{00}(k)$ for $k^0 \sim |\vec{k}| \ll T$

In this limit, we obtain the hard thermal loop (HTL) expression for the longitudinal gluon polarization tensor,

$$\begin{aligned}
\mathrm{Re}\left[\Pi^{\mathrm{R}}_{00}(k)\right]_{\mathrm{thermal}} = \mathrm{Re}\left[\Pi^{\mathrm{A}}_{00}(k)\right]_{\mathrm{thermal}} &= m_D^2\left(1 - \frac{k^0}{2|\vec{k}|}\ln\left|\frac{k^0+|\vec{k}|}{k^0-|\vec{k}|}\right|\right) \\
\mathrm{Im}\left[\Pi^{\mathrm{R}}_{00}(k)\right]_{\mathrm{thermal}} = -\mathrm{Im}\left[\Pi^{\mathrm{A}}_{00}(k)\right]_{\mathrm{thermal}} &= m_D^2\,\frac{k^0}{|\vec{k}|}\,\frac{\pi}{2}\,\theta(-k^2)
\end{aligned}$$

where m_D is the Debye mass:

$$m_D^2 = \frac{g^2T^2}{3}\left(N_c + \frac{n_f}{2}\right)$$

◦ Braaten Pisarski NPB 337(90)569

- The Debye mass m_D is a dynamically generated, temperature dependent, scale. In the weak-coupling regime ($g \ll 1$), m_D is smaller than T. It effectively plays the role of a screening mass for the longitudinal gluons. This can be better seen under the condition $k^0 \ll |\vec{k}| \sim m_D \ll T$; in this case, the gluon polarization diagrams need to be resummed, giving the HTL resummed propagator

$$-iD_{00}^{\mathrm{R,A}}(k) = \frac{1}{\vec{k}^2 + m_D^2} \mp i\frac{\pi}{2}\,\frac{k^0}{|\vec{k}|}\,\frac{m_D^2}{\left(\vec{k}^2 + m_D^2\right)^2}$$

  The Fourier transform of the real part of the propagator is the Yukawa potential $\sim e^{-m_D r}/r$.

- The gluon polarization and therefore the gluon propagator develops an imaginary part. This may be traced back to the scattering of particles in the medium carrying momenta of order T with space-like gluons, a phenomenon also known as Landau damping.

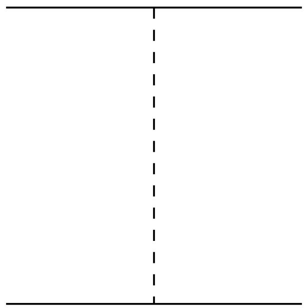
$\Pi_{00}(k)$ for $|\vec{k}| \gg T \gg k^0$

$$\left[\Pi^{\mathrm{R}}_{00}(k)\right]_{\mathrm{thermal}} = \left[\Pi^{\mathrm{A}}_{00}(k)\right]_{\mathrm{thermal}} = -\frac{N_c\, g^2\, T^2}{18}$$

At leading order, the result is real and does not depend on k.

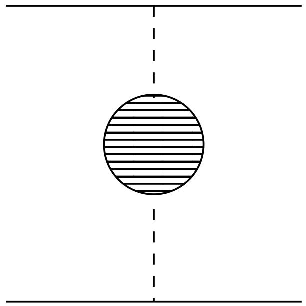
Tree level potential

- $Q\bar{Q}$ bound states for in£nitely heavy quarks in a thermal bath are characterized by the scales: $1/r$, E, T, m_D, ...
- we assume the following hierarchy: $1/r \gg T \gg m_D \gg E$;
- under this condition the momentum regions $\sim 1/r$, T and m_D all contribute to the potential;
- it is convenient to perform the calculation in Coulomb gauge.



$$V_s \equiv [\mathbf{V_s}(r)]_{11} = -C_F \frac{\alpha_{\rm s}}{r}$$

One-loop thermal corrections to the potential



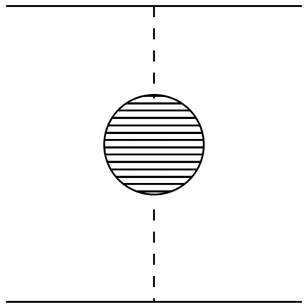
The one-loop correction to the potential is

$$\delta V_s \equiv [\delta \mathbf{V_s}(r)]_{11} = \mu^{4-d} \int \frac{d^{d-1}k}{(2\pi)^{d-1}}\, e^{-i\vec{k}\cdot\vec{r}}\, g^2\, C_F\, \left[i\delta \mathbf{D}_{00}(0,\vec{k}) \right]_{11}$$

where

$$\begin{aligned}
[\delta \mathbf{D}_{00}(k)]_{11} &= \frac{\delta D^{\mathrm{R}}_{00}(k) + \delta D^{\mathrm{A}}_{00}(k)}{2} + \left(\frac{1}{2} + n_{\mathrm{B}}(k^0) \right) \left(\delta D^{\mathrm{R}}_{00}(k) - \delta D^{\mathrm{A}}_{00}(k) \right) \\
\delta D^{\mathrm{R,A}}_{00}(k) &= -\frac{i}{\vec{k}^4}\, \Pi^{\mathrm{R,A}}_{00}(k)
\end{aligned}$$

One-loop thermal corrections to the potential



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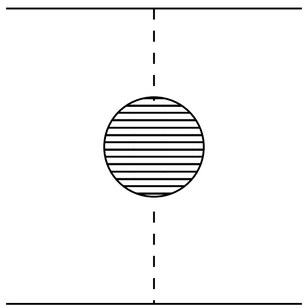
- The fourth-component of the momentum in the longitudinal gluon has been set to zero because we expand in E: E does not identify a momentum region that contributes to the potential.

