System-Environment Quantum Discord for general dynamics

César A. Rodríguez-Rosario

Department of Chemistry and Chemical Biology at Harvard University

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Conclusion

System-Environments correlations, *classical* vs. *quantum* (as in *q discord*), each yield different OQS dynamics.

Traditional Open Quantum Systems

Assumes:

- Specific Environment, *un*correlated to System
- Specific System-Environment coupling

Find: System dynamical properties

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Our Approach

Assumes:

- Families of System-Environment *Correlations*
- *Any* System-Environment Coupling
  - non-Markov, strong or weak, non-equilibrium

Find: System dynamical properties

Outline:

System-Environment Quantum Discord for general dynamics

1. Not-Completely Positive Maps

Rodríguez-Rosario, Modi, Kuah, Shaji, Sudarshan ’07

Kuah, Modi, Rodríguez-Rosario, Sudarshan ’07

Rodríguez-Rosario, Sudarshan ’08

2. Assignment Maps and No-Broadcasting

Rodríguez-Rosario, Modi, Aspuru-Guzik ‘09

3. Decoherence Rate and Classicality

Rodríguez-Rosario, Kimura, Imai, Aspuru-Guzik ‘10

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Dynamical Maps for Density Matrices

[Sudarshan et al. 1961]

$$\mathfrak{B}\left(\rho\right) = \rho'$$

- Preserve Hermiticity

$$\mathfrak{B}\left(\rho\right) = \sum_{\alpha} \lambda_\alpha C_\alpha \, \rho \, C_\alpha^\dagger = \rho' \qquad \text{with } \lambda_\alpha \text{ a real number}$$

- Preserve Trace

$$\sum_{\alpha} \lambda_\alpha C_\alpha^\dagger C_\alpha = \mathbb{I}$$

- Preserve Positivity

$$\forall \text{ non-negative } R, \quad \mathfrak{B}\left(R\right) = R', \text{ where } R' \text{ is non-negative}$$

  - subclass: complete positivity $\quad \lambda_\alpha \geq 0$

(Standard) Open Quantum Systems

$$\begin{array}{ccc} \rho^{\mathcal{SE}} = \rho^{\mathcal{S}} \otimes \rho^{\mathcal{E}} & \longleftrightarrow & U\,\rho^{\mathcal{SE}}\,U^{\dagger} \\ \mathrm{Tr}_{\mathcal{E}} \downarrow & & \downarrow \\ \rho^{\mathcal{S}} & \longrightarrow & \mathfrak{B}\left(\rho^{\mathcal{S}}\right) = \mathrm{Tr}_{\mathcal{E}}\left[U\rho^{\mathcal{SE}}U^{\dagger}\right] = \rho^{\mathcal{S}\,\prime} \end{array}$$

Stinespring: *Uncorrelated* System-Environment lead to CP Map

$$\mathfrak{B}\left(\rho^{\mathcal{S}}\right) = \mathrm{Tr}_{\mathcal{E}}\left[U\rho^{\mathcal{S}} \otimes \rho^{\mathcal{E}}U^{\dagger}\right] = \sum_{\alpha} \lambda_{\alpha} C_{\alpha}\, \rho^{\mathcal{S}}\, C_{\alpha}^{\dagger}, \text{ where } \lambda_{\alpha} \geq 0$$

Correlated System-Environment?

- Example: $\rho^{\mathcal{SE}} = \frac{1}{4}(\mathbb{I}^{\mathcal{S}} \otimes \mathbb{I}^{\mathcal{E}} + a_j \sigma_j^{\mathcal{S}} \otimes \mathbb{I}^{\mathcal{E}} + c_{23} \sigma_2^{\mathcal{S}} \otimes \sigma_3^{\mathcal{E}}),$ evolved by $U = e^{-itH_{tot}}, H_{tot} = \omega \sum_j \sigma_j^{\mathcal{S}} \otimes \sigma_j^{\mathcal{E}}$

$$\begin{array}{ccc} \rho^{\mathcal{SE}} & \longleftrightarrow & U\,\rho^{\mathcal{SE}}\,U^{\dagger} \\ \mathrm{Tr}_{\mathcal{E}} \downarrow & & \downarrow \\ \rho^{\mathcal{S}} & \rightarrow & \mathfrak{B}\left(\rho^{\mathcal{S}}\right) = \rho^{\mathcal{S}\,\prime} \end{array}$$

$$\rho^{\mathcal{S}\,\prime} = \mathrm{Tr}_{\mathcal{E}}[U\rho^{\mathcal{SE}}U^{\dagger}] = \frac{1}{2}\left[\mathbb{I} + \cos^2(2\omega t)\, a_j \sigma_j + c_{23}\cos(2\omega t)\sin(2\omega t)\,\sigma_1\right]$$

Correlated System-Environment?

