An improved excursion set theory approach to halo mass function, non-Gaussianities and bias

based on

MM and A. Riotto, ApJ 711, 907 (2010) arXiv:0903.1249
717, 515 (2010) 0903.1250
717, 515 (2010) 0903.1251
MNRAS, 405, 1244 (2010) 0910.5125
A. De Simone, MM and A. Riotto, 1007.1903
C.-P. Ma, MM, A. Riotto and J. Zhang 1007.

M. Maggiore, Benasque 2010

UNIVERSITÉ DE GENÈVE

Plan of the talk

- Excursion set theory
- Path integral formulation of excursion set theory
  - non-Markovian evolution with smoothing scale
  - non-Gaussianities in the primordial fluctuations
- Improvement in the collapse model
  - ellipsoidal barrier
  - diffusing barrier

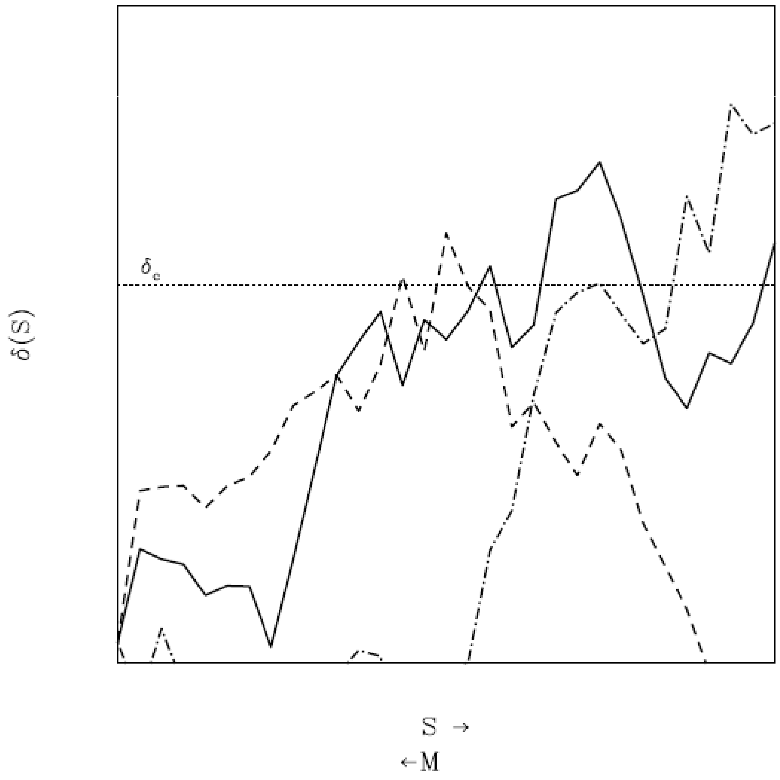
- Results of the formalism
  - halo mass function for spherical collapse
    - Gaussian fluctuation (MR1)
    - Non-Gaussian fluctuations (MR3,MR4)
  - ellipsoidal collapse
    - A more formal derivation of the SMT mass function
    - Ellipsoidal collapse + Non-Gaussianities

      (De Simone, MM, Riotto)
  - halo bias (Ma, MM,Riotto,Zhang)

Excursion set theory

Bond, Cole, Efstathiou and Kaiser (1991)
Peacock and Heavens (1990)

- study the evolution of $\delta(R)$ as a function of R
  - at $R=\infty$, $\delta(R)=0$. Lowering R, $\delta(R)$ evolves stochastically
  - use $S=\sigma^2(R)$ as "time". At $R=\infty$, $S=0$. As R decreases, S increases



First-passage time problem

What are the equations governing this stochastic motion ?

$$\delta(R) \equiv \delta(\mathbf{x}=0, R) = \int \frac{d^3k}{(2\pi)^3} \tilde{W}(k,R)\tilde{\delta}_{\mathbf{k}}$$

$$\frac{\partial \delta(R)}{\partial R} = \int \frac{d^3k}{(2\pi)^3} \frac{\partial \tilde{W}(k,R)}{\partial R} \tilde{\delta}_{\mathbf{k}} \equiv \zeta(R)$$

"noise"

$$\zeta(R)\frac{dS}{dR} = \eta(S),$$

$$\frac{\partial \delta(S)}{\partial S} = \eta(S)$$

For sharp k-space filter: $\langle \eta(S_1)\eta(S_2)\rangle = \delta_D(S_1 - S_2)$

Langevin eq. with Dirac-delta noise !

→ $\Pi(\delta,S)$ satisfies the Fokker-Planck eq.

$$\frac{\partial \Pi}{\partial S} = \frac{1}{2}\frac{\partial^2 \Pi}{\partial \delta^2}$$

to take into account that we are interested in the first-passage problem, Bond et al impose an "absorbing barrier boundary condition",

$$\Pi(\delta = \delta_c, S) = 0$$

The solution is

$$\Pi(\delta, S) = \frac{1}{\sqrt{2\pi S}}\left[e^{-\delta^2/(2S)} - e^{-(2\delta_c-\delta)^2/(2S)}\right]$$

The first-crossing rate is

$$\mathcal{F}(S) = -\frac{\partial}{\partial S}\int_{-\infty}^{\delta_c} d\delta \, \Pi(\delta, S)$$

With standard manipulations the halo mass function is then:

$$\frac{dn}{dM} = f(\sigma)\frac{\bar{\rho}}{M^2}\frac{d\ln\sigma^{-1}}{d\ln M}$$

$$f(\sigma) = 2\sigma^2\mathcal{F}(\sigma^2) \qquad\qquad (S \equiv \sigma^2),$$

$$f_{\mathrm{PS}}(\sigma) = \left(\frac{2}{\pi}\right)^{1/2}\frac{\delta_c}{\sigma}\,e^{-\delta_c^2/(2\sigma^2)}$$

Excursion set theory is very elegant, but its original formulation suffers of two type of problems:

- at the technical level: the evolution with S is Markovian only if δ is smoothed with a sharp filter in k-space. However, with this filter it is not possible to associate a mass to the smoothing radius !