One-loop thermal corrections to the potential



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Momentum regions that contribute to the potential are

(1) $|\vec{k}| \sim 1/r$; h.o. corrections are suppressed by Tr, $m_D r$, Er

(2) $|\vec{k}| \sim T$; h.o. corrections are suppressed by m_D/T, E/T

(3) $|\vec{k}| \sim m_D$; h.o. corrections are suppressed by E/m_D.

$|\vec{k}| \sim 1/r$

The £rst momentum region is $|\vec{k}| \sim 1/r$. The relevant thermal contribution to the longitudinal gluon polarization tensor comes from the region $|\vec{k}| \gg T \gg k^0$, which, substituted in the expression of the potential, gives

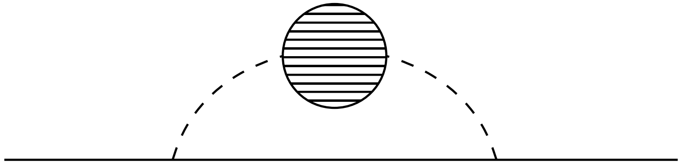
$$\delta V_s = \int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}\cdot\vec{r}} \left(-C_F \frac{4\pi\alpha_{\rm s}}{\vec{k}^4} \right) \frac{N_c\, g^2\, T^2}{18} = \frac{\pi}{9}\, N_c C_F\, \alpha_{\rm s}^2\, r\, T^2$$

$|\vec{k}| \sim T$

The second momentum region is $|\vec{k}| \sim T$. Since $T \ll 1/r$, we may expand $e^{-i\vec{k}\cdot\vec{r}}$

$$\delta V_s = \mu^{4-d} \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \left(1 - \frac{(\vec{k}\cdot\vec{r})^2}{2} + \ldots\right) g^2\, C_F\, \left[i\delta \mathbf{D}_{00}(0,\vec{k})\right]_{11} .$$

- The £rst term in the expansion is a mass correction and cancels against twice



- The second term gives

$$\begin{aligned}
\delta V_s &= \left[-\frac{3}{2}\zeta(3)\, C_F\, \frac{\alpha_s}{\pi}\, r^2\, T\, m_D^2 + \frac{2}{3}\zeta(3)\, N_c C_F\, \alpha_s^2\, r^2\, T^3\right] \\
&+ i\left[\frac{C_F}{6}\alpha_s\, r^2\, T\, m_D^2\, \left(\frac{1}{\epsilon} + \gamma_E + \ln\pi - \ln\frac{T^2}{\mu^2} + \frac{2}{3} - 4\ln 2 - 2\frac{\zeta'(2)}{\zeta(2)}\right)\right. \\
&\qquad \left. + \frac{4\pi}{9}\ln 2\; N_c C_F\, \alpha_s^2\, r^2\, T^3\right]
\end{aligned}$$

$|\vec{k}| \sim m_D$

The third momentum region is $|\vec{k}| \sim m_D$. The contribution to the potential is like the previous one, but now $|\vec{k}| \sim m_D \ll T$ and the correct expression for the longitudinal gluon propagator is the HTL resummed propagator.

- The £rst term in the expansion corresponds to a mass correction, which this time comes from the scale m_D and cancels against twice the contribution of the static quark self energy, when the loop momentum is of order m_D and a HTL resummed gluon propagator is used.

- The second term gives

$$\delta V_s = \frac{C_F}{6}\,\alpha_{\rm s}\,r^2 m_D^3 - i\,\frac{C_F}{6}\,\alpha_{\rm s}\,r^2\,T\,m_D^2\left(\frac{1}{\epsilon} - \gamma_E + \ln\pi + \ln\frac{\mu^2}{m_D^2} + \frac{5}{3}\right)$$

The potential for $1/r \gg T \gg m_D \gg E$

The £nal expression of the potential reads

$$
\begin{aligned}
V_s(r) \quad = \quad & -C_F \frac{\alpha_{\rm s}(1/r)}{r} \\
& + \frac{\pi}{9} N_c C_F \, \alpha_{\rm s}^2 \, r \, T^2 - \frac{3}{2}\zeta(3) \, C_F \, \frac{\alpha_{\rm s}}{\pi} \, r^2 \, T \, m_D^2 + \frac{2}{3}\zeta(3) \, N_c C_F \, \alpha_{\rm s}^2 \, r^2 \, T^3 \\
& \qquad\qquad\qquad\qquad + \frac{C_F}{6} \, \alpha_{\rm s} \, r^2 m_D^3 + \dots \\
& + i \left[\frac{C_F}{6} \alpha_{\rm s} \, r^2 \, T \, m_D^2 \, \left(2\gamma_E - \ln \frac{T^2}{m_D^2} - 1 - 4\ln 2 - 2\frac{\zeta'(2)}{\zeta(2)} \right) \right. \\
& \qquad\qquad\qquad\qquad \left. + \frac{4\pi}{9} \ln 2 \; N_c C_F \, \alpha_{\rm s}^2 \, r^2 \, T^3 \right] + \dots
\end{aligned}
$$

- The potential is £nite;
- the term $\ln T^2/m_D^2$ is a remenant of the fact that an IR divergence at the scale T has canceled against an UV divergence at the scale m_D;
- the imaginary part stems from the scattering of soft space-like gluons with hard particles, i.e. the Landau-damping phenomenon. It induces the thermal decay of the static bound state.

Exercises

(1) Derive m_D in the QED case. This corresponds to calculating the hard thermal loop part of the photon polarization.

(2) Discuss the situation $1/r \gg E \gg T \gg m_D$.

