# DECOUPLING APPROXIMATION DIFFUSION AND HOMOGENIZATION FOR THE LINEAR BOLTZMANN EQUATION

Marjolaine Puel

Institut de Mathématiques de Toulouse

Joint work with N. Ben Abdallah and M. Vogelius

email : puel@math.univ-toulouse.fr

# The linear Boltzmann equation

$$\partial_t f(t,x,v) + v\cdot\nabla f(t,x,v) = \int_V \sigma(x,v,v')f(t,x,v')dv' - \Sigma(x,v)f(t,x,v) = Q(f). \quad (1)$$

$f \geq 0, \quad t \in \mathbf{R}, \quad x \in \mathbf{R}^d, \quad v \in V$ compact set of $\mathbf{R}^d$.

(Ref Dautray Lions (88), Bardos, Golse, Perthame, Sentis (88))

Mass conservation: $\int_V \sigma(x,v,v')dv = \Sigma(x,v')$

$\longrightarrow$ 1 belongs to the kernel of $Q^*$.

Maxwellian profile: we define $M$ by

$$Q(M) = 0 \quad \text{and} \quad \int_V M(x,v)dv = 1 \quad \forall x \in \mathbf{R}^d$$

# Two Scaling: mean free path and size of the cell

$\epsilon$: the mean free path = average time between two collisions

$\alpha$: the size of the cell, we assume that $\sigma$ is periodic of period $\alpha$

The Boltzmann equation becomes

$$\partial_t f + \frac{1}{\epsilon} v \cdot \nabla f + \frac{1}{\epsilon^2} \Sigma(\frac{x}{\alpha}, v) f = \frac{1}{\epsilon^2} \int \sigma(\frac{x}{\alpha}, v, v') f(x, v') dv' \qquad (2)$$

Ref: $\epsilon \to 0, \alpha$ fixed Diffusion approximation, Dautray Lions (88)

$\alpha \to 0, \epsilon$ fixed Homogenization, Dumas Golse (2000)

$\epsilon = 0, \alpha \to 0$ Homogenization of the diffusion equation

Poupaud, Goudon Poupaud, Capdeboscq

$\epsilon = \alpha \to 0$ Homogenization + diffusion approximation

Allaire Bal and (99-02) Goudon Mellet (01-03)

$\epsilon << \alpha \to 0$ $M(v)$, Sentis (80)

# Heuristic I: diffusion approximation

Diffusion Limit: Formal expansion in $\epsilon$ ($\alpha = 1$) $f = f^0 + \epsilon f^1 + \epsilon^2 f^2$

$$\begin{aligned} \text{order} \quad \epsilon^{-2} \quad &: \quad Q(f^0) = 0 \\ \epsilon^{-1} \quad &: \quad Q(f^1) = v \cdot \nabla_x f^0 \\ \epsilon^{0} \quad &: \quad Q(f^2) = v \cdot \nabla_x f^1 + \partial_t f^0 \end{aligned}$$

$$\begin{aligned} \Longrightarrow \quad f^0 &= n(t,x) M(x,v) \\ f^1 &= \chi \cdot \nabla_x n + \lambda n \quad \text{with} \quad Q(\chi) = vM, \ \text{and } Q(\lambda) = v \cdot \nabla M \\ f^2 &= \dots \end{aligned}$$

Compatibility conditions:

Non drift condition + Diffusion equation

$$\int_V vM dv = 0 \qquad \partial_t n - \mathrm{div}(D(x) \cdot \nabla n) + \mathrm{div}(U(x) n) = 0$$

where $D(x) = -\int_V v \otimes \chi dv$ and $U(x) = \int_V v \lambda dv$.

# Heuristic I: homogenization I

## Homogenization process

- Reintroduce $\alpha$, the diffusion equation for $n$ becomes

$$\partial_t n^\alpha - \mathrm{div}(D(\frac{x}{\alpha}) \cdot \nabla n^\alpha) + \frac{1}{\alpha}\mathrm{div}(U(\frac{x}{\alpha})n^\alpha) = 0$$

- Make the formal two scale expansion introducing

the fast variable $y = \frac{x}{\alpha}$

$$n^\alpha(t,x) = n^0(t,x,y) + \alpha n^1(t,x,y) + \alpha^2 n^2(t,x,y)$$

Recall $\nabla n = \sum_{i=0}^{2} \alpha^i (\nabla_x n^i + \frac{1}{\alpha}\nabla_y n^i)$

