

# The isoperimetric problem in the plane with a piecewise constant density

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# The Isoperimetric Problem

We look for the **least perimeter set** in  $\mathbb{R}^2$  enclosing a prescribed quantity of area

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- **Existence** is not guaranteed
- In case of existence  $\rightarrow$  **isoperimetric region**

# Density setting

We shall use a **density**  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^+$  to **weight** the area and the perimeter:

For  $\Omega \subset \mathbb{R}^2$ ,

$$\text{area}(\Omega) = \int_{\Omega} f, \quad P(\Omega) = \int_{\partial\Omega} f$$

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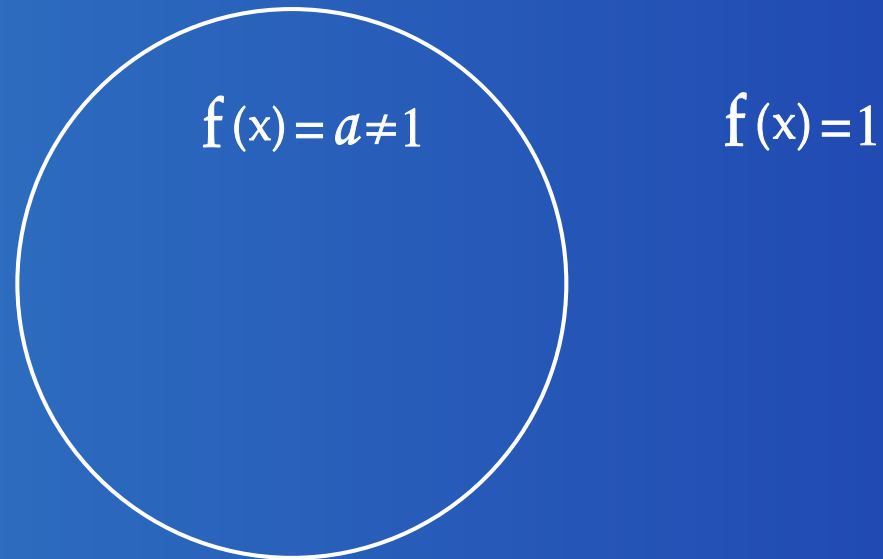
For  $\Omega \subset \mathbb{R}^2$ ,

$$\text{area}(\Omega) = \int_{\Omega} f, \quad P(\Omega) = \int_{\partial\Omega} f$$

- We will consider **piecewise constant** densities (examples of discontinuous ones)

# Piecewise constant densities

Ball density in  $\mathbb{R}^2$ :



# Piecewise constant densities

Strip density in  $\mathbb{R}^2$ :

$$\frac{\{x = 1\}}{\text{-----}} \quad f(x) = a > 1$$

$$\frac{\{x = -1\}}{\text{-----}} \quad f(x) = 1$$

$$f(x) = a > 1$$

# Piecewise constant densities

Half-plane density in  $\mathbb{R}^2$ :