(3) Discuss the situation $1/r \gg T \gg E \gg m_D$.

Lecture II

1. EFTs

2. pNRQCD at $\mu < T$

3. pNRQCD at $\mu < m_D$

- Aim of the lecture is to derive the same potential of the previous one in an EFT framework. This means making the expansions in Tr, $m_D r$, Er, m_D/T, E/T and E/m_D explicit at the Lagrangian level.

Effective Field Theories

Whenever a system H, described by $\mathcal{L}$, is characterized by 2 scales $\Lambda \gg \lambda$, observables may be calculated by expanding one scale with respect to the other.
An effective field theory makes the expansion in λ/Λ explicit at the Lagrangian level.

The EFT Lagrangian, $\mathcal{L}_{\rm EFT}$, suitable to describe H at scales lower than Λ is de£ned by
(1) a cut off $\Lambda \gg \mu \gg \lambda$;
(2) by some degrees of freedom that exist at scales lower than μ

$\Rightarrow$ $\mathcal{L}_{\rm EFT}$ is made of all operators O_n that may be built from the effective degrees of freedom and are consistent with the symmetries of $\mathcal{L}$.

Effective Field Theories

$$\mathcal{L}_{\rm EFT} = \sum_n c_n(\Lambda/\mu) \frac{O_n(\mu,\lambda)}{\Lambda^n}$$

- Since $\langle O_n \rangle \sim \lambda^n$ the EFT is organized as an expansion in λ/Λ.

- The EFT is renormalizable order by order in λ/Λ.

- The matching coef£cients $c_n(\Lambda/\mu)$ encode the non-analytic behaviour in Λ. They are calculated by imposing that $\mathcal{L}_{\rm EFT}$ and $\mathcal{L}$ describe the same physics at any £nite order in the expansion: matching procedure. Since we are interested in integrating out only the large scale, we may expand in the matching in the low scale: the matching is a one-scale problem.

- If $\Lambda \gg \Lambda_{\rm QCD}$ then $c_n(\Lambda/\mu)$ may be calculated in perturbation theory.

Weak coupling

In the weak coupling regime:

- $v \sim \alpha_{\rm s} \ll 1$; valid for tightly bound states: $\Upsilon(1S)$, J/ψ, ...
- $T \gg gT \sim m_D$.

Effects due to the scale $\Lambda_{\rm QCD}$ will not be considered.

Scales for non-relativistic bound states at £nite T

Quarkonium in a medium is characterized by different energy and momentum scales:

- the scales of the bound state (v is the relative heavy-quark velocity):
  - m (mass),
  - mv (momentum transfer, inverse distance),
  - mv^2 (kinetic energy, binding energy, potential V), ...

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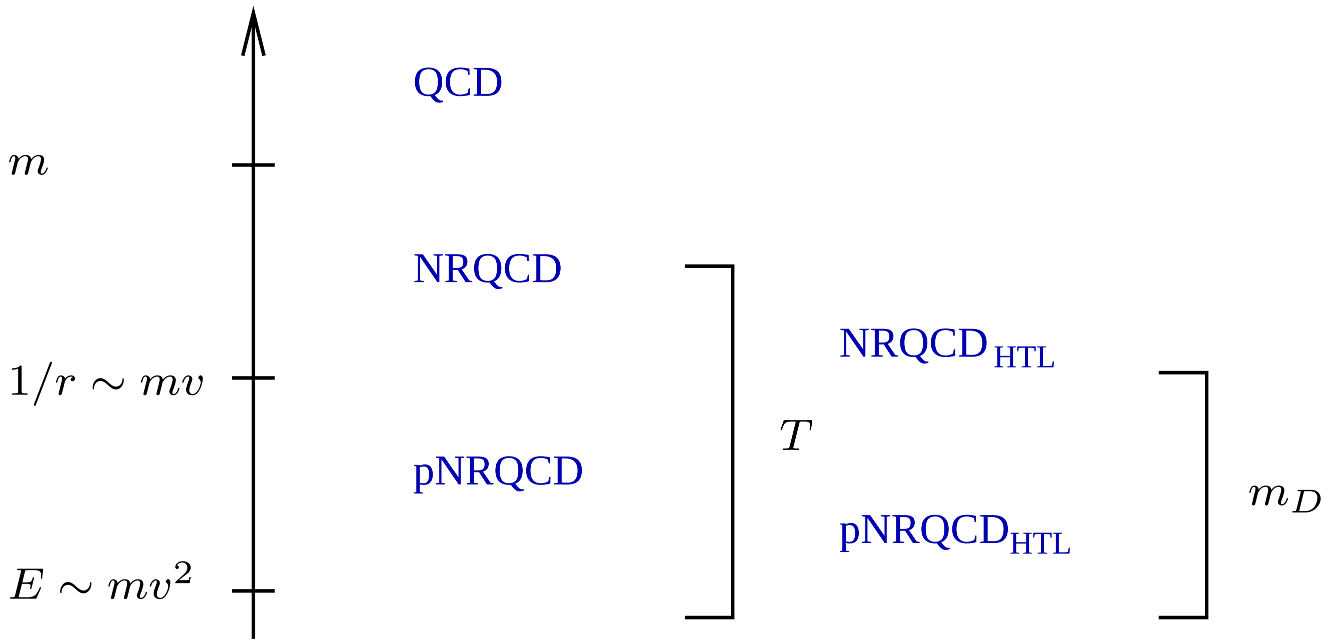
- the thermodynamical scales:
  - T (temperature),
  - m_D (Debye mass, i.e. screening of the chromoelectric interactions), ...