# Heuristic I: homogenization II

- at the order $\alpha^{-2}$

$$-\mathrm{div}_y(D(y)\nabla_y n^0) + \mathrm{div}_y(U(y)n^0) = L(n^0) = 0$$

- at the order $\alpha^{-1}$

$$\begin{aligned} -\mathrm{div}_y(D(y)\nabla_x n^0) + \mathrm{div}_x(D(y)\nabla_y n^0) + \nabla_x(U(y)n^0) & \\ -\mathrm{div}_y(D(y)\nabla_y n^1) + \mathrm{div}_y(U(y)n^1) &= 0 \end{aligned}$$

- at the order $\alpha^0$

$$\begin{aligned} \partial_t n^0 - \mathrm{div}_x(D(y)\nabla_x n^0) & \\ -\mathrm{div}_y(D(y)\nabla_x n^1) - \mathrm{div}_x(D(y)\nabla_y n^1) + \nabla_x(U(y)n^1) & \\ -\mathrm{div}_y(D(y)\nabla_y n^2) + \mathrm{div}_y(U(y)n^2) &= 0. \end{aligned}$$

# Heuristic I: homogenization III

First equation implies

$$n^0(t, x, y) = \tilde{n}(t, x)\rho(y)$$

with $L(\rho) = -\mathrm{div}_y(D(y)\nabla_y \rho) + \mathrm{div}_y(U(y)\rho) = 0.$

The second equation gives

$$L(n^1) = H(y) \cdot \nabla_x \tilde{n}$$

The third equation can be writen

$$L(n^2) = F(n^0, n^1)$$

Note that 1 belongs to the Ker $Q^*$.

# Heuristic I: homogenization IV

Compatibility conditions: $(1 \in KerL^*)$

For the second equation

$$\int_Y \mathrm{div}_x(D(y)\nabla_y n^0) + \nabla_x(U(y)n^0) = 0$$

and for the third equation

$$\partial_t \tilde{n} - \mathrm{div}_x([\int\int \rho(y)D(y)]\nabla_x \tilde{n}) - \mathrm{div}_x \int\int (v \cdot \nabla \chi^* - \chi^* v \cdot \nabla M) L^{-1} H(y) \cdot \nabla_x \tilde{n} = 0.$$

Let us introduce $\theta_{-1}$ solution to

$$L^*(\theta_{-1}) = \int_V (v \cdot \nabla_y \chi^* - \chi^* v \cdot \nabla_y M)$$

# Formal result

$$f(t,x,v) \sim n(t,x,\frac{x}{\alpha})M(\frac{x}{\alpha},v) = \rho(\frac{x}{\alpha})\tilde{n}(t,x)M(\frac{x}{\alpha},v)$$

- with $\rho$ periodic in $y$ satisfying

$$-\mathrm{div}_y(D(y)\nabla_y\rho(y)) + \mathrm{div}_y(U(y)\rho(y)) = 0 \text{ with } \int_Y \rho(y) = 1$$

where $D(y) = -\int_V \chi^*(y,v)\otimes v dv$ and $U(y) = \int_V v\lambda(y,v)dv$

- and $\tilde{n}(t,x)$ is the solution of the diffusive equation

$\partial_t\tilde{n} - \mathrm{div}_x(D\nabla_x\tilde{n}) = 0$ with a diffusion coefficient given by

$$D = -\int_Y\int_V (\rho v\otimes\chi + \rho M\chi^*\cdot\nabla_y\theta_{-1}\otimes v + v\cdot\nabla_y(\rho M)\theta_{-1}\otimes\chi^*)\, dvdy$$

# Heuristic II

Start from Allaire Bal or Goudon Mellet and introduce $\eta = \dfrac{\epsilon}{\alpha}$

We obtain

$$f \sim n^\eta(t,x) F^\eta(y,v)$$

$F^\eta$ solution of $\quad \eta v \cdot \nabla_y F^\eta - Q(F^\eta) = 0$

$n^\eta$ solution of $\quad \partial_t n^\eta + \nabla_x (D^\eta \nabla_x n) = 0$

$$\text{with} \quad D^\eta = \int_Y \int_V \chi^{*\eta} \otimes v F^\eta dv dy$$

where $\chi^{*\eta}$ is defined by $\quad \eta v \cdot \nabla_y \chi^{*\eta} + Q^*(\chi^{*\eta}) = v.$