  With any other filter the evolution is non-Markovian

  (important also for non-Gaussianities)

- at the physical level: the spherical collapse model is an oversimplification of the complex dynamics of halo formation

our formulation of the problem

- consider a stochastic process $\delta(S)$ defined by

  $\langle\, \delta(S_1)\, \delta(S_2)\rangle_c$, $\langle\delta(S_1)\, \delta(S_2)\, \delta(S_3)\rangle_c$, ...

- consider an ensemble of trajectories all starting at ``time'' $S=0$ from $\delta(0)= \delta_0$ and follow them for a time S

- discretize time S, $S_k=k\,\varepsilon$, $(k=0,\dots,n)$, $S_n=S$

- a trajectory is defined by the collection of values δ_k such that $\delta(S_k)= \delta_k$

- the probability density in the space of trajectories is

$$W(\delta_0;\delta_1,\ldots,\delta_n;S_n) = \langle \delta_D(\delta(S_1)-\delta_1)\ldots\delta_D(\delta(S_n)-\delta_n)\rangle$$

using $\quad \delta_D(x) = \int_{-\infty}^{\infty}\frac{d\lambda}{2\pi}e^{-i\lambda x} \quad$ we get

$$W(\delta_0;\delta_1,\ldots,\delta_n;S_n) = \int \mathcal{D}\lambda\ e^{i\lambda_i\delta_i}\langle e^{-i\lambda_i\delta(S_i)}\rangle$$

$$e^Z \equiv \langle e^{-i\sum_i \lambda_i\delta(S_i)}\rangle$$

Z is the generating functional of connected correlators!

$$\begin{aligned} Z &= -\frac{1}{2}\lambda_i\lambda_j\langle\delta(S_i)\delta(S_j)\rangle_c \\ &\quad +\frac{(-i)^3}{3!}\lambda_i\lambda_j\lambda_k\langle\delta(S_i)\delta(S_j)\delta(S_k)\rangle_c + \ldots \end{aligned}$$

- define

$$\Pi_\epsilon(\delta_0;\delta_n;S_n) = \int_{-\infty}^{\delta_c} d\delta_1 \ldots \int_{-\infty}^{\delta_c} d\delta_{n-1}\, W(\delta_0;\delta_1,\ldots,\delta_n;S_n)$$

this is the probability of arriving in δ_n through trajectories that never exceeded a threshold δ_c

- first crossing rate:

$$\mathcal{F}(S) = -\frac{\partial}{\partial S}\int_{-\infty}^{\delta_c} d\delta\, \Pi_\epsilon(\delta_0;\delta,S)$$

- The problem is reduced to the evaluation of a path-integral with boundaries
- First-principle approach. No ad hoc “absorbing barrier boundary condition”

1. Markovian case + Gaussian fluctuations

$$\langle \delta_i \delta_j \rangle = \mathrm{min}(S_i, S_j) \qquad \delta_i \equiv \delta(S_i)$$

all higher connected correlators are zero.
We can then compute explicitly:

$$\begin{aligned} W^{\mathrm{gm}} &= \int \mathcal{D}\lambda \; e^{i\lambda_i\delta_i - (1/2)\lambda_i\lambda_j\langle\delta_i\delta_j\rangle} \\ &= \frac{1}{(2\pi\epsilon)^{n/2}} \mathrm{exp}\{-\frac{1}{2\epsilon}\sum_{i=0}^{n-1}(\delta_{i+1}-\delta_i)^2\} \end{aligned}$$

From this, we can prove that, in the continuum limit, $\Pi(\delta,S)$ satisfies the FP equation and the b.c. $\Pi(\delta,S=0)$ if $\delta > \delta_c$

We recover the standard results from excursion set theory

2. Non-Markovian case + Gaussian fluctuations

For top-hat filter in cordinate space:

$$\langle \delta_i \delta_j \rangle = \mathrm{min}(S_i, S_j) + \Delta(S_i, S_j)$$

$$\Delta(S_i, S_j) \simeq \kappa \frac{S_i(S_j - S_i)}{S_j} \qquad (S_i < S_j)$$

$$\kappa = \lim_{R' \to \infty} \frac{\langle \delta(R')\delta(R) \rangle - \langle \delta^2(R') \rangle}{\langle \delta^2(R') \rangle} \simeq 0.4$$

κ measures the amount of non-Markovianity

In this case:

$$W = \int \mathcal{D}\lambda \; e^{i\lambda_i\delta_i - (1/2)\lambda_i\lambda_j[\mathrm{min}(S_i,S_j)+\Delta(S_i,S_j)]}$$

Expand perturbatively in Δ.
A flavor of the computation: the correction to Π is

$$\frac{1}{2}\int_{-\infty}^{\delta_c} d\delta_1 \ldots d\delta_{n-1} \sum_{i,j=1}^{n} \Delta_{ij}\partial_i\partial_j W^{\mathrm{gm}}(\delta_0;\delta_1,\ldots,\delta_n;S_n)$$

$$\partial_i \equiv \frac{\partial}{\partial\delta_i}$$

Consider for instance the terms with i<n, j=n. We then need

$$\sum_{i=1}^{n-1} \Delta_{in} \int_{-\infty}^{\delta_c} d\delta_1 \ldots d\delta_{n-1} \partial_i W^{\mathrm{gm}}(\delta_0;\delta_1,\ldots,\delta_n;S_n)$$

$$\sum_{i=1}^{n-1} \Delta_{in} \int_{-\infty}^{\delta_c} d\delta_1 \ldots d\delta_{n-1} \partial_i W^{\mathrm{gm}}(\delta_0; \delta_1, \ldots, \delta_n; S_n)$$

$$= \sum_{i=1}^{n-1} \Delta_{in} \int_{-\infty}^{\delta_c} d\delta_1 \ldots \widehat{d\delta_i} \ldots d\delta_{n-1} W^{\mathrm{gm}}(\delta_0; \ldots, \delta_i = \delta_c, \ldots, \delta_n; S_n)$$

$$= \sum_{i=1}^{n-1} \Delta_{in} \Pi_\epsilon^{\mathrm{gm}}(\delta_0; \delta_c; S_i) \Pi_\epsilon^{\mathrm{gm}}(\delta_c; \delta_n; S_n - S_i)$$

$$= \frac{1}{\epsilon} \int_0^{S_n} dS_i \Delta(S_i, S_n)\, \Pi_\epsilon^{\mathrm{gm}}(\delta_0; \delta_c; S_i) \Pi_\epsilon^{\mathrm{gm}}(\delta_c; \delta_n; S_n - S_i)$$