If these scales are hierarchically ordered (if the bound state is non relativistic: $v \ll 1$; in the weak coupling regime $T \gg m_D$) then we may expand physical observables in the ratio of the scales. If we separate/factorize explicitly the contributions from the different scales at the Lagrangian level this amounts to substituting QCD with a hierarchy of EFTs, which are equivalent to QCD order by order in the expansion parameters.

EFTs



We assume that bound states exist for

- $T \ll m$
- $1/r \sim mv \gtrsim m_D$

We neglect smaller thermodynamical scales.

Scales

We will analyze

(1) the static limit: $m \to \infty$;

(2) a situation where the relevant scales, which are $1/r$, T, m_D and E, are hierarchically ordered as

$$1/r \gg T \gg m_D \gg E$$

QCD/NRQCD for $m \to \infty$

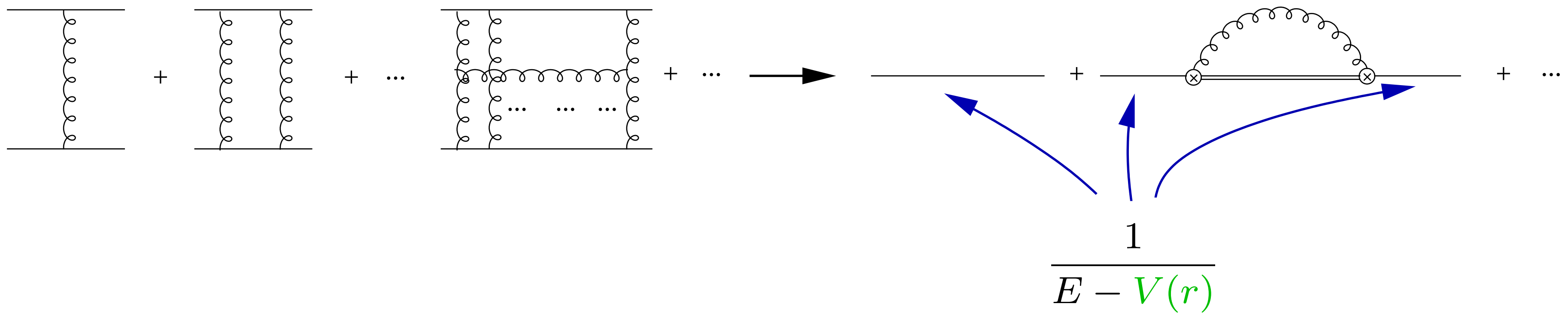
$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} + \sum_{i=1}^{n_f} \bar{q}_i \, i\not{D} q_i + \psi^\dagger i D_0 \psi + \chi^\dagger i D_0 \chi$$

ψ (χ) is the field that annihilates (creates) the static (anti)quark.

- $1/m$ corrections may be systematically included in NRQCD, leading to a Lagrangian systematically organized as a double expansion in $1/m$ and $\alpha_s(m)$: $\mathcal{L} = \sum_n c(\alpha_s(m/\mu)) \times O_n(\mu, \lambda)/m^n$. Similarly $1/m$ corrections may be systematically included in pNRQCD.
- The relevant scales for static $Q\bar{Q}$ pairs at £nite T are: $1/r$, E, ... T, m_D, ...

pNRQCD for $m \to \infty$: integrating out $1/r$

pNRQCD is obtained by integrating out modes associated with the scale $\frac{1}{r}$



NRQCD pNRQCD

- The Lagrangian is organized as an expansion in r:

$$\mathcal{L} = \sum_n V(r\mu', r\mu) \times O_n(\mu', \lambda)\, r^n$$

- A crucial observation is that in the situation $1/r \gg T$, we may set $T = 0$ in the construction of the EFT that follows by integrating out $1/r$. As a consequence, the EFT turns out to be described by the same Lagrangian as pNRQCD at $T = 0$.

pNRQCD for $m \to \infty$: d.o.f. and power counting

- Degrees of freedom:
  - Q-$\bar{Q}$ states, with energy $\sim$ scales lower than $1/r$ and momentum $\lesssim 1/r$
    $\Rightarrow$ (i) singlet S (ii) octet O
  - Gluons with energy and momentum $\sim$ scales lower than $1/r$

- Power counting:

  $p \sim \frac{1}{r}$;
  all gauge fields are multipole expanded: $A(R, r, t) = A(R, t) + \mathbf{r} \cdot \boldsymbol{\nabla} A(R, t) + \dots$
  and scale like $($scales lower than $1/r)^{\mathrm{dimension}}$.

  Non-analytic behaviour in $r \to$ matching coef£cients V

pNRQCD for $m \to \infty$: the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F^{a}_{\mu\nu} F^{\mu\nu\, a} + \sum_{i=1}^{n_f} \bar{q}_i \, i \not{D} q_i + \int d^3 r \, \mathrm{Tr} \left\{ \mathrm{S}^\dagger \left(i\partial_0 - V_s \right) \mathrm{S} + \mathrm{O}^\dagger \left(iD_0 - V_o \right) \mathrm{O} \right\}$$

- LO in r

$$\theta(T)\, e^{-iTV_s} \qquad \theta(T)\, e^{-iTV_o} \left(e^{-i \int dt\, A^{\mathrm{adj}}} \right)$$

pNRQCD for $m \to \infty$: the Lagrangian

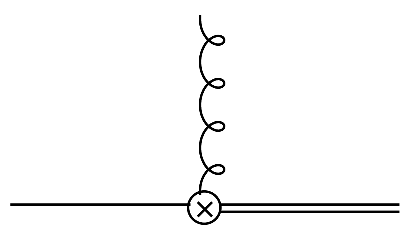
$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu\, a} + \sum_{i=1}^{n_f} \bar{q}_i \, i \not{D} q_i + \int d^3 r \mathrm{Tr} \left\{ \mathrm{S}^\dagger \left(i\partial_0 - V_s \right) \mathrm{S} + \mathrm{O}^\dagger \left(iD_0 - V_o \right) \mathrm{O} \right\}$$

