Lectures on Quark Flavor Physics: Exercises

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1. **Yukawa couplings, CKM matrix, and unitarity triangles:**

   a) Show that flavor non-diagonal kinetic terms in the Standard Model Lagrangian can always be diagonalized and brought into standard form by field redefinitions. To this end, study the Lagrangian

   $$\mathcal{L}_{\text{kinetic}} = \bar{Q}_L\, Z_Q\, i\not{D}\, Q_L + \bar{u}_R\, Z_u\, i\not{D}\, u_R + \bar{d}_R\, Z_d\, i\not{D}\, d_R\,,$$

   where all fields are 3-component vectors in generation space, and Z_A are non-negative, hermitian 3×3 matrices.

   b) Show that an arbitrary complex matrix Y can be diagonalized by a biunitary transformation:

   $$W^\dagger\, Y\, U = \lambda\,,$$

   where U, W are unitary matrices, and λ is a real, diagonal matrix with non-negative eigenvalues. (*Hint:* Consider the matrices $YY^\dagger$ and $Y^\dagger Y$.)

   c) Derive the number of mixing angles and physical (i.e., observable) phases of the CKM matrix for the Standard Model with N fermion generations.

   d) Show that the Jarlskog determinant J defined as

   $$\text{Im}\left(V_{ij}V_{kl}V_{il}^*V_{kj}^*\right) = J\sum_{m,n}\epsilon_{ikm}\,\epsilon_{jln} \quad (i\neq k,\, j\neq l)$$

   is invariant under phase redefinitions of the quark fields, and calculate its value in terms of the Wolfenstein parameters to leading nontrivial order in λ.

   e) Show that all unitarity triangles have the same area $J/2$.

2. **Matching of Wilson coefficients in the effective weak Hamiltonian:**

   Assume that, in addition to its standard interactions, the Z^0 boson has a small flavor-changing coupling to left-handed b and s quarks:

   $$\mathcal{L}_Z = \frac{g_2}{\cos\theta_W}Z^\mu\left\{\sum_f \bar{f}\,\gamma_\mu\left(T_f^3\,\frac{1-\gamma_5}{2} - Q_f\sin^2\theta_W\right)f + \left(\varepsilon_{bs}\,\bar{s}\,\gamma_\mu\,\frac{1-\gamma_5}{2}\,b + \text{h.c.}\right)\right\},$$

   where $|\varepsilon_{bs}|\ll 1$. The sum in the first term is over all Standard Model fermions. T_f^3 is the third component of weak isospin, Q_f the electric charge in units of e, g_2 the SU(2) gauge coupling, and θ_W the weak mixing angle.

   Calculate the contributions to the Wilson coefficients C_{3-10} in the effective weak Hamiltonian for $b\to s\bar{q}q$ transitions arising from tree-level Z-boson exchange, working to first order in ε_{bs}. Recall that $m_Z\cos\theta_W = m_W$ and $G_F/\sqrt{2} = g_2^2/8m_W^2$. Use the fact that $T_f^3 = 0$ for right-handed quarks, while $T_f^3 = Q_f - Y$ with $Y = 1/6$ for left-handed quarks.

3. **OPE for inclusive B-meson decays:**

Calculate the leading contribution (in powers of α_s and $1/m_b$) to the inclusive $\bar{B} \to X_s\gamma$ decay rate using the optical theorem,

$$\Gamma(\bar{B} \to X_s\gamma) = \text{Disc}\,\frac{\langle \bar{B}|\,\boldsymbol{T}\,|\bar{B}\rangle}{2m_B}\,, \qquad \boldsymbol{T} = i\int d^4x\, T\,\{\mathcal{H}_{\text{eff}}(x)\,\mathcal{H}_{\text{eff}}(0)\}\,,$$

by evaluating the discontinuity of the one-loop diagram with two insertions of the dipole operator $Q_{7\gamma}$ in the effective weak Hamiltonian.

a) Show that the Feynman rule (in momentum space) for the dipole operator

$$Q_{7\gamma} = -\frac{em_b}{4\pi^2}\,\bar{s}\,\sigma_{\mu\nu}\,\frac{1+\gamma_5}{2}\,F^{\mu\nu}\,b$$

is

$$-\frac{em_b}{4\pi^2}\,[\gamma_\mu, \not{q}]\,\frac{1+\gamma_5}{2}\,,$$

where q is the outgoing photon momentum and μ its polarization index.