In the continuum limit: $\Pi_{\epsilon=0}^{\mathrm{gm}}(\delta_0; \delta_n = \delta_c; S) = 0$

we need to know how it approaches zero:

$$\Pi_\epsilon^{\mathrm{gm}}(\delta_0; \delta_c; S) = \sqrt{\epsilon} \frac{1}{\sqrt{\pi}} \frac{\delta_c - \delta_0}{S^{3/2}} e^{-(\delta_c - \delta_0)^2/2S}$$

The remaining integral can be computed analytically.
Terms with i,j <n are more difficult (cancellation of divergences)

- we are able to computing everything analytically, and in the end

$$\frac{dn}{dM} = f(\sigma)\frac{\bar{\rho}}{M^2}\frac{d\ln\sigma^{-1}}{d\ln M}$$

$$f(\sigma) \;=\; \boxed{(1-\kappa)}\left(\frac{2}{\pi}\right)^{1/2}\frac{\delta_c}{\sigma}\,e^{-\delta_c^2/(2\sigma^2)} + \boxed{\frac{\kappa}{\sqrt{2\pi}}\frac{\delta_c}{\sigma}\Gamma\left(0,\frac{\delta_c^2}{2\sigma^2}\right)}$$

3. Non-gaussianities

- example: bispectrum

$$W = \int \mathcal{D}\lambda \; e^{i\lambda_i\delta_i - (1/2)\lambda_i\lambda_j\langle\delta_i\delta_j\rangle + (1/3!)\lambda_i\lambda_j\lambda_k\langle\delta_i\delta_j\delta_k\rangle}$$

- expand perturbatively in the bispectrum and compute with the same technique

Note that we have the can compute the full dependence on

$$< \delta(S_i)\, \delta(S_j)\, \delta(S_k) >$$

while NG extension of Press-Schechter theory only depends on the cumulant $<\delta^3(S)>$

Our result for the mass function (MR3)

$$f(\sigma) = (1-\kappa)\left(\frac{2}{\pi}\right)^{1/2}\frac{\delta_c}{\sigma}e^{-\delta_c^2/(2\sigma^2)}\left[1+\frac{\sigma^2}{6\delta_c}h_{\rm NG}(\sigma)\right] + \frac{\kappa}{\sqrt{2\pi}}\frac{\delta_c}{\sigma}\Gamma\left(0,\frac{\delta_c^2}{2\sigma^2}\right)$$

$$h_{\rm NG}(\sigma) = \frac{\delta_c^4}{\sigma^4}\mathcal{S}_3(\sigma) - \frac{\delta_c^2}{\sigma^2}\left(2\mathcal{S}_3(\sigma)+\mathcal{U}_3(\sigma)-\frac{d\mathcal{S}_3}{d\ln\sigma}\right) - \left(\mathcal{S}_3+\mathcal{U}_3+\mathcal{V}_3+\frac{d(\mathcal{S}_3+\mathcal{U}_3)}{d\ln\sigma}\right)+\mathcal{O}\left(\frac{\sigma^2}{\delta_c^2}\right)$$

$$\mathcal{S}_3 = \frac{1}{S^2}\langle\delta^3(S)\rangle\,, \qquad \mathcal{U}_3 = \frac{3}{S}\left[\frac{d}{dS_1}\langle\delta(S_1)\delta^2(S)\rangle\right]_{S_1=S}$$

$$\mathcal{V}_3 = \frac{9}{2}\left[\frac{d^2}{dS_1^2}\langle\delta(S_1)\delta^2(S)\rangle + 12\frac{d^2}{dS_1dS_2}\langle\delta(S_1)\delta(S_2)\delta(S)\rangle\right]_{S_1=S_2=S}$$

The dependence on <δ^3(S)> agrees with that found from NG extension of Press-Schechter theory (LoVerde, Miller, Shandera & Verde 2008)

Improving the physical model for collapse

The spherical collapse model is a poor approximation to the complex dynamics of halo formation. Various improvements can be implemented in the excursion set theory framework

- ellipsoidal collapse → moving barrier B(S)   Sheth &Tormen (1999)

  Sheth, Mo &Tormen (2001)

- diffusing barrier model: the critical threshold for collapse is treated as a stochastic variable, to take into account at an effective level the randomness of the process

  (MR2; see also Audit et al (1997), Lee & Shandaring (1998), Sheth, Mo and Tormen (2001) for earlier related ideas)

The diffusing barrier

- Realistic halo formation proceed through a mixture of smooth accretion, tidal effects with the environment, and violent episodes of merging and fragmentation
- Further intrinsic randomness related to the actual definition of halos in N-body simulations and in observations

In MR2 we propose that at least some of this complexity can be accounted for, at an effective level, by promoting the critical value for collapse to a stochastic variable, that fluctuates over an average value given by the spherical or ellipsoidal collapse model

We considered a barrier that performs a random walk around the constant value δ_c, with a diffusion coefficient D_B

- repeating our computation with such a diffusing barrier

$$f(\sigma) = (1-\tilde{\kappa})\left(\frac{2}{\pi}\right)^{1/2}\frac{a^{1/2}\delta_c}{\sigma}e^{-a\delta_c^2/(2\sigma^2)} + \frac{\tilde{\kappa}}{\sqrt{2\pi}}\frac{a^{1/2}\delta_c}{\sigma}\Gamma\left(0,\frac{a\delta_c^2}{2\sigma^2}\right)$$