**Idea:** study the expansion with respect to $\eta$ of the solutions $F^\eta$ and $\chi^{*\eta}$ of two cell equations.

# Study of the cell equations I

**Expansion of $F^\eta$ solution of** $\quad \eta v \cdot \nabla_y F^\eta - Q(F^\eta) = 0$**:**

$$F^\eta(y,v) = \rho(y)M(y,v) + \eta\left(\chi(y,v)\cdot\nabla_y\rho(y) + \lambda(y,v)\rho(y)\right) + \eta^2 F^2(y,v) + R^\eta(y,v)$$

where $Q(\chi) = v$ and $Q(\lambda) = v \cdot \nabla_y M$

- at the order $\eta^{-1}$, $Q(F^0) = 0 \Longrightarrow F^0 = \rho(y)M(y,v)$
- at the order $\eta^0$, $\quad Q(F^1) = v \cdot \nabla(F^0) = \rho(y)v \cdot \nabla_y M + Mv \cdot \nabla_y \rho$
- at the order $\eta^1$, $\quad Q(F^2) = -v \cdot \nabla(F^1) \Longrightarrow$ same eq. on $\rho$

# Study of the cell equations II

**Expansion of $\chi^{*\eta}$ solution of** $\eta v \cdot \nabla_y \chi^{*\eta} + Q^*(\chi^{*\eta}) = v$**:**

$$\chi^{\eta *} = \frac{1}{\eta}\theta_{-1}(y) + \chi^* + \chi^* \cdot \nabla\theta_{-1}(y) + \theta_0(y) + r^{\eta}_{\chi^*}$$

- at the order $\eta^{-2}$, $Q^*(\chi^{-1*}) = 0 \Longrightarrow \chi^{-1} = \theta_{-1}(y)$
- at the order $\eta^{-1}$, $Q^*(\chi^{0*}) = v - v \cdot \nabla(\chi^{-1*})$
- at the order $\eta^{0}$, $Q^*(\chi^{1^*}) = -v \cdot \nabla(\chi^{0*})$

under compatibility conditions that give the expression of $\theta_{-1}$

which is the same as above

# Theorem

Assume $0 < c_0 \leq \sigma \in C^\infty$,

$\sigma(-y, v, v') = \sigma(y, v, v'), \sigma(y, -v, -v') = \sigma(y, v, v')$

$\Longrightarrow$ compatibility conditions

$$\int_{\mathbf{R}^d} \int_V f^{\epsilon\eta}(x, v) dv dx \leq C.$$

Then $f^{\epsilon,\eta}$ two scale converges ( at the scale 1 and $\alpha = \frac{\epsilon}{\eta}$) towards

$$\mathcal{F}(t, x, y, v) = n(t, x, y) M(y, v) = \rho(y) \tilde{n}(t, x) M(y, v)$$

with $\rho$ solution of

$$-\mathrm{div}_y(D(y) \nabla_y \rho(y)) + \mathrm{div}_y(U(y)\rho(y)) = 0 \text{ with } \int_Y \rho(y) = 1$$

and $\tilde{n}(t, x)$ is the solution of the diffusive equation

$$\partial_t \tilde{n} - \mathrm{div}_x(D \nabla_x \tilde{n}) = 0 \quad \text{with the diffusion coefficient } D$$

# Idea of the proof I: two scale convergence

**Definition Two scale convergence**

*A sequence of positive measures $\mu_{\epsilon\eta}$ is said to two scale converge toward $\mu$ if and only if for any test function $\phi \in \mathcal{C}^0_{c,\#}([0,T], \mathbf{R}^d, \mathbf{R}^d, V)$, we have*

$$\int_0^\infty \int_{\mathbf{R}^d} \int_V \phi(t, x, \frac{\eta x}{\epsilon}, v) \mu_{\epsilon\eta}(dt, dx, dv) \longrightarrow_{\epsilon\eta\to 0} \int_0^\infty \int_{\mathbf{R}^d} \int_Y \int_V \phi(t, x, y, v) \mu(dt, dx, dy, dt)$$

Apply Banach Alaoglu to $\mu_{\epsilon\eta}(t, x, y, v) = f^{\epsilon,\eta}(t, x, v)\delta(y - \frac{\eta x}{\epsilon})$ which is a bounded sequence of measure and you obtain that $f^{\epsilon,\eta}(t, x, v)$ two scale converges towards $\mathcal{F}(t, x, y, v)$.

