

The Hardy inequality and the asymptotic behaviour of the heat equation in twisted domains

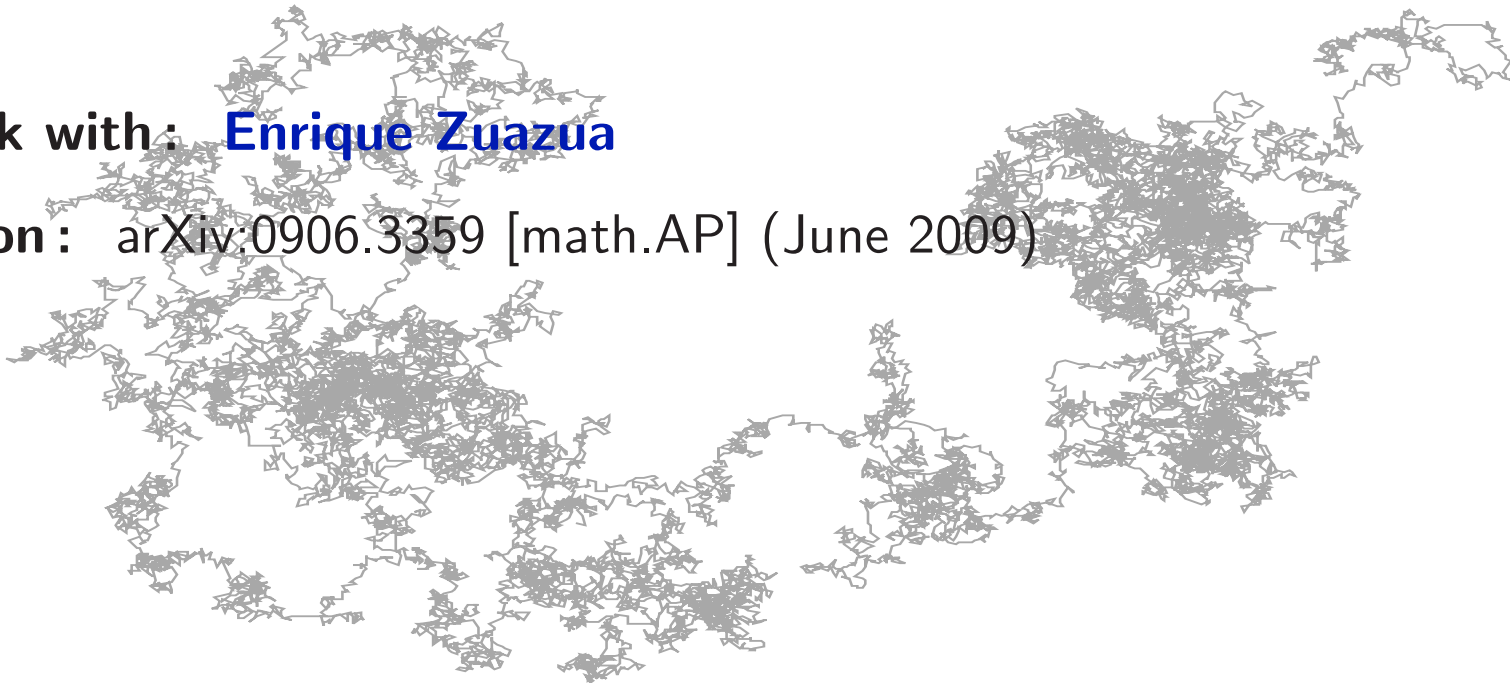
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Joint work with: [Enrique Zuazua](#)