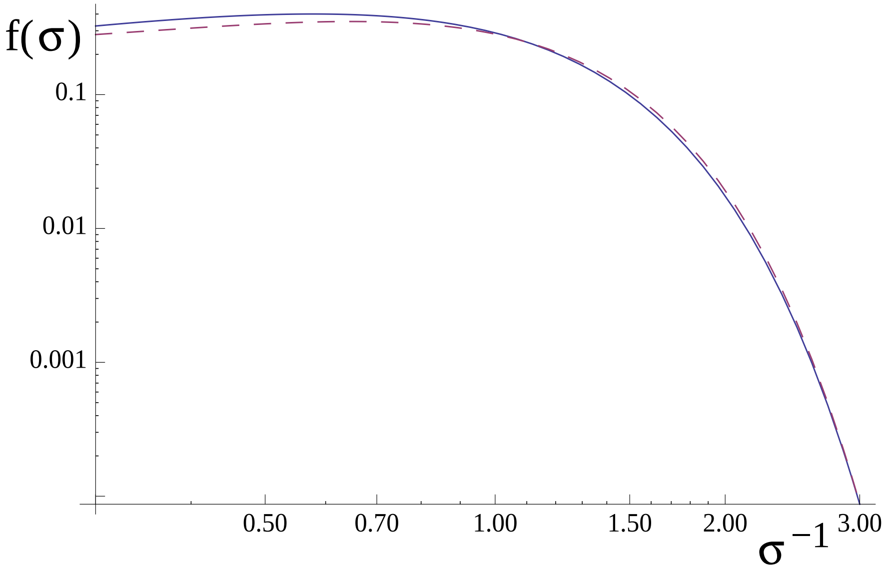
with $a=1/(1+D_B)$

$$(\tilde{\kappa} \equiv \kappa/(1+D_B))$$

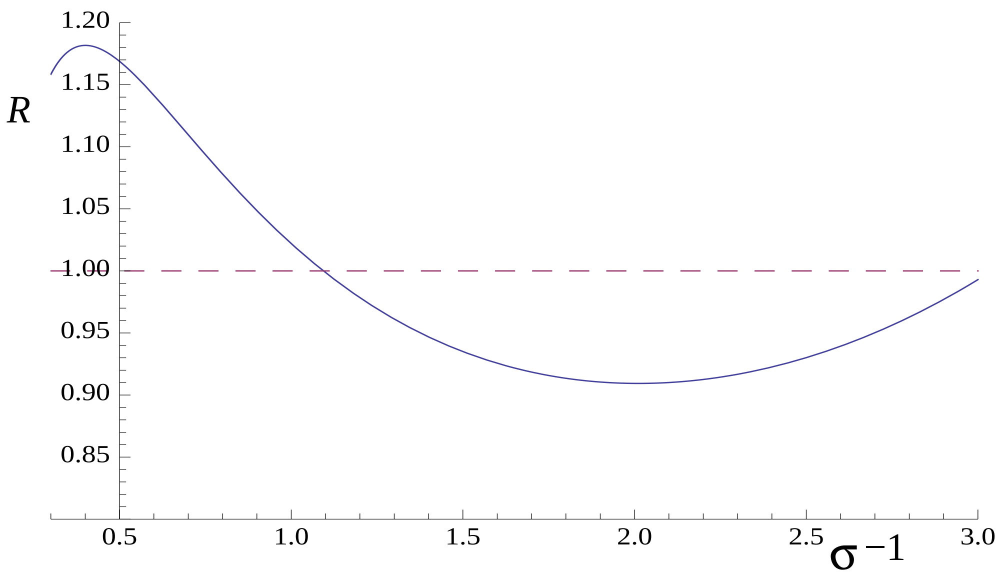
It works as if the barrier for collapse were lower, $\delta_c \rightarrow a^{1/2}\,\delta_c$
(the same holds for our NG mass function)

This is just the replacement usually performed in the literature to fit N-body data. Our result therefore offer a physical interpretation for this replacement

From recent N-body simulation (Robertson et al 2008) we deduce $D_B \approx 0.25$ and $a \approx 0.80$, in excellent agreement with the slope of the mass function from the same N-body simulation



Comparison of our mass function (non-Markovian corrections and $a=1/(1+D_B)$ with the fitting function of Tinker et al



Relative error < 10%

for $\sigma^{-1} \geq 1$

Ellipsoidal collapse in the path integral formulation

De Simone, MM, Riotto

- For a wide range of moving barriers B(S) ,

Sheth & Tormen (2002) show that the empirical formula

$$\mathcal{F}_{\rm ST}(S) = \frac{e^{-B^2(S)/2S}}{\sqrt{2\pi}\,S^{3/2}} \sum_{p=0}^{5} \frac{(-S)^p}{p!} \frac{\partial^p B(S)}{\partial S^p}$$

fits well the first-crossing rate obtained generating numerically a large set of random walks.

It also reproduces the exact result for constant barrier and for linear barrier (Sheth 1998)

- a puzzle: the truncation to p=5 works well empirically. However

$$\sum_{p=0}^{\infty} \frac{(-S)^p}{p!} \frac{\partial^p B(S)}{\partial S^p}$$

is the Taylor expansion of $B(S_0\text{-}S)$ in $S_0=S$, so it resums to B(0) !
Even if it works well, it cannot be fundamentally correct!

(see also Lam & Sheth 2009)