- LO in r

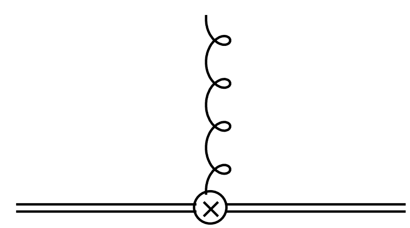
$\theta(T)\, e^{-iTV_s}$ $\quad$ $\theta(T)\, e^{-iTV_o} \left(e^{-i \int dt\, A^{\mathrm{adj}}} \right)$

$$+V_A \mathrm{Tr} \left\{ \mathrm{O}^\dagger \mathbf{r} \cdot g\mathbf{E}\, \mathrm{S} + \mathrm{S}^\dagger \mathbf{r} \cdot g\mathbf{E}\, \mathrm{O} \right\} + \frac{V_B}{2} \mathrm{Tr} \left\{ \mathrm{O}^\dagger \mathbf{r} \cdot g\mathbf{E}\, \mathrm{O} + \mathrm{O}^\dagger \mathrm{O} \mathbf{r} \cdot g\mathbf{E} \right\}$$

- NLO in r



$O^\dagger \mathbf{r} \cdot g\mathbf{E}\, \mathrm{S}$

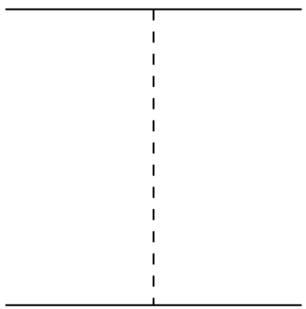


$O^\dagger \{ \mathbf{r} \cdot g\mathbf{E},\, \mathrm{O} \}$

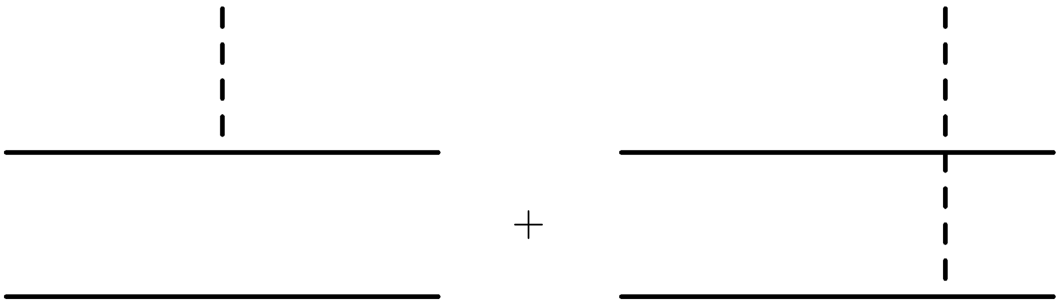
$+ \cdots$

pNRQCD for $m \to \infty$: matching at $\mathcal{O}(\alpha_s)$

(a)



(b)



(a) The potentials V_s and V_o are the Coulomb potentials:
$V_s(r) = -C_F \dfrac{\alpha_s}{r}$ and $V_o(r) = \dfrac{\alpha_s}{2N_c\, r}$.

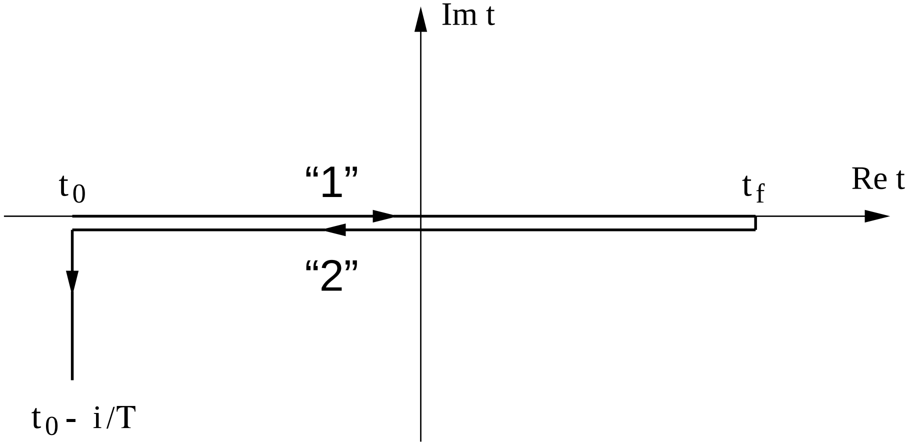
(b) $V_A = V_B = 1$.

The pNRQCD partition function

The pNRQCD partition function is

$$Z = \mathrm{Tr}\left\{e^{-H_{\mathrm{pNRQCD}}/T}\right\} = \int_{\mathrm{p.}} \mathcal{D}A\,\mathcal{D}S\,\mathcal{D}O\,e^{-\int_0^{1/T} dt\,\int d^3x \mathcal{L}}$$

In real-time formalism the contour of the time integration is modified in order to allow for real time; a consequence is the doubling of the fields discussed in the previous lecture.