- **Example:** $\rho^{\mathcal{SE}} = \frac{1}{4}(\mathbb{I}^{\mathcal{S}} \otimes \mathbb{I}^{\mathcal{E}} + a_j \sigma_j^{\mathcal{S}} \otimes \mathbb{I}^{\mathcal{E}} + c_{23} \sigma_2^{\mathcal{S}} \otimes \sigma_3^{\mathcal{E}})$, evolved by $U = e^{-itH_{tot}}$, $H_{tot} = \omega \sum_j \sigma_j^{\mathcal{S}} \otimes \sigma_j^{\mathcal{E}}$

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Eigenvals of $\mathfrak{B}$:

$$\lambda_{1,2} = \frac{1}{2}\left[1 - \cos^2(2\omega t) \pm c_{23}\cos(2\omega t)\sin(2\omega t)\right],$$ Negative!!!

$$\lambda_{3,4} = \frac{1}{2}\left[1 + \cos^2(2\omega t) \pm \cos(2\omega t)\sqrt{4\cos^2(2\omega t) + c_{23}^2\sin^2(2\omega t)}\right]$$

Which States give CP?

Relationship to SE correlations?

Challenge:

- Trivial couplings give CP Maps independent of correlations

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Solution: Change the question!

- Which states give CP *independent* of coupling?

Classical vs. Quantum Correlations

- Separable vs. Entangled [Werner '89]:

$$\begin{array}{ll} \text{Classical:} & \rho^{\mathsf{XY}} = \sum_j p_j \eta_j^{\mathsf{X}} \otimes \tau_j^{\mathsf{Y}}, \quad p_j \geq 0 \\ \text{Quantum:} & \quad \text{the rest} \end{array}$$

Classical vs. Quantum Correlations

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Classical: $\rho^{\mathsf{XY}} = \sum_j p_j \eta_j^{\mathsf{X}} \otimes \tau_j^{\mathsf{Y}}, \quad p_j \geq 0$
Quantum: the rest

- **Quantum Discord** [Ollivier, Zurek (2001); Henderson, Vedral (2001)]:

Classical (zero discord): $\rho^{\mathsf{XY}} = \sum_j \Pi_j^{\mathsf{X}} \otimes \mathbb{I}^{\mathsf{Y}} \, \rho^{\mathsf{XY}} \, \Pi_j^{\mathsf{X}} \otimes \mathbb{I}^{\mathsf{Y}} = \sum_j p_j \Pi_j^{\mathsf{X}} \otimes \rho_j^{\mathsf{Y}}$
Quantum (non-zero discord): the rest

Π_j is a one-dimensional orthonormal projector: $\Pi_j = |j\rangle\langle j|$

Q: What kind of system-environment correlations give CP maps for any coupling?

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A: Zero Discord!

Theorem:

If: $$\rho^{\mathcal{S}\mathcal{E}} = \sum_j p_j \Pi_j^{\mathcal{S}} \otimes \rho_j^{\mathcal{E}},$$

Then: $$\mathfrak{B}\left(\rho^{\mathcal{S}}\right) = \sum_\alpha \lambda_\alpha C_\alpha \rho^{\mathcal{S}} C_\alpha^\dagger$$

$$\lambda_\alpha \geq 0$$

Proof: (1)

$$
\begin{array}{ccc}
\rho^{\mathcal{SE}} = \mathfrak{A}\left(\rho^{\mathcal{S}}\right) = \rho^{\mathcal{SE}} & \longleftrightarrow & \mathfrak{U}\left(\rho^{\mathcal{SE}}\right) = U\,\rho^{\mathcal{SE}}\,U^{\dagger} \\
\uparrow & & \downarrow \\
\rho^{\mathcal{S}} & \longrightarrow & \mathfrak{B}\left(\rho^{\mathcal{S}}\right) = \mathrm{Tr}_{\mathcal{E}}\left[\mathfrak{U}\left(\mathfrak{A}\left(\rho^{\mathcal{S}}\right)\right)\right] = \rho^{\mathcal{S}\,\prime}
\end{array}
$$

$$\mathfrak{B} = \mathrm{Tr}_{\mathcal{E}} \circ \mathfrak{U} \circ \mathfrak{A}$$

To Blame

Tr_{E} and $\mathcal{U}$ are linear and completely positive maps

Need: Assignment Map $\mathfrak{A}$ for Zero Discord

Proof: (2)

Need: Assignment Map $\mathfrak{A}$ for Zero Discord

$$\mathfrak{A}\left(\rho^{\mathcal{S}}\right) = \sum_j \mathrm{Tr}\ \left[\Pi_j \rho^{\mathcal{S}}\right] \Pi_j \otimes \rho_j^{\mathcal{E}}$$

This Assignment is CP, therefore,
the full dynamical map $\mathfrak{B}$ is also CP

Zero Discord leads to CP for any coupling!