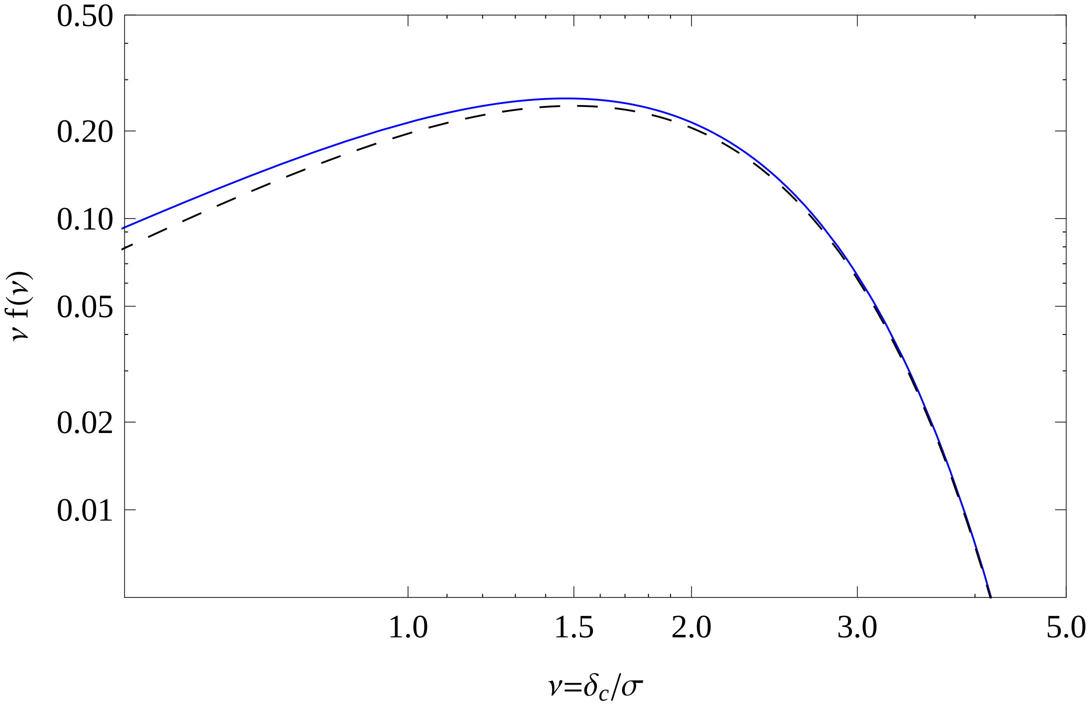
Our path integral formulation is well suited for computing the first-crossing rate with moving barrier from first principles:

$$\Pi_\epsilon(\delta_0;\delta_n;S_n) = \int_{-\infty}^{B(S_1)} d\delta_1 \ldots \int_{-\infty}^{B(S_{n-1})} d\delta_{n-1}\, W(\delta_0;\delta_1,\ldots,\delta_n;S_n)$$

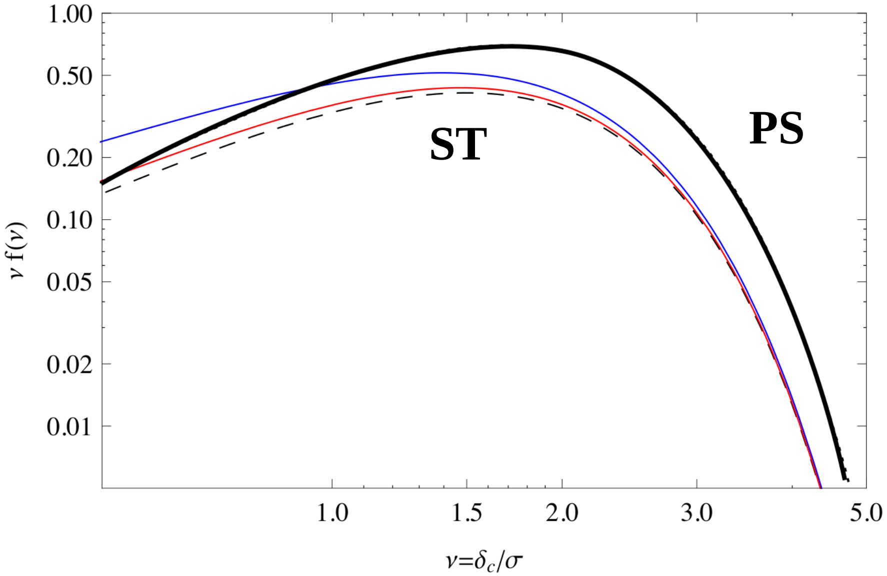
Shifting the integration variables $\delta_i \rightarrow \delta_i - B(S_i)$ we reduce to constant barrier + new terms in W, which depends on the derivatives of B(S) $\rightarrow$ Apply our perturbative expansion

- we perform an expansion in derivatives (appropriate for the ellipsoidal collapse barrier). (De Simone,MM, Riotto, 1007.1903)

$$\begin{aligned}\mathcal{F}(S) &= \frac{B(S)}{\sqrt{2\pi}S^{3/2}}e^{-B^2(S)/2S} \\ &-\frac{B'(S)}{\sqrt{2\pi S}}e^{-B^2(S)/2S} \\ &+\frac{B''(S)}{4\pi}\left\{\sqrt{2\pi S}e^{-B^2(S)/2S}-\pi B(S)\mathrm{Erfc}\left[\frac{B(S)}{2S}\right]\right\}\end{aligned}$$