Real-time static quark-antiquark propagator

- Free static quark-antiquark propagator:

$$\mathbf{S}^{(0)}_{\bar{Q}Q}(p) \quad = \quad \begin{pmatrix} \dfrac{i}{p^0+i\epsilon} & 0 \\ 2\pi\delta(p^0) & \dfrac{-i}{p^0-i\epsilon} \end{pmatrix} = \mathbf{U}^{(0)} \begin{pmatrix} \dfrac{i}{p^0+i\epsilon} & 0 \\ 0 & \dfrac{-i}{p^0-i\epsilon} \end{pmatrix} \mathbf{U}^{(0)}$$

where

$$\mathbf{U}^{(0)} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Similar to the quark propagator, but quark-antiquark fields are bosons.

Since $[\mathbf{S}^{(0)}_{\bar{Q}q}(p)]_{12} = 0$, the static quark-antiquark fields labeled “2” never enter in any physical amplitude, i.e. any amplitude that has the physical fields, labeled “1”, as initial and £nal states.

Real-time potential

- Static quark-antiquark potential:

$$\mathbf{V} \quad = \quad \begin{pmatrix} V & 0 \\ -2i\,\mathrm{Im}\,V & -V^* \end{pmatrix} = [\mathbf{U}^{(0)}]^{-1} \begin{pmatrix} V & 0 \\ 0 & -V^* \end{pmatrix} [\mathbf{U}^{(0)}]^{-1}$$

Hence the sum of all insertions of a potential exchange between a free quark and antiquark amounts to the full propagator:

$$\mathbf{U}^{(0)} \begin{pmatrix} \dfrac{i}{p^0 - V + i\epsilon} & 0 \\ 0 & \dfrac{-i}{p^0 - V^* - i\epsilon} \end{pmatrix} \mathbf{U}^{(0)} = \mathbf{S}^{(0)}_{\bar{Q}Q}(p) \sum_{n=0}^{\infty} \left[(-i\mathbf{V}(r))\, \mathbf{S}^{(0)}_{\bar{Q}Q}(p) \right]^n$$

pNRQCD$_{\rm HTL}$: integrating out T

Integrating out T from pNRQCD modifies pNRQCD into pNRQCD$_{\rm HTL}$.

- In the Yang–Mills sector, the Lagrangian gets an additional hard thermal loop (HTL) part coming from contributions of order T to the gluon self-energy:

$$\delta\mathcal{L}_{\rm HTL} = \frac{m_D^2}{2}\,{\rm Tr}\left\{\int \frac{d\Omega_k}{4\pi}\,F_{\mu\alpha}\,\frac{\hat{K}^\alpha \hat{K}_\beta}{(\hat{K}_\nu D^\nu)^2}\,F^{\beta\mu}\right\},$$

where $\hat{K} = (1, \hat{k})$. This modifies the gluon propagator.

E.g. the longitudinal propagator at $k^0 = 0$ becomes

$$\frac{i}{\vec{k}^2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \to \frac{i}{\vec{k}^2 + m_D^2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \pi\,\frac{T}{|\vec{k}|}\,\frac{m_D^2}{\left(\vec{k}^2 + m_D^2\right)^2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

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pNRQCD$_{\mathrm{HTL}}$: integrating out T

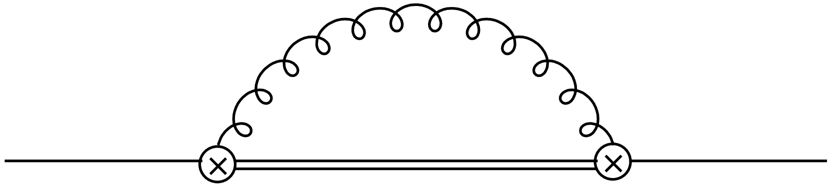
- In the singlet sector the potential gets an additional thermal correction δV_s to the Coulomb potential:

$$\delta \mathcal{L}_{\mathrm{singlet}} = -\int d^3r \; \mathrm{Tr}\left\{ \mathrm{S}^\dagger \, \delta V_s \, \mathrm{S} \right\}$$

where

$$\delta V_s \equiv [\delta \mathbf{V_s}(r)]_{11} = -ig^2 \, \frac{T_F}{N_c} \, \frac{r^2}{3} \int_0^\infty dt \, e^{-it\Delta V} \left[\langle \vec{E}^a(t) \phi(t,0)^{\mathrm{adj}}_{ab} \vec{E}^b(0) \rangle_T \right]_{11}$$

$$\Delta V = V_o - V_s$$



Chromoelectric correlator

The chromoelectric correlator $\langle \vec{E}^a(t)\phi(t,0)^{\rm adj}_{ab}\vec{E}^b(0)\rangle_T$ enters into the potential; $\phi(t,0)^{\rm adj}_{ab}$ is a Wilson line in the adjoint representation connecting $(t,\vec{0})$ to $(0,\vec{0})$.

At zeroth order in $\alpha_{\rm s}$, the thermal part of $\langle \vec{E}^a(t)\phi(t,0)^{\rm adj}_{ab}\vec{E}^b(0)\rangle_T$ is

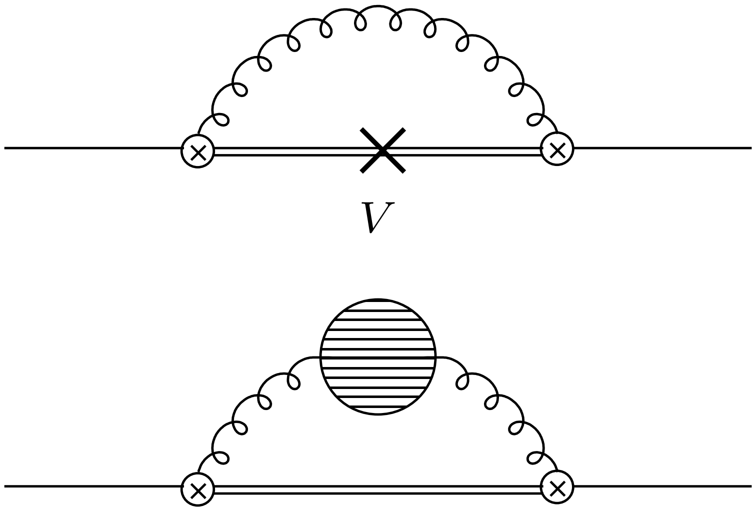
$$\langle \vec{E}^a(t)\phi(t,0)^{\rm adj}_{ab}\vec{E}^b(0)\rangle_T\Big|_{\text{thermal part}} = (N_c^2-1)\int \frac{d^3k}{(2\pi)^3}\,2|\vec{k}|\cos(|\vec{k}|t)\,n_{\rm B}(|\vec{k}|)\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$$