b) Prove the following identity for Dirac matrices in 4 dimensions:

$$[\gamma_\mu, \not{q}]\,\gamma^\alpha\,[\gamma^\mu, \not{q}] = 16q^\alpha\not{q} - 4q^2\gamma^\alpha$$

c) Evaluate the discontinuity of the partonic $b \to s\gamma$ forward scattering amplitude by using the Cutkosky rule to replace the cut propagators with

$$\text{Disc}\,\frac{1}{q^2+i\epsilon}\,\frac{1}{(p-q)^2+i\epsilon} = (2\pi)\,\delta(q^2)\,\theta(q^0)\,(2\pi)\,\delta((p-q)^2)\,\theta(p^0-q^0)\,.$$

Use Feynman gauge, set the strange-quark mass to zero, and work in the b-quark rest frame, where the on-shell b-quark momentum is $p^\mu = (m_b, 0, 0, 0)$. First show that

$$\int \frac{d^4q}{(2\pi)^4}\,(2\pi)\,\delta(q^2)\,\theta(q^0)\,(2\pi)\,\delta((p-q)^2)\,\theta(p^0-q^0) = \int \frac{d\Omega_q}{32\pi^2}\,, \qquad q^0 = |\vec{q}| = \frac{m_b}{2}$$

in that frame, where $d\Omega_q$ is the measure for the angular integration over the direction of the vector $\vec{q}$. Then use the results form parts a) and b).

d) Make the replacement

$$\bar{u}(p)\,\ldots\,u(p) \to \frac{\langle \bar{B}|\,\bar{b}\,\ldots\,b\,|\bar{B}\rangle}{2m_B}$$

for the b-quark spinor product and evaluate the resulting matrix element to obtain the total hadronic $\bar{B} \to X_s\gamma$ decay rate.

4. **Exclusive $\boldsymbol{B}$-meson decays:**

The time-dependent rate for the decay of a $\bar{B}^0$ meson into a CP eigenstate f is

$$\Gamma_{\bar{B}^0\to f}(t) \propto 1 + \frac{|A|^2 - |\bar{A}|^2}{|A|^2 + |\bar{A}|^2}\cos(\Delta m\, t) - \frac{2\,\mathrm{Im}(A^*\bar{A}\,e^{2i\beta})}{|A|^2 + |\bar{A}|^2}\sin(\Delta m\, t)\,,$$

where A and $\bar{A}$ are the decay amplitudes for $\bar{B}^0 \to f$ and $B^0 \to f$, respectively, and 2β is the weak phase of the B^0–$\bar{B}^0$ mixing amplitude. The rate for the decay of a B^0 meson into the same state f is given by the analogous expression with $A \leftrightarrow \bar{A}$ and $\beta \to -\beta$.

a) Show that the time-dependent CP asymmetry has the general form

$$A_{\mathrm{CP}}(t) \equiv \frac{\Gamma_{\bar{B}^0\to f}(t) - \Gamma_{B^0\to f}(t)}{\Gamma_{\bar{B}^0\to f}(t) + \Gamma_{B^0\to f}(t)} = \mathcal{C}\cos(\Delta m\, t) - \mathcal{S}\sin(\Delta m\, t)\,,$$

where the coefficient $\mathcal{C}$ is given by the direct CP asymmetry of the decay $\bar{B}^0 \to f$.

b) The general expression for the decay rate A in the Standard Model is

$$A = A_1\,e^{i\delta_1} + A_2\,e^{i\delta_2}\,e^{-i\gamma} \propto 1 + r\,e^{i\delta}\,e^{-i\gamma}\,,$$

where $r = A_2/A_1$ and $\delta = \delta_2 - \delta_1$. Here A_i are real amplitudes, and δ_i are CP-conserving strong rescattering phases. Obtain an exact expression for the coefficient $\mathcal{S}$ in terms of r, δ, β, and γ.

c) In rare, penguin-dominated hadronic B decays the coefficient r is numerically very small. Expand your result from part b) to first order in r and simplify the answer as much as possible. Using the experimental values $\gamma \approx 68^\circ$ and $\beta \approx 22^\circ$, as well as the QCD prediction that the strong phase shift δ is small in the heavy-quark limit, show that the measured coefficient $\mathcal{S}$ is always larger than $\sin 2\beta$.