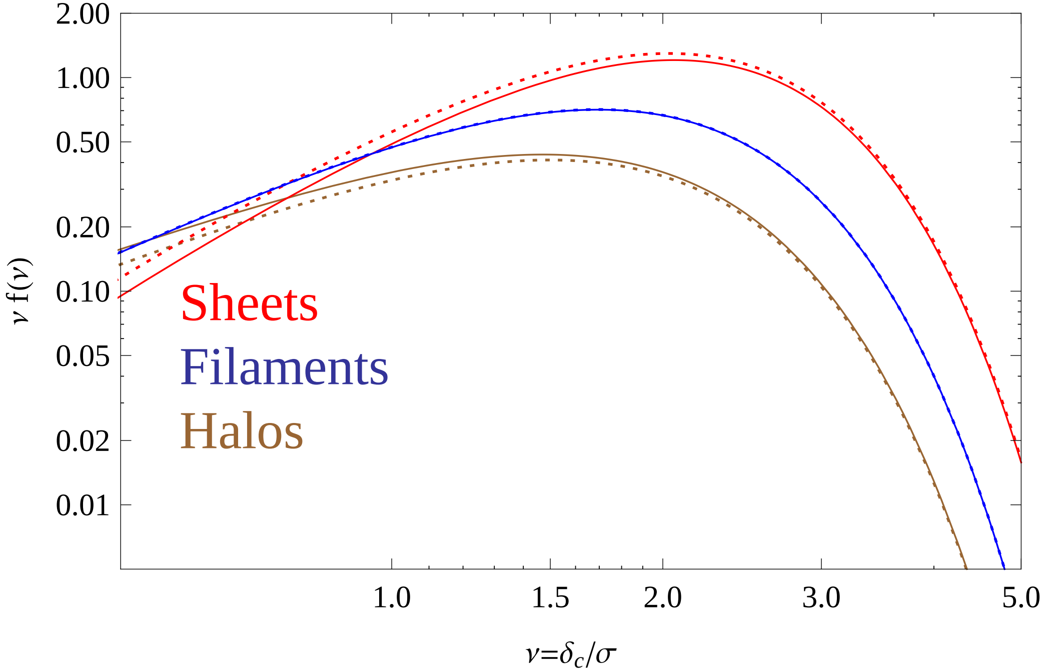


Comparison with the ST first-crossing rate



Using a different expansion, which resums an infinite number of term

Works well also for other barriers



Ellipsoidal barrier + Non-Gaussianities

- Motivations: the complete halo mass function is usually computed hoping that

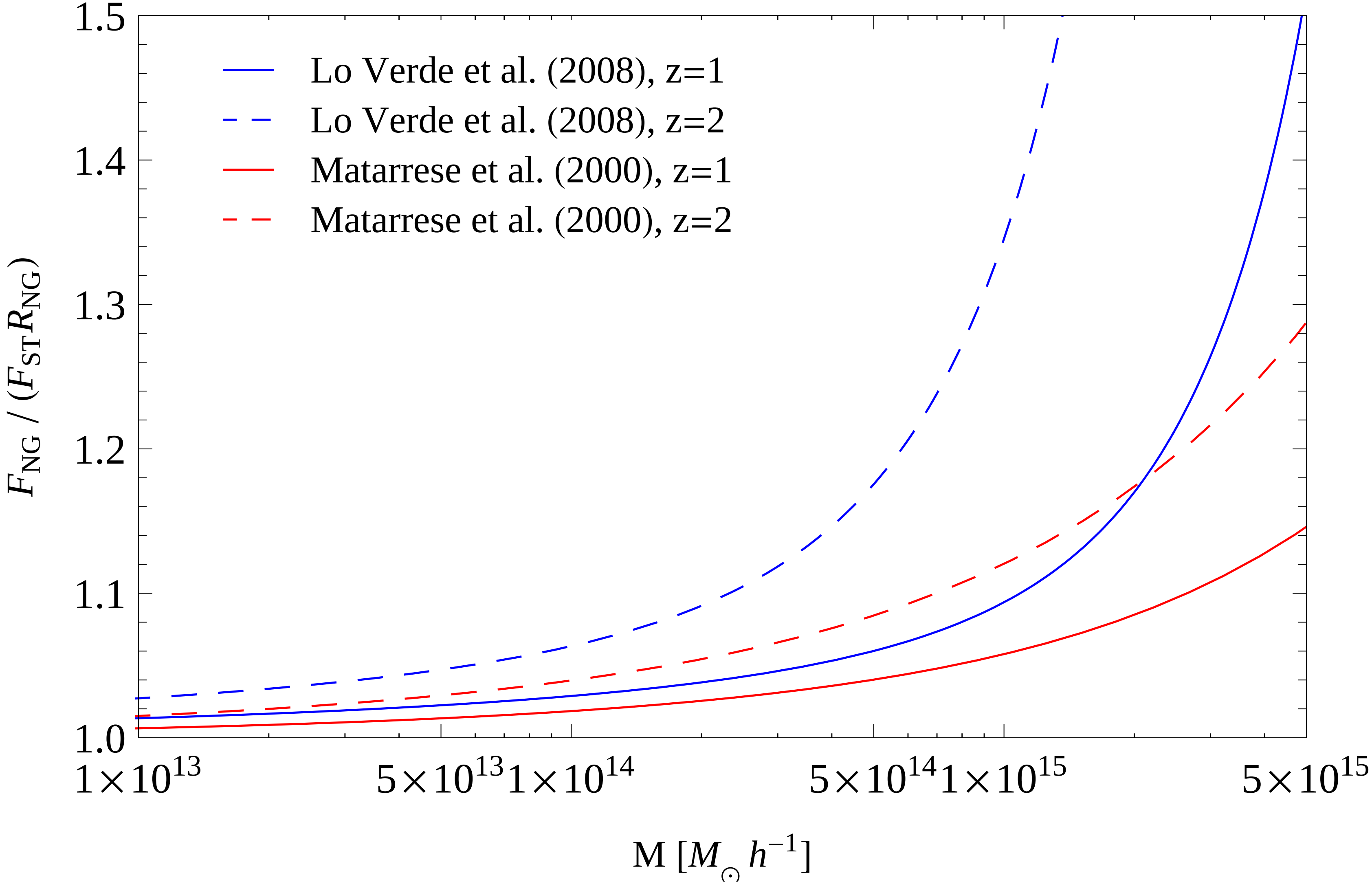
$$\frac{f_{\mathrm{NG}}(\sigma)}{f_{\mathrm{Gau}}(\sigma)} = \left[\frac{f_{\mathrm{NG}}(\sigma)}{f_{\mathrm{Gau}}(\sigma)}\right]_{\mathrm{PS}} \equiv R(\sigma)$$

then one takes

$$f_{\mathrm{NG}}(\sigma) = R(\sigma) f_{\mathrm{ST}}(\sigma)$$

With our formalism we can compute directly the first-crossing rate with ellipsoidal barrier + NG and compare with the above ansatz. Full analytic result, see De Simone,MM, Riotto 2010 (including a saddle point improvement suggested in

D'Amico, Musso,Noreña & Paranjape 2010)



Lo Verde et al. (2008), z=1
Lo Verde et al. (2008), z=2
Matarrese et al. (2000), z=1
Matarrese et al. (2000), z=2
$F_{NG} / (F_{ST} R_{NG})$
M $[M_\odot \, h^{-1}]$
1.0
1.1
1.2
1.3
1.4
1.5
1×10^{13}
5×10^{13}
1×10^{14}
5×10^{14}
1×10^{15}
5×10^{15}

Non-Markovian corrections to halo bias

(Ma,MM,Riotto, Zhang)

- Halo bias can be obtained from excursion set theory + spherical collapse :