Rodríguez-Rosario, Modi, Kuah, Shaji, Sudarshan ’07

- What about the converse? Shabani, Lidar ’09???

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Assignment Maps

$$\rho^{\mathcal{S}\mathcal{E}} = \mathfrak{U}\left(\rho^{\mathcal{S}\mathcal{E}}\right) = U\,\rho^{\mathcal{S}\mathcal{E}}\,U^\dagger$$

$$\mathfrak{A}\left(\rho^{\mathcal{S}}\right) = \rho^{\mathcal{S}\mathcal{E}} \quad \uparrow \qquad \longleftrightarrow \qquad \downarrow$$

$$\rho^{\mathcal{S}} \quad \rightarrow \quad \mathfrak{B}\left(\rho^{\mathcal{S}}\right) = \mathrm{Tr}_{\mathcal{E}}\left[\mathfrak{U}\left(\mathfrak{A}\left(\rho^{\mathcal{S}}\right)\right)\right] = \rho^{\mathcal{S}\,\prime}$$

$$\mathfrak{B} = \mathrm{Tr}_{\mathcal{E}} \circ \mathfrak{U} \circ \mathfrak{A}$$

To Blame

Tr_{E} and $\mathcal{U}$ are linear and completely positive maps

Debate: Interpretation SE Correlations

- (Pechukas '94) Assignment maps, SE correlations problematic
- (Alicki '95) Properties of Assignments

$$
\begin{array}{ll}
i. & \text{linear: } \mathfrak{A}(a\,\rho_1^{\mathcal{S}} + b\,\rho_2^{\mathcal{S}}) = a\,\mathfrak{A}(\rho_1^{\mathcal{S}}(+b\,\mathfrak{A}(\rho_2^{\mathcal{S}}), \\
ii. & \text{consistent: } \mathrm{Tr}_{\mathcal{E}}\left[\mathfrak{A}(\rho^{\mathcal{S}})\right] = \rho^{\mathcal{S}}, \\
iii. & \text{positive: } \mathfrak{A}(\rho^{\mathcal{S}}) \geq 0 \text{ for all } \rho^{\mathcal{S}}.
\end{array}
$$

- (Pechukas '95) *If* correlations, *Then* non-linear QM *?????*

Which properties should be saved?

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- (Pechukas '95) *If* correlations, *Then* non-linear QM *?????*

Which properties should be saved?

- (Rodríguez-Rosario, Modi, Sudarshan '07) zero-discord lead to CP
- (Rodríguez-Rosario, Modi, Aspuru-Guzik '09) big picture, saves linearity!

Generalized Linear Assignment Map

Define a set of linearly independent projectors $\mathbf{P}_j$ that span $\mathcal{S}$ such that for any $\rho^{\mathcal{S}} = \sum_j q_j \mathbf{P}_j$ where q_j can be negative

Example (qubit states): $\mathbf{P}_{x+} = \frac{1}{2}\left(\mathbb{I} + \sigma_x\right)$, $\mathbf{P}_{y+} = \frac{1}{2}\left(\mathbb{I} + \sigma_y\right)$, $\mathbf{P}_{z+} = \frac{1}{2}\left(\mathbb{I} + \sigma_z\right)$, $\mathbf{P}_{x-} = \frac{1}{2}\left(\mathbb{I} - \sigma_x\right)$

$$\mathbf{P}_i \rightarrow \mathfrak{A}\left[\mathbf{P}_i\right] = \mathbf{P}_i \otimes \tau_i.$$

$\{\tau_i\}$ are Hermitian and Trace$= 1$

Special case (from before!): $\Pi_i \rightarrow \mathfrak{A}\left[\Pi_i\right] = \Pi_i^{\mathcal{S}} \otimes \rho_i^{\mathcal{E}}$

Properties of Assignment Maps and Discord

Correlations	Linear	Consistent	Positive
None	***yes***	***yes***	***yes***
Classical	***yes***	***no***	***yes***
Quantum	***yes***	***yes***	***no***

- Recall: Each term in a classically correlated state is orthogonal to each other

No-cloning, No-Broadcasting and Assignments

$$\mathfrak{A}_B\left[\sum_i p_i \Pi_i\right] = \sum_i p_i \Pi_i \otimes \Pi_i$$

Zero Discord (symmetrically) can be broadcast

$$\mathfrak{A}_{NB}\left[\sum_i q_i \mathbf{P}_i\right] = \sum_i q_i \mathbf{P}_i \otimes \mathbf{P}_i$$

rest: no broadcast

- Linearity ⟹ No-Cloning Thm
- Consistency ⟹ No-Broadcasting Thm

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Assumes:

- Specific Environment, *un*correlated to System
- Specific System-Environment coupling (Markov approx?, weak coupling?)