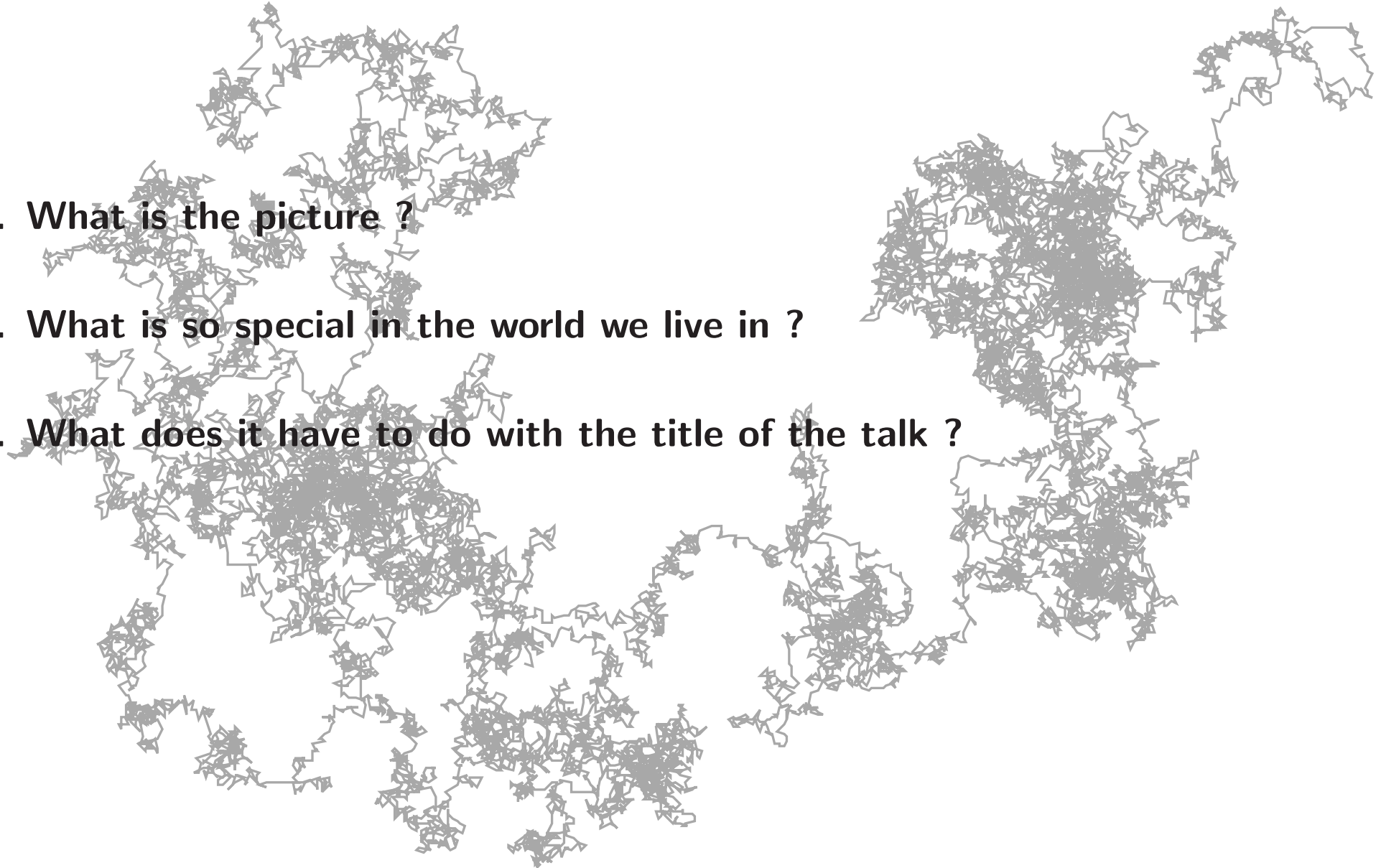
Preprint on: [arXiv:0906.3359](https://arxiv.org/abs/0906.3359) [math.AP] (June 2009)



Outline

3 fundamental questions

1. **What is the picture ?**
2. **What is so special in the world we live in ?**
3. **What does it have to do with the title of the talk ?**



The uniqueness of \mathbb{R}^3

- Brownian motion is **recurrent** in \mathbb{R}^1 and \mathbb{R}^2 ,
i.e. the Brownian particle visits every region infinitely many times.
- Brownian motion is **transient** in \mathbb{R}^d with $d \geq 3$,
i.e. the particle will escape from any bounded region after some time forever.