Cole & Kaiser 1989, Bond et al 1991, Mo & White 1996

compute the first crossing rate for

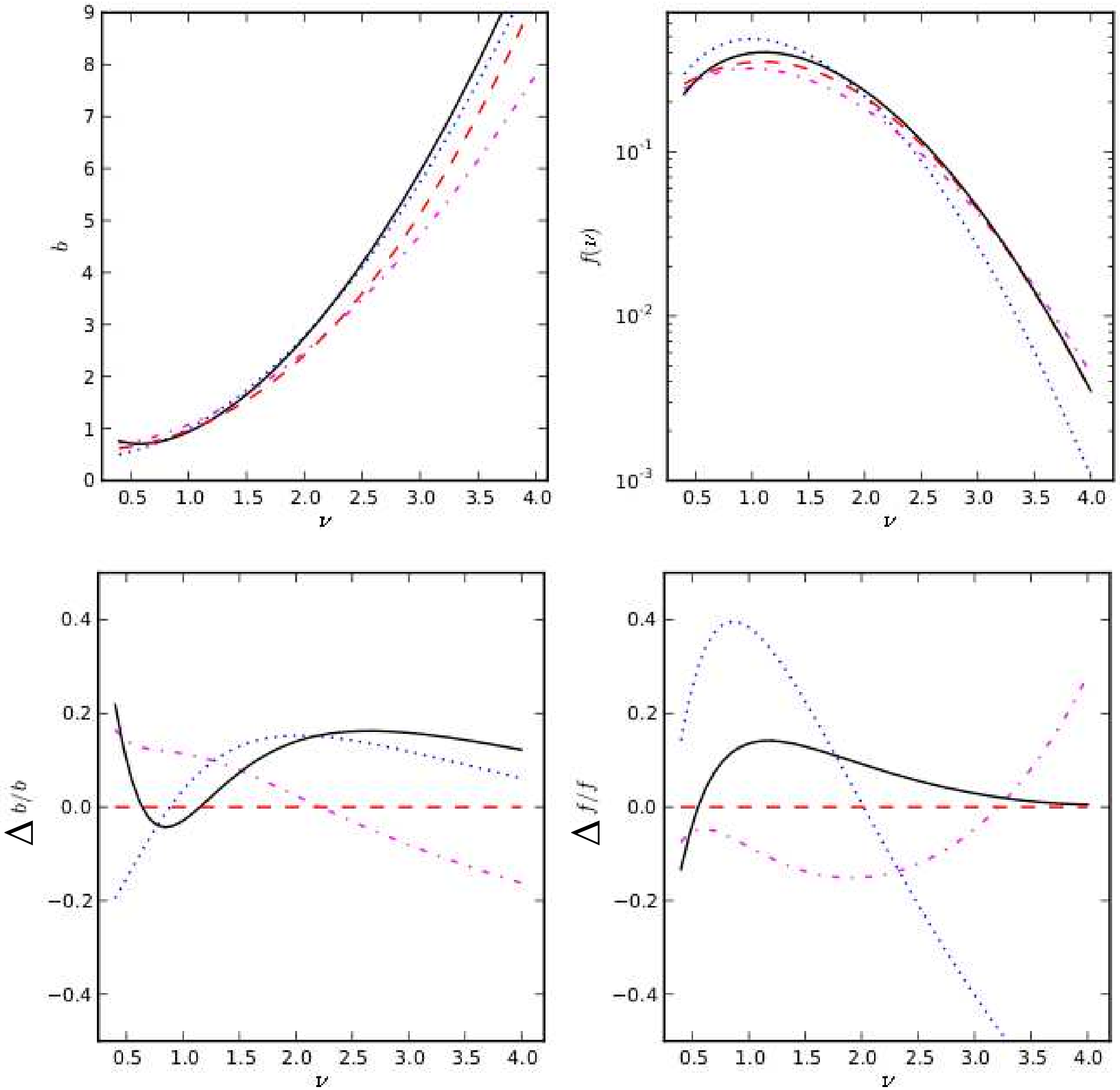
trajectories that starts from a value $\delta_0 \neq 0$ at S=0

$$b_h(\nu) = 1 + \frac{\nu^2 - 1}{\delta_c} \qquad (\nu=\delta_c/\sigma)$$

- With our path integral formalism we can compute the non-Markovian corrections due to the filter

$$b_h(\nu) = 1 + \frac{1}{\delta_c}\frac{1}{1-\kappa+(\kappa/\nu^2)}\left[(\nu^2-1) + \frac{\kappa}{2}\left(1 - e^{\nu^2/2}\Gamma(0,\nu^2/2)\right)\right]$$

- Combining it with the diffusing barrier model, $\kappa \rightarrow a\kappa$, $\delta_c \rightarrow a^{1/2}\delta_c$
- Fit to mass function and bias taking a and κ as free parameters.



Comparing with the N-body results of Tinker et al 2008,2010 we get a=0.815 and κ =0.21

Tinker et al

our result

ellipsoidal collapse (ST)

spherical collapse

With only 2 free parameters, matches N-body results to ≈15%

Conclusions

- Excursion set theory, after some technical and physical improvements, works quite well, and catches a large part of the complicated physics of halo formation and bias
- Better justification of the standard ST result for ellipsoidal collapse
- Consistent derivation of the effect of non-Gaussianities

thank you!

backup slides

Can we reproduce the markovian result?

- for a gaussian process

$$W(x_0; x_1, \ldots, x_n; t_n) = \int \mathcal{D}\lambda \; e^{i\lambda_i x_i - \frac{1}{2}\lambda_i \lambda_j \langle \xi_i \xi_j \rangle}$$

- take $\quad \dot{\xi} = \eta \,, \qquad \langle \eta(t)\eta(t') \rangle = \delta(t - t')$

- then

$$\begin{aligned} \langle \xi(t_i)\xi(t_j) \rangle &= \int_0^{t_i} dt \int_0^{t_j} dt' \langle \eta(t)\eta(t') \rangle \\ &= \mathrm{min}(t_i, t_j) = \epsilon \, \mathrm{min}(i, j) \end{aligned}$$

and

$$W = \frac{1}{(2\pi\epsilon)^{n/2}} \exp\{-\frac{1}{2\epsilon} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2\}$$

Wiener measure !