- The thermal contribution to the potential induced by the leading chromoelectric correlator, which would be of order $g^2\,r^2\,T^3$, vanishes.
- We will consider corrections $a)$ of order $g^2\,r^2\,T^3\times V/T$, $b)$ of order $g^2\,r^2\,T^3\times(m_D/T)^2$. Note that corrections $c)$ induced by higher order terms in the multipole expansion would be of order $g^2\,r^2\,T^3\times(m_D/T)^2\times(rT)$, hence they are suppressed. Remember that $g^2(T)\approx(m_D/T)^2$.

Integrating out T: real part of δV_s

a)

b)

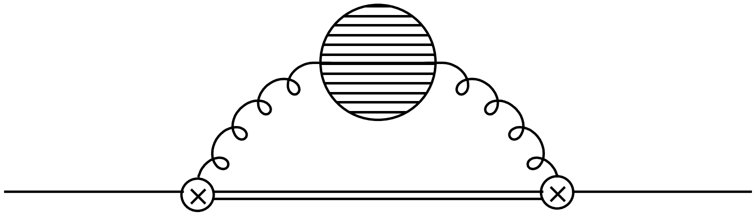


$$\mathrm{Re}\,\delta V_s(r) = \frac{\pi}{9} N_c C_F\, \alpha_s^2\, r\, T^2 \qquad a) \quad \sim g^2 r^2 T^3 \times \frac{V}{T}$$

$$-\frac{3}{2}\zeta(3)\, C_F\, \frac{\alpha_s}{\pi}\, r^2\, T\, m_D^2 + \frac{2}{3}\zeta(3)\, N_c C_F\, \alpha_s^2\, r^2\, T^3 \qquad b) \quad \sim g^2 r^2 T^3 \times \left(\frac{m_D}{T}\right)^2$$

- Contribution $a)$ corresponds to the $|\vec{k}| \sim 1/r$ momentum region contribution to the static potential calculated in Lecture I;
- Contribution $b)$ corresponds to the $|\vec{k}| \sim T$ momentum region contribution to the static potential calculated in Lecture I.

Integrating out T: imaginary part of δV_s



Landau-damping contribution

$$\mathrm{Im}\,\delta V_s(r) = +\frac{C_F}{6}\alpha_{\rm s}\,r^2\,T\,m_D^2\left(\frac{1}{\epsilon}+\gamma_E+\ln\pi-\ln\frac{T^2}{\mu^2}+\frac{2}{3}-4\ln 2-2\frac{\zeta'(2)}{\zeta(2)}\right)$$
$$+\frac{4\pi}{9}\ln 2\;N_cC_F\,\alpha_{\rm s}^2\,r^2\,T^3 \qquad \sim g^2r^2T^3\times\left(\frac{m_D}{T}\right)^2$$

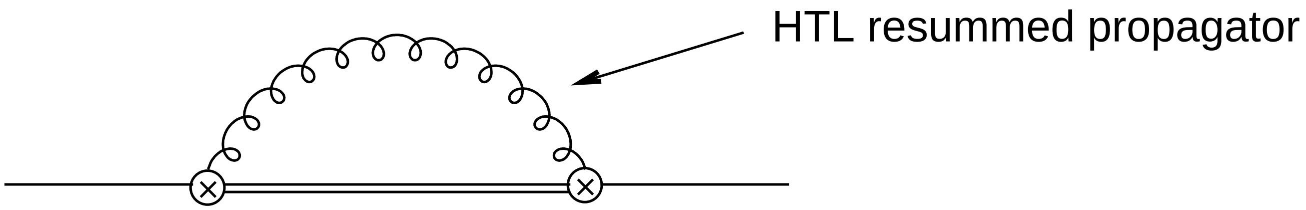
- This contribution corresponds to the $|\vec{k}| \sim T$ momentum region contribution to the static potential calculated in Lecture I.
- Divergences appear in the imaginary part of the potential at order $g^2r^2T^3 \times \left(\frac{m_D}{T}\right)^2$. They cancel in physical observables against loop corrections from lower energy scales: in our case against contributions coming from the scale m_D.

pNRQCD$_{\rm HTL}$: integrating out m_D

Integrating out m_D from pNRQCD$_{\rm HTL}$ modifies the singlet sector adding an additional thermal correction δV_s to the singlet potential

$$\delta\mathcal{L}_{\rm singlet} = -\int d^3r \,{\rm Tr}\left\{{\rm S}^\dagger\, \delta V_s\, {\rm S}\right\}$$

where $\delta V_s = -ig^2\, \dfrac{T_F}{N_c}\, \dfrac{r^2}{3} \displaystyle\int_0^\infty dt\, e^{-it\Delta V} \left[\langle \vec{E}^a(t)\phi(t,0)^{\rm adj}_{ab}\vec{E}^b(0)\rangle_T\right]_{11}$;
the relevant diagram is now



Integrating out m_D: δV_s

$$\mathrm{Re}\,\delta V_s(r) \;\sim\; g^2 r^2 T^3 \times \left(\frac{m_D}{T}\right)^3$$

$$\mathrm{Im}\,\delta V_s(r) \;=\; -\frac{C_F}{6}\,\alpha_\mathrm{s}\,r^2\,T\,m_D^2\left(\frac{1}{\epsilon} - \gamma_E + \ln\pi + \ln\frac{\mu^2}{m_D^2} + \frac{5}{3}\right)$$

- This contribution corresponds to the $|\vec{k}| \sim m_D$ momentum region contribution to the static potential calculated in Lecture I.