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Remark. The Brownian motion (Wiener process) can be rigorously constructed via the heat kernel associated with the diffusion equation $u_t - \frac{1}{2}\Delta u = 0$.

Analytic background of transience

[Grigor'yan 1999]

Theorem. Let M be a complete non-compact Riemannian manifold.

The following properties are **equivalent**:

1. Brownian motion on M is **transient**.
2. M is **non-parabolic**, *i.e.* \exists non-constant positive superharmonic function.
3. $-\Delta$ is **subcritical** in M , *i.e.* \exists Green function, finite off the pole.
4. \forall precompact open subset of M has **positive capacity**.
5. $-\Delta$ has **no virtual bound state**, *i.e.* $-\Delta - \varepsilon V \geq 0$ for all small ε . [$V \in C_0^\infty(M)$]
6. \exists **Hardy inequality** for $-\Delta$, *i.e.* $-\Delta \geq w(\cdot)$ with some positive w .

Remark. $M = \mathbb{R}^d$ ($d \geq 3$): $w(x) = \left(\frac{d-2}{2}\right)^2 \frac{1}{|x|^2}$ (classical HI)



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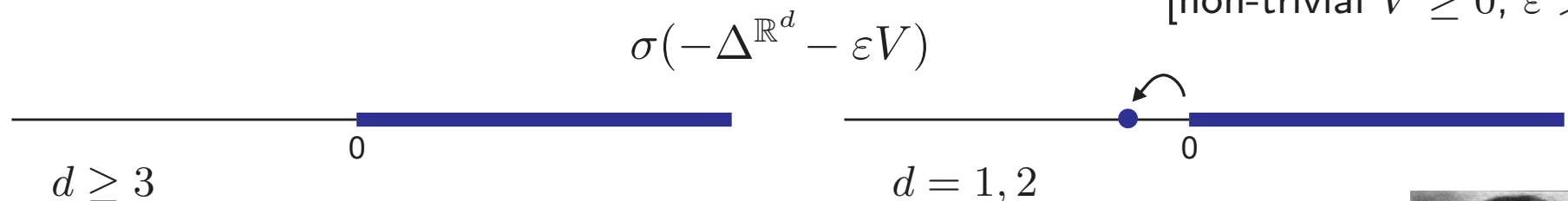
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Remark. 5 reflects the **spectral-stability** effect of transience.

[non-trivial $V \geq 0, \varepsilon > 0$]



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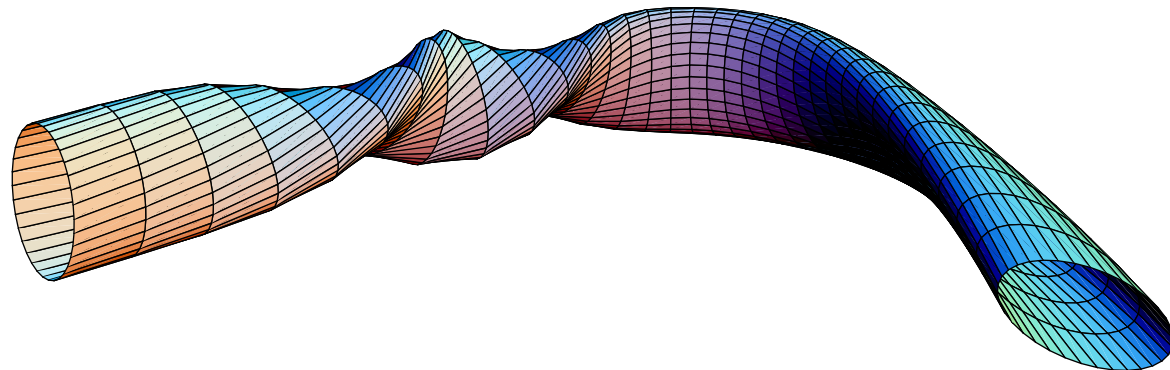
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Natural candidates: TUBES