next we must compute Π

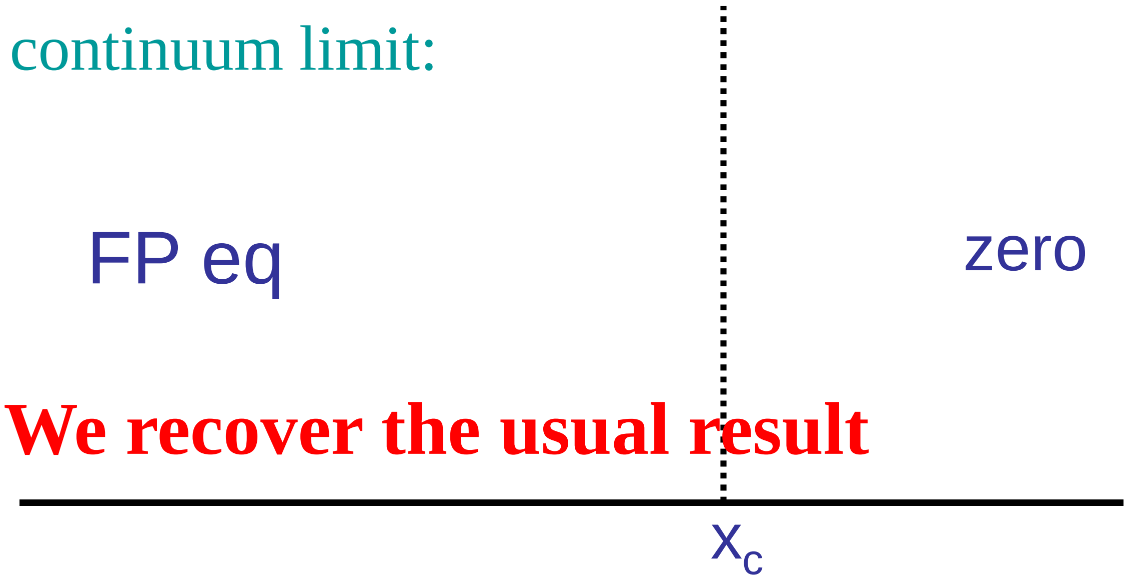
$$\Pi_\epsilon(x_0;x;t+\epsilon) = \int_{x-x_c}^{\infty} d(\Delta x)\,\Psi_\epsilon(\Delta x)\Pi_\epsilon(x_0;x-\Delta x;t)$$

$$\Psi_\epsilon(\Delta x) = (2\pi\epsilon)^{-1/2} e^{-(\Delta x)^2/(2\epsilon)}$$

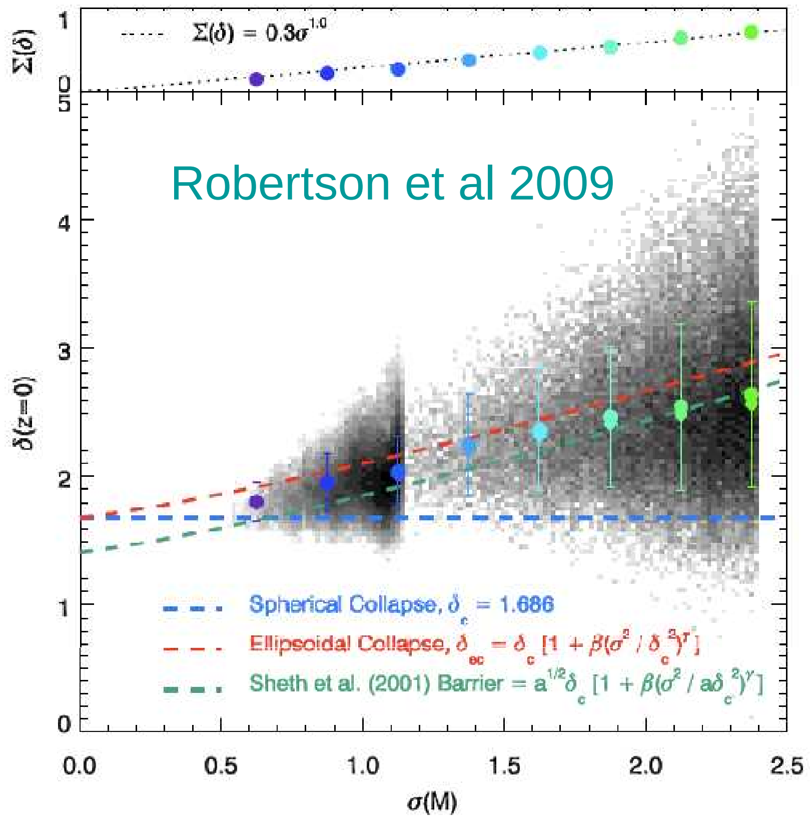
(generalized Chapman-Kolmogorov).

From the study of this integral equation we get:

In the continuum limit:

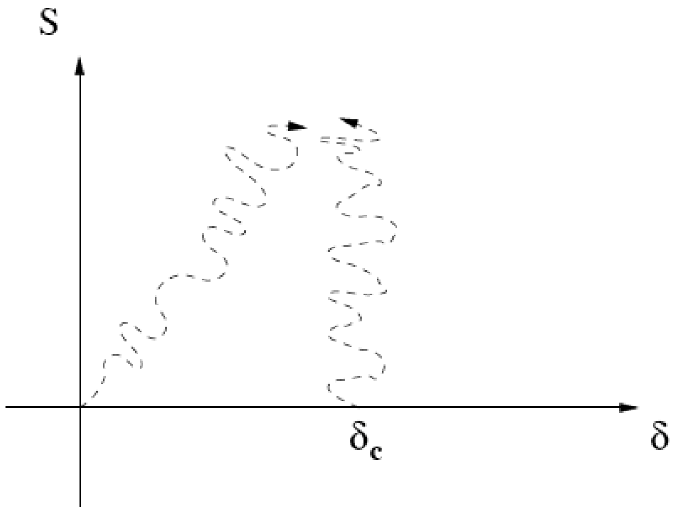


We recover the usual result



$$\langle (B - \langle B \rangle)^2 \rangle \simeq 0.3 \delta_c \sigma(R)$$

We deduce from this that the barrier diffuses, and its diffusion coefficient is $D_B = (0.3\delta_c)^2$



The first-passage problem with diffusing barrier is a classical problem (e.g. Redner 2001).

- Effective diffusion coefficient

$$D = 1 + D_B$$