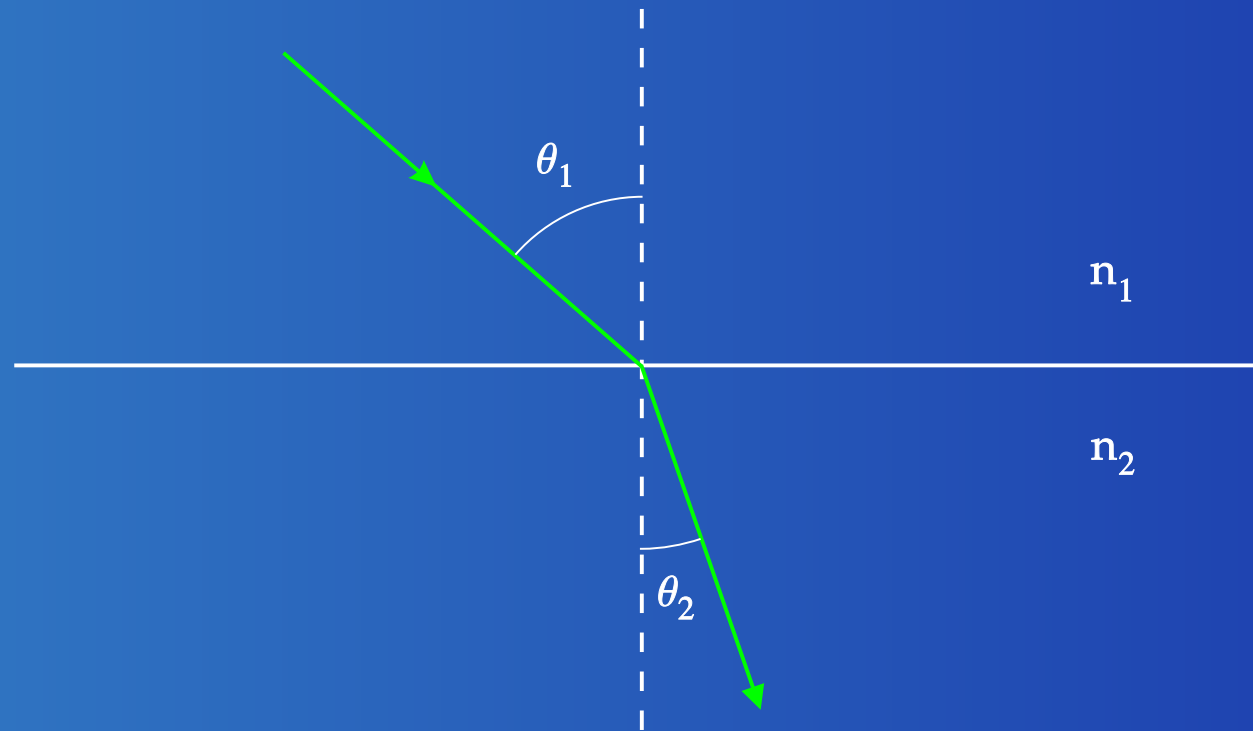
$$\{x=0\}$$

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$$f(x) = a > 1$$

$$f(x) = 1$$

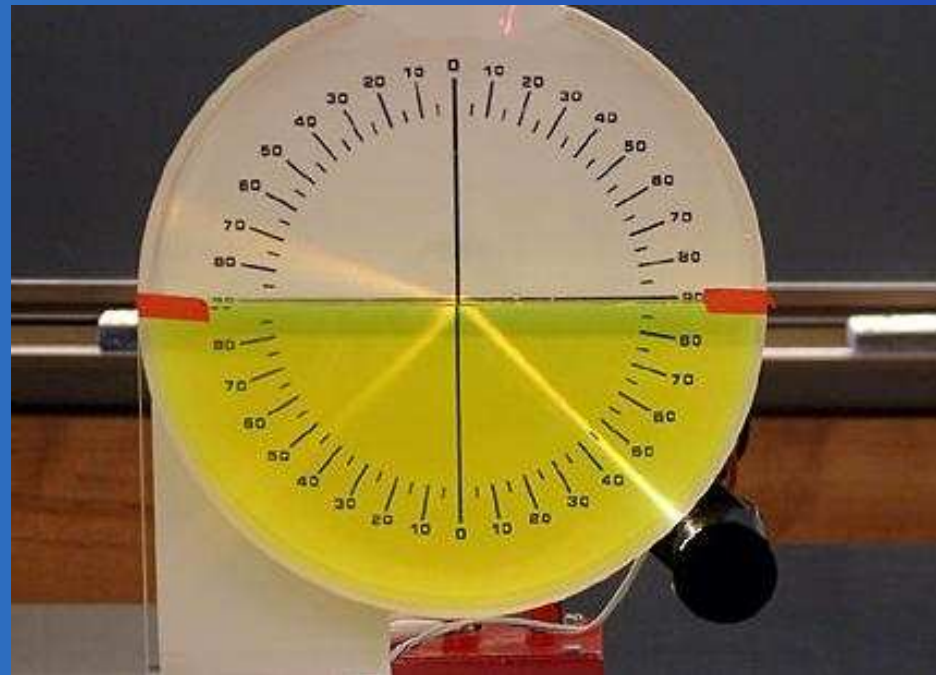
# Snell's law in Optics



$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$$

# Snell's law in Optics

Experimentally:



# Properties of the isoperimetric regions

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- 1)  $\Sigma - \Gamma$  is composed by curves with constant geodesic curvature
- 2) Pieces of  $\Gamma$  may be part of  $\Sigma$

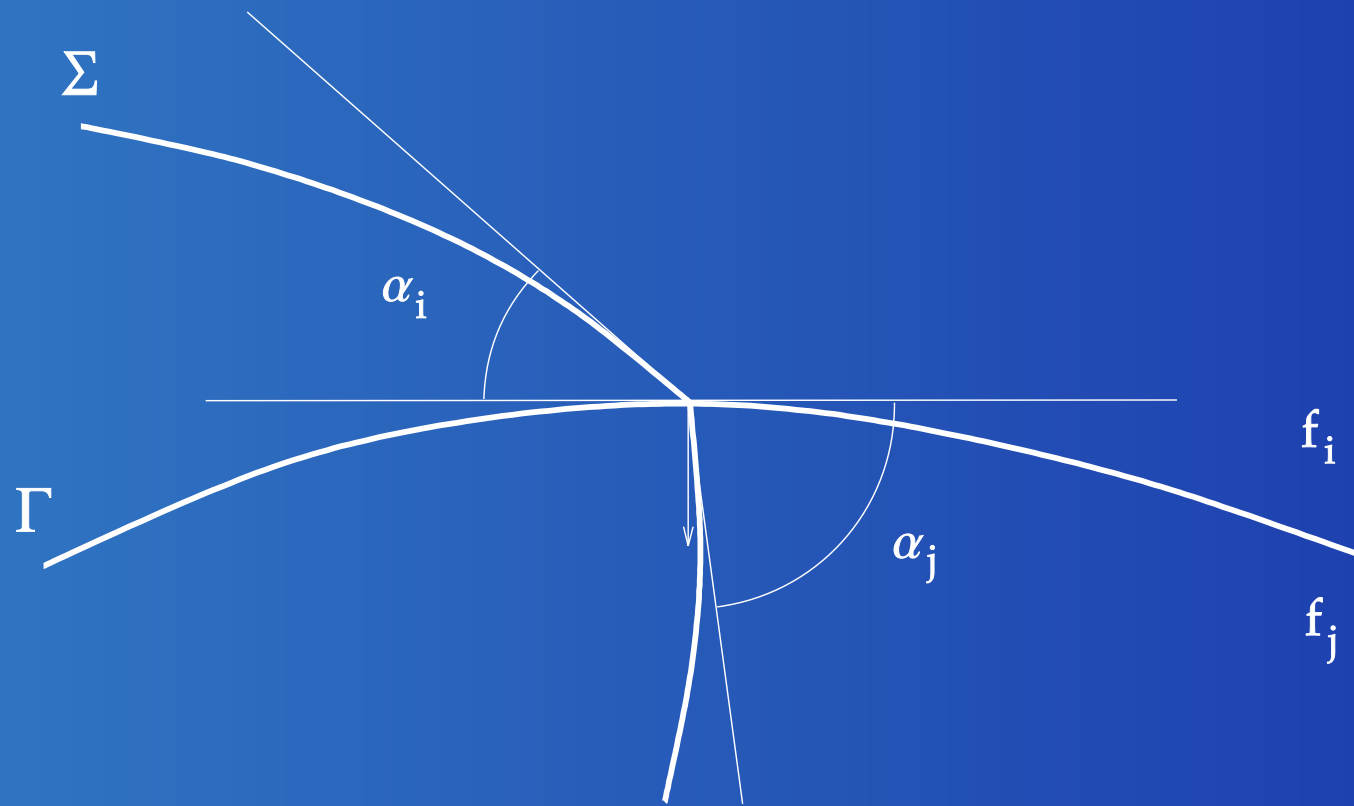
# Properties of the isoperimetric regions