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study of *fine* properties of the transience

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Conjecture. Better time-decay of $S_D^\Omega(t)$ iff $-\Delta_D^\Omega - \lambda_1 \geq w(\cdot) > 0$ holds.

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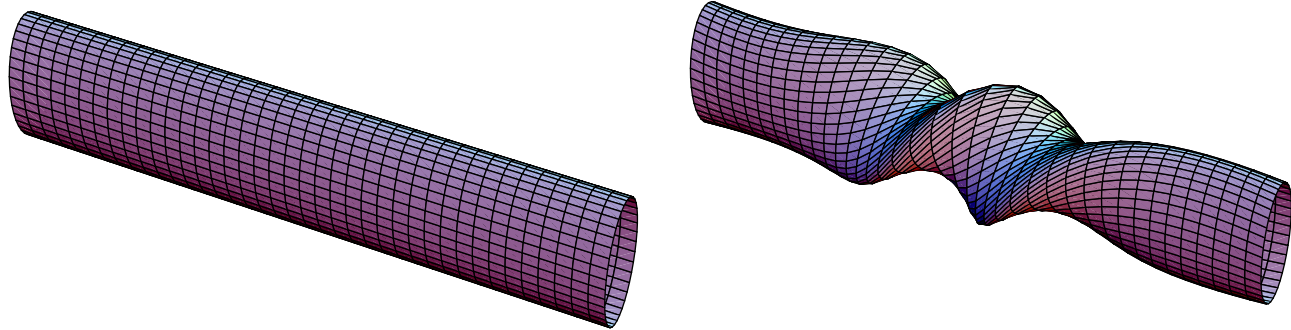
2. polynomial decay-rate inherited from 'dimensional' properties of Ω

e.g. $\|S_D^{\mathbb{R}^d}(t)\|_{L^2(\mathbb{R}^d, K) \rightarrow L^2(\mathbb{R}^d)} \sim t^{-d/4}$ as $t \rightarrow \infty$, $K(x) := e^{|x|^2/4}$

The Hardy inequality in twisted tubes

$\Omega_0 := \mathbb{R} \times \omega$ where ω is a bounded domain in \mathbb{R}^2

$\Omega_\theta := \{ (x_1, x_2 \cos \theta(x_1) + x_3 \sin \theta(x_1), -x_2 \sin \theta(x_1) + x_3 \cos \theta(x_1)) \mid x \in \Omega_0 \}$