# Idea of the proof II: oscillating test functions I

For any oscillating test function $\phi(t,x,y,v)$ periodic in $y$, we have

$$\lim_{\epsilon\eta\to 0}\int_0^T\int_{\mathbf{R}^d}\int_V f^{\epsilon\eta}(t,x,v)Q^*(\phi(t,x,\eta\frac{x}{\epsilon},v))dvdxdt = 0.$$

Then $\forall\psi$ s.t. $\int_V \psi M dv = 0$, $\int_0^\infty\int_{\mathbf{R}^d}\int_Y\int_V \psi\mathcal{F} dvdydxdt = 0.$

$$\Longrightarrow \mathcal{F}(t,x,y,v) = n(t,x,y)M(y,v)$$

# Idea of the proof II: oscillating test functions II

Using the peculiar test function

$$\phi(t,x,\eta\frac{x}{\epsilon},v)=\psi(t,x,\eta\frac{x}{\epsilon})+\eta z(t,x,\eta\frac{x}{\epsilon},v)+\epsilon w(t,x,\eta\frac{x}{\epsilon},v),$$

with $Q^*(z)=-v\cdot\nabla_y\psi$, and $Q^*(w)=-v\cdot\nabla_x\psi$, we obtain

$$\frac{d}{dt}\int_V\int_{\mathbf{R}^d}f^{\epsilon\eta}\phi=\int_V\int_{\mathbf{R}^d}f^{\epsilon\eta}(\partial_t\psi+\eta\partial_t z+\epsilon\partial_t w)+\partial_t f^{\epsilon\eta}(\psi+\eta z+\epsilon w)$$

$$=\int_V\int_{\mathbf{R}^d}f^{\epsilon\eta}(\partial_t\psi+\eta\partial_t z+\epsilon\partial_t w+\frac{\eta}{\epsilon}v\cdot\nabla_x z+v\cdot\nabla_x w+\frac{\eta^2}{\epsilon^2}v\cdot\nabla_y z+\frac{\eta}{\epsilon}v\cdot\nabla_y w),$$

# Idea of the proof II: oscillating test functions III

Passing to the limit + $z = -\chi^* \cdot \nabla_y \psi$

$$\int_0^T \int_{\mathbf{R}^d} \int_Y \int_V n(t,x,y) M(y,v) v \cdot \nabla_y (-\chi^*(y,v) \cdot \nabla_y \psi(t,x,y)) dt dx dy dv = 0$$

Passing the derivative on $n$

$$\int \mathrm{div}_y [-v \cdot \nabla_y n(t,x,y) M(y,v) \chi^*(y,v) - n(t,x,y) v \cdot \nabla_y M(y,v) \chi^*(y,v)] \psi(t,x,y)) dt dx$$

which is nothing but $\quad L(n) = 0 \quad \Longrightarrow \quad n(t,x,y,v) = \rho(y) \tilde{n}(t,x)$

## Idea of the proof II: oscillating test functions IV

Finnally, to obtain the diffusion equation for $\tilde{n}$, take

$$\phi(t,x,\eta\frac{x}{\epsilon},v) = \varphi(t,x) + \epsilon\psi(t,x,\eta\frac{x}{\epsilon},v)$$

with $\eta v \cdot \nabla_y \psi^\eta + Q^*(\psi^\eta) = -v \cdot \nabla_x \varphi$ i.e. $\psi^\eta = -\chi^{\eta *} \cdot \nabla_x \phi$

and replace $\chi^{*\eta}$ by its expansion

$$\chi^{*\eta}(y,v) = \frac{1}{\eta}\theta_{-1}(y) + \chi^*(y,v) - \chi^*(y,v) \cdot \nabla_y \theta_{-1}(y).$$

## Remark on the proof: the entropy equation

Thanks to the classical dissipation property, we have

$$\int_V \int_Y [\partial_t f \frac{f}{M} + v \cdot \nabla f \frac{f}{M}] dvdy = \int_V \int_Y [Q(f) \frac{f}{M}] dvdy \leq 0$$

$$\nabla_y M \neq 0 \longrightarrow \text{ no } L^2 \text{ bound}$$

$$L^1 \text{framework}$$

$\longrightarrow$ We need $C^0$ convergence for $\chi^{*\eta}$