$\Omega \equiv$  isoperimetric region,  $\partial\Omega = \Sigma$

$\Gamma \equiv$  set of discontinuities of  $f$

- 1)  $\Sigma - \Gamma$  is composed by curves with constant geodesic curvature
- 2) Pieces of  $\Gamma$  may be part of  $\Sigma$
- 3)  $\Sigma$  may not be smooth: when it crosses transversally  $\Gamma$ , a corner is formed according to Snell's law

# Snell's law



$$f_i \cos(\alpha_i) = f_j \cos(\alpha_j)$$

# Snell's law: Proof

$p \in \Sigma \cap \Gamma$ , local variation of  $\Sigma$  preserving the area

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$$0 = P'(0) = \int_{\Sigma} \langle \nabla \psi, \nu_{\Sigma} \rangle u f - \int_{\Sigma} H f u + \sum_{i=1}^k f_i(p) \langle X(p), \nu_{\Sigma_i}(p) \rangle,$$

where  $f = e^{\psi}$ ,  $u = \langle X, \nu_{\Sigma} \rangle$ ,  $H \equiv$  geod. curv.,

$f_i = f|_{\Omega_i}$ ,  $\Sigma_i = \Sigma \cap \Omega_i$

# Snell's law: Proof

$$p \in \Sigma_i \cap \Sigma_j \cap \Gamma$$

$X \equiv$  variational field,  $\nu \equiv$  normal vector

$$0 = f_i(p) \langle X(p), \nu_{\Sigma_i}(p) \rangle + f_j(p) \langle X(p), \nu_{\Sigma_j}(p) \rangle$$

$$f_i(p) \langle X(p), \nu_{\Sigma_i}(p) \rangle = -f_j(p) \langle X(p), \nu_{\Sigma_j}(p) \rangle$$

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$$f_i(p) \langle X(p), \nu_{\Sigma_i}(p) \rangle = -f_j(p) \langle X(p), \nu_{\Sigma_j}(p) \rangle$$

In particular, taking  $X(p)$  tangent to  $\Gamma$ ,  $|X(p)| = 1$ ,

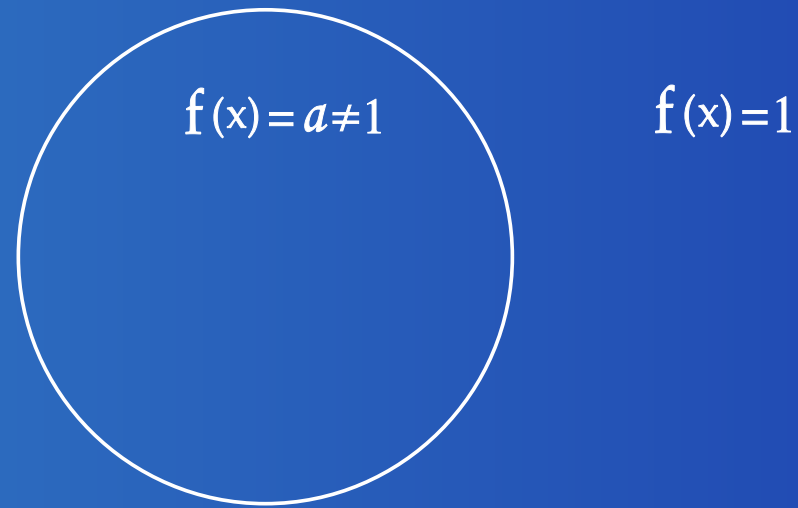
$$f_i(p) \cos(\alpha_i) = f_j(p) \cos(\alpha_j)$$

The boundary of an isoperimetric region  $\rightsquigarrow$

- curves with constant geodesic curvature
- possibly pieces of  $\Gamma$
- Snell's law is satisfied

We focus on  $\left\{ \begin{array}{l} \text{ball density} \\ \text{strip density} \\ \text{half-plane density} \end{array} \right.$

# Ball density



- Existence
- Two different cases  $\left\{ \begin{array}{l} a > 1 \\ a < 1 \end{array} \right.$

# Ball density with $a > 1$

Isoperimetric regions are:

- For areas  $v \leq v_0$ , balls of type a)
- For areas  $v_0 \leq v \leq a\pi$ , sets of type b)
- For areas  $v \geq a\pi$ , balls of type c)



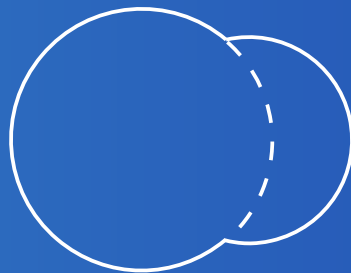
# Ball density with $a < 1$

Isoperimetric regions are:

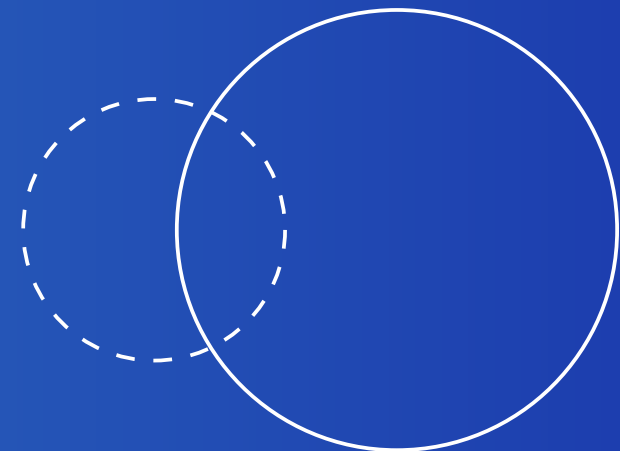
- For areas  $v \leq a\pi$ , balls of type a)
- For areas  $a\pi \leq v \leq v_1$ , sets of type b)
- For areas  $v_1 \leq v \leq v_2$ , sets of type b) or c)
- For areas  $v \geq v_2$ , balls of type c)



a)



b)



c)

# Ball density with $a < 1$

Isoperimetric regions are:

- For areas  $v \leq a\pi$ , balls of type a)
  - For areas  $a\pi \leq v \leq v_1$ , sets of type b)
  - For areas  $v_1 \leq v \leq v_2$ , sets of type b) or c)
  - For areas  $v \geq v_2$ , balls of type c)
- We believe  $v_1 = v_2$ , but we have not proved it

# Strip density

$$\frac{\{x = 1\}}{f(x) = a > 1}$$

$$\frac{\{x = -1\}}{f(x) = 1}$$

$$f(x) = a > 1$$

- Existence
- Vertical symmetry

# Strip density

Isoperimetric regions are:

- For areas  $v \leq \pi$ , balls of type i)
- For areas  $\pi \leq v \leq v_0$ , sets of type ii)
- For areas  $v_0 \leq v \leq v_1$ , sets of type iii) or iv)
- For areas  $v \geq v_1$ , sets of type iii)



# Strip density

Isoperimetric regions are:

- For areas  $v \leq \pi$ , balls of type i)
  - For areas  $\pi \leq v \leq v_0$ , sets of type ii)
  - For areas  $v_0 \leq v \leq v_1$ , sets of type iii) or iv)
  - For areas  $v \geq v_1$ , sets of type iii)
- In most cases, type iv) does not appear

# Half-plane density

$$f(x) = a > 1$$

$$\{x = 0\}$$

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$$f(x) = 1$$

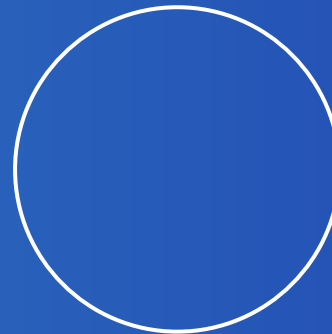
# Half-plane density

Isoperimetric regions are balls in  $\{x < 0\}$

$$f(x) = a > 1$$

$\{x = 0\}$

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$$f(x) = 1$$

- Also true in  $\mathbb{R}^n$ ,  $n > 2$

# Summary

- Density → New definitions of area and perimeter
- Piecewise Constant Density → Corners may appear
- Particular densities → Different isoperimetric regions