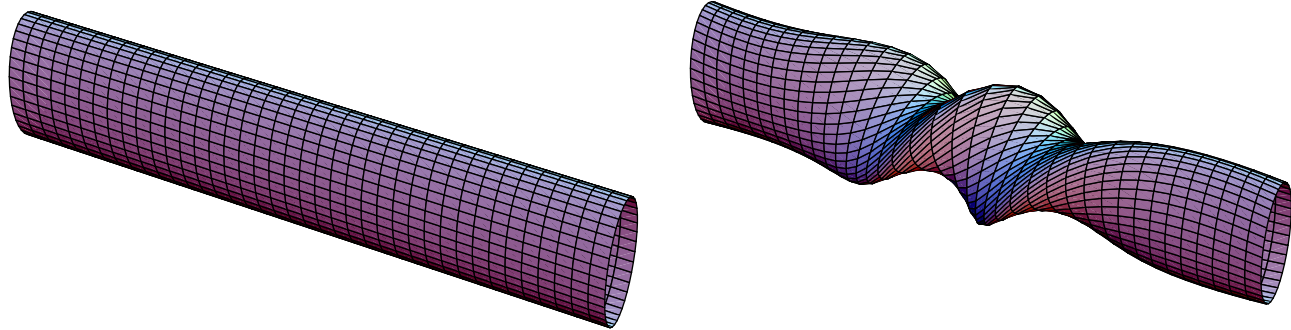
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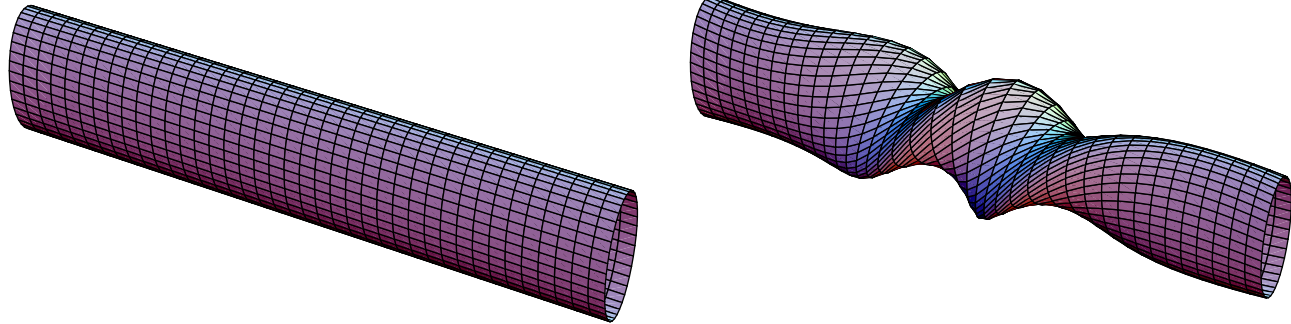
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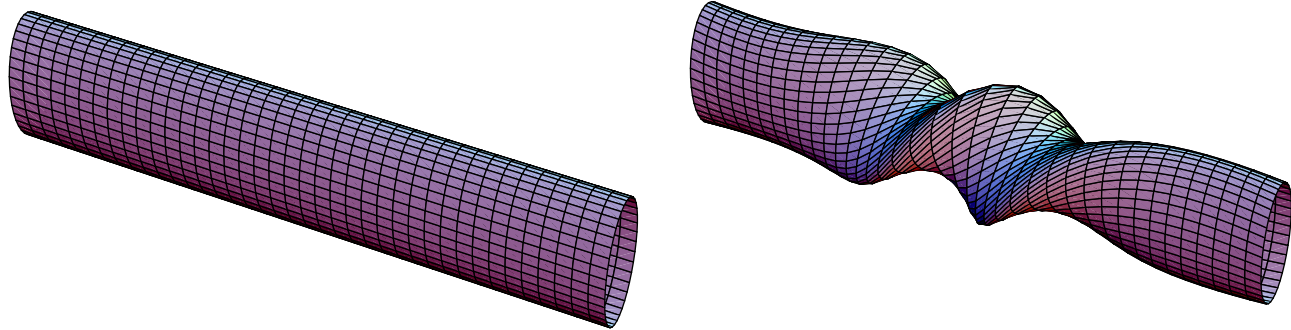
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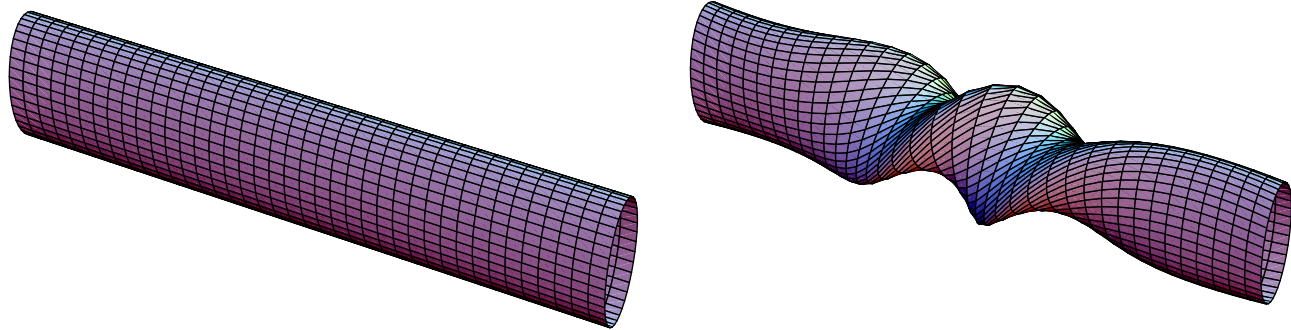
$$-\Delta_D^{\Omega_\theta} - \lambda_1 \geq \frac{c_H}{1 + x_1^2}$$