The potential for $1/r \gg T \gg m_D \gg E$

The £nal expression of the potential agrees with the one found in Lecture I:

$$
\begin{aligned}
V_s(r) \quad = \quad & -C_F \frac{\alpha_s(1/r)}{r} \\
& +\frac{\pi}{9} N_c C_F \alpha_s^2 \, r \, T^2 - \frac{3}{2}\zeta(3)\, C_F \, \frac{\alpha_s}{\pi} \, r^2 \, T \, m_D^2 + \frac{2}{3}\zeta(3)\, N_c C_F \, \alpha_s^2 \, r^2 \, T^3 \\
& \qquad\qquad\qquad\qquad\qquad\qquad + \frac{C_F}{6} \, \alpha_s \, r^2 m_D^3 + \dots \\
& +i \left[\frac{C_F}{6} \alpha_s \, r^2 \, T \, m_D^2 \, \left(2\gamma_E - \ln \frac{T^2}{m_D^2} - 1 - 4\ln 2 - 2\frac{\zeta'(2)}{\zeta(2)} \right) \right. \\
& \qquad\qquad\qquad\qquad\qquad\qquad \left. + \frac{4\pi}{9} \ln 2 \; N_c C_F \, \alpha_s^2 \, r^2 \, T^3 \right] + \dots
\end{aligned}
$$

- The potential is £nite;
- the term $\ln T^2/m_D^2$ is a remenant of the fact that an IR divergence at the scale T has canceled against an UV divergence at the scale m_D;
- the imaginary part stems from the scattering of soft space-like gluons with hard particles, i.e. the Landau-damping phenomenon. It induces the thermal decay of the static bound state.

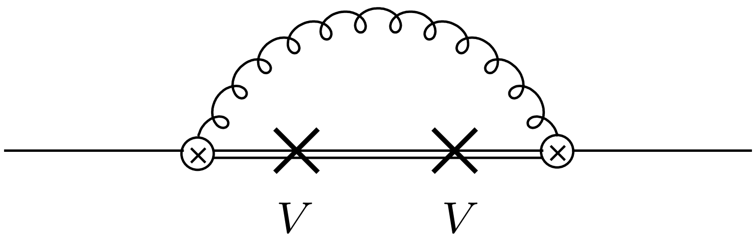
Static quark antiquark at $1/r \gg T \gg m_D \gg V$: energy and width

The potential is £nite because it provides the following physical observables:

$$\delta E \;=\; \frac{\pi}{9}\, N_c C_F\, \alpha_{\rm s}^2\, r\, T^2 \;-\; \frac{3}{2}\zeta(3)\, C_F\, \frac{\alpha_{\rm s}}{\pi}\, r^2\, T\, m_D^2 + \frac{2}{3}\zeta(3)\, N_c C_F\, \alpha_{\rm s}^2\, r^2\, T^3$$

$$\Gamma \;=\; \frac{C_F}{3}\alpha_{\rm s}\, r^2\, T\, m_D^2 \left(-2\gamma_E + \ln\frac{T^2}{m_D^2} + 1 + 4\ln 2 + 2\frac{\zeta'(2)}{\zeta(2)} \right) - \frac{8\pi}{9}\ln 2\; N_c C_F\, \alpha_{\rm s}^2\, r^2\, T^3$$

Singlet to octet break up



Singlet to octet break up contribution

$$\mathrm{Im}\,\delta V_s(r) \;=\; -\frac{N_c^2 C_F}{6}\,\alpha_{\rm s}^3\,T \qquad\qquad \sim g^2 r^2 T^3 \times \left(\frac{V}{T}\right)^2$$

- This contribution to the thermal width comes from the thermal break up of a quark-antiquark color singlet state into a color octet state and it is different from the thermal width induced by the Landau damping phenomenon discussed so far. It is speci£c of QCD, while the Landau damping would also show up in QED. Having assumed $m_D \gg V$, the term due to the singlet to octet break up is parametrically suppressed by $(V/m_D)^2$ with respect to the imaginary gluon self-energy contributions.

Conclusions

- It is possible to treat the real-time evolution of a static quark-antiquark pair in a medium of gluons and light quarks at £nite temperature in a framework that makes close contact with modern effective field theories for non-relativistic bound states at zero temperature. In other words, we have shown how to include/factorize thermodynamical scales in the framework of non-relativistic EFTs.
- In the speci£c example of the weak-coupling static potential, we have shown the equivalence of the EFT approach to a direct calculation in QCD.

Exercises

(1) Show that at one loop the HTL Lagrangian for static quarks gets contributions only from the gluon polarization.

(2) Show that at leading order the thermal part of the gluon condensate in the weak-coupling regime gives back the Stefan–Boltzmann law:

$$\langle \vec{E}^a(0) \cdot \vec{E}^a(0) \rangle_T \Big|_{\text{thermal part}} = (N_c^2 - 1)\, T^4\, \frac{\pi^2}{15} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

(3) Show that the thermal contribution to the potential induced by the chromoelectric correlator at leading order vanishes.