where $c_H = c_H(\dot{\theta}, \omega) \geq 0$ is *positive* if, and only if, Ω_θ is twisted.

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↪ **boom:** Briet, Exner, Fraas, Raikov, Sacchetti, Soccorsi, ...

The improved decay rate

$$\left\{ \begin{array}{ll} u_t - \Delta u - \lambda_1 u = 0 & \text{in } \Omega_\theta \times (0, \infty) \\ u = 0 & \text{in } \partial\Omega_\theta \times (0, \infty) \\ u = u_0 & \text{in } \Omega_\theta \times \{0\} \end{array} \right. \quad \boxed{S_D^{\Omega_\theta}(t) := e^{(\Delta_D^{\Omega_\theta} + \lambda_1)t} : u_0 \mapsto u(t)}$$

heat semigroup

decay rate

$$\Gamma(\Omega_\theta, K) := \sup \left\{ \Gamma \mid \exists C_\Gamma > 0, \forall t \geq 0, \|S_D^{\Omega_\theta}(t)\|_{L^2(\Omega_\theta, K) \rightarrow L^2(\Omega_\theta)} \leq C_\Gamma (1+t)^{-\Gamma} \right\}$$

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Theorem (D.K., Zuazua 2009).

$$\Gamma(\Omega_\theta, K) \begin{cases} = 1/4 & \text{if } \Omega_\theta \text{ is untwisted} \\ \geq 3/4 & \text{if } \Omega_\theta \text{ is twisted} \end{cases}$$

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Proof.

≥ self-similarity transformation,* weighted Sobolev spaces, ...

= no twisting $\|S(t)\|_{L^2(\Omega_0, K) \rightarrow L^2(\Omega_0)} \sim t^{-1/4}$ as $t \rightarrow \infty$ q.e.d.

*[Escobedo, Kavian 1987]

Self-similarity transformation

1. Straightening of the tube

$\mathcal{L}_\theta : \Omega_0 \rightarrow \Omega_\theta$ (curvilinear coordinates)

$$u_t - (\partial_1 - \dot{\theta} \partial_\tau)^2 u - \Delta' u - \lambda_1 u = 0 \quad \text{in } \Omega_0$$

$$\partial_\tau := x_3 \partial_2 - x_2 \partial_3, \quad \Delta' := \partial_2^2 + \partial_3^2$$

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2. Changing the time $\tilde{u}(y_1, y_2, y_3, s) = e^{s/4} u(e^{s/2} y_1, y_2, y_3, \underbrace{e^s - 1}_t)$

$$\tilde{u}_s - \frac{1}{2} y_1 \partial_1 \tilde{u} - (\partial_1 - \sigma_s \partial_\tau)^2 \tilde{u} - e^s \Delta' \tilde{u} - \lambda_1 e^s \tilde{u} - \frac{1}{4} \tilde{u} = 0 \quad \sigma_s(y_1) := e^{s/2} \dot{\theta}(e^{s/2} y_1)$$

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3. Changing the space $L^2(\Omega_0, K), \quad K(y) = e^{y_1^2/4} \implies$ compactness !

4. Asymptotic analysis As $s \rightarrow \infty,$ $|\sigma_s(y_1)| \rightarrow \|\dot{\theta}\|_{L^1(\mathbb{R})} \delta(y_1)$
 $\mathbb{R} \times e^{-s/2} \omega \rightarrow \mathbb{R}$

$$\varphi_s - \frac{1}{2} y_1 \varphi_{y_1} - \varphi_{y_1 y_1} - \frac{1}{4} \varphi = 0 \quad + \quad \text{Dirichlet b.c. at } y_1 = 0 \text{ iff } \Omega_\theta \text{ twisted}$$

Conclusions

Moral :

→ straight and twisted tubes have the same spectrum



→ Hardy inequality **iff** the tube is twisted

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→ improved decay rate **iff** the tube is twisted

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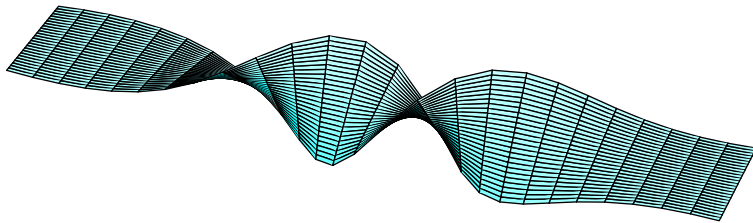


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Possible extensions: (other twisted systems)



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→ Hardy inequality **iff** the tube is twisted

$$-\Delta_D^{\Omega_\theta} - \lambda_1 \geq w(\cdot)$$

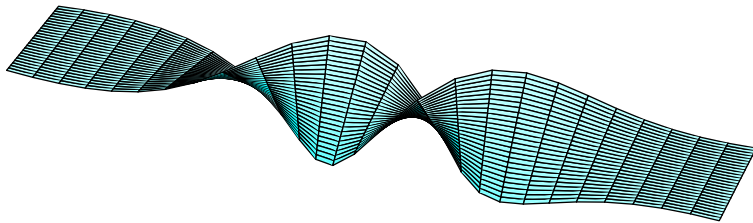


→ improved decay rate **iff** the tube is twisted

$$\|e^{(\Delta_D^{\Omega_\theta} + \lambda_1)t}\| \leq (1+t)^{-(1/4+\gamma)}$$

⇒ fine effect of transience, faster cool down

Possible extensions: (other twisted systems)



Open problems :

¿ better topology than $L^2(\Omega_\theta, K) \rightarrow L^2(\Omega_\theta)$? **NB** $K(x) = e^{x^2/4}$

¿ $\Gamma(\Omega_\theta, K) = 3/4$ if Ω_θ is twisted ? (optimality)

¿ direct proof of the equivalence \Updownarrow ? (energy methods fail)

¿ general quasi-cylindrical domains ? (\exists Hardy inequality \Leftrightarrow improved decay rate)

Our conjecture

[D.K., Zuazua 2009]

Let Ω be an open connected subset of \mathbb{R}^d . Let H and H_+ be two self-adjoint operators in $L^2(\Omega)$ such that $\inf \sigma(H) = \inf \sigma(H_+) = 0$. Assume that there is a positive smooth function $\varrho : \Omega \rightarrow \mathbb{R}$ such that $H_+ \geq \varrho$, while $H - V$ is a negative operator for any non-negative non-trivial $V \in C_0^\infty(\Omega)$. Then there exists a positive function $K : \Omega \rightarrow \mathbb{R}$ such that

$$\lim_{t \rightarrow \infty} \frac{\|e^{-H_+ t}\|_{L^2(\Omega, K) \rightarrow L^2(\Omega)}}{\|e^{-H t}\|_{L^2(\Omega, K) \rightarrow L^2(\Omega)}} = 0.$$