Finds: System decoherence properties

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Finds: System decoherence properties

Definitions

- Finite system-environment:

$$H_{tot} := H^{\mathcal{S}} \otimes \mathbf{I}^{\mathcal{E}} + \mathbf{I}^{\mathcal{S}} \otimes H^{\mathcal{E}} + H_{int}$$

- Full dynamics (no Markov approximation):

$$\frac{d}{dt}\rho_t^{\mathcal{S}\mathcal{E}}\,|_{t=\tau} = -i\left[H_{tot},\, \rho_\tau^{\mathcal{S}\mathcal{E}}\right]$$

- purity of the system:

$$\mathbf{P} := \mathrm{tr}\left\{\rho^2\right\}$$

(Exact) Purity Rate

$$\left[\frac{d}{dt}\mathbf{P}_t^{\mathcal{S}}\right]_{t=\tau} = \mathrm{tr}_{\mathcal{S}}\Big\{2\rho_\tau^{\mathcal{S}}\Big(-i\,\mathrm{tr}_{\mathcal{E}}\left[H_{tot},\rho_\tau^{\mathcal{SE}}\right]\Big)\Big\} = -2i\,\mathrm{tr}_{\mathcal{SE}}\Big\{\rho_\tau^{\mathcal{S}}\otimes\mathbb{I}^{\mathcal{E}}\left[H_{int},\rho_\tau^{\mathcal{SE}}\right]\Big\}$$

Note that the dependence on $H^{\mathcal{S}}$ and $H^{\mathcal{E}}$ vanishes. $H^{\mathcal{S}}$ vanishes since $\mathrm{tr}_{\mathcal{SE}}\left\{\rho^{\mathcal{SE}}\left[\rho^{\mathcal{S}}\otimes\mathbb{I}^{\mathcal{E}}, H^{\mathcal{S}}\otimes\mathbb{I}^{\mathcal{E}}\right]\right\} = \mathrm{tr}_{\mathcal{S}}\left\{[\rho^{\mathcal{S}},H^{\mathcal{S}}]\mathrm{tr}_{\mathcal{E}}\rho^{\mathcal{SE}}\right\}$. Using the cyclic property again, $\mathrm{tr}_{\mathcal{S}}\left\{[\rho^{\mathcal{S}},H^{\mathcal{S}}]\mathrm{tr}_{\mathcal{E}}\rho^{\mathcal{SE}}\right\} = \mathrm{tr}_{\mathcal{S}}\left\{H^{\mathcal{S}}[\rho^{\mathcal{S}},\rho^{\mathcal{S}}]\right\} = 0$. The term $H^{\mathcal{E}}$ vanishes in a similar manner.

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$$\left[\frac{d}{dt}\mathbf{P}_t^{\mathcal{S}}\right]_{t=\tau} = 2i\mathrm{tr}_{\mathcal{SE}}\left\{H_{int}\left[\rho_\tau^{\mathcal{S}}\otimes\mathbb{I}^{\mathcal{E}},\rho_\tau^{\mathcal{SE}}\right]\right\}$$

$$\mathfrak{C}\left(\rho^{\mathcal{SE}}\right) = \left[\rho^{\mathcal{S}} \otimes \mathbf{I}^{\mathcal{E}}, \rho^{\mathcal{SE}}\right]$$

"Equilibrium"

Sufficient and necessary condition for
purity rate to be zero under *any* Hamiltonian

Theorem:

$$\forall\ H_{tot}, \quad \left[\frac{d}{dt}\mathbf{P}_t^{\mathcal{S}}\right]_{t=\tau} = 0 \quad \Leftrightarrow \quad \mathfrak{C}\left(\rho^{\mathcal{SE}}\right) = 0$$

Lazy States: $\mathfrak{C}\left(\rho^{\mathcal{SE}}\right) = \left[\rho^{\mathcal{S}} \otimes \mathbf{I}^{\mathcal{E}}, \rho^{\mathcal{SE}}\right] = 0$

- *Not* necessarily eigenstates of Hamiltonian
- Dynamical system properties from *structure* of the total state
- $\frac{d}{dt}\mathbf{P}|_{t=\tau} = 0$ analogous $\frac{d}{dt}|\psi\rangle|_{t=\tau} = 0$

Lazy States as Generalized "Classical" Correlations

(for the "right" basis in Discord)

Quantum Discord

- classically correlated states are invariant under a set of measurements on one of it subspaces

$$\sum_j \Pi_j^{\mathcal{S}} \otimes \mathbf{I}^{\mathcal{E}} \rho^{\mathcal{SE}} \Pi_j^{\mathcal{S}} \otimes \mathbf{I}^{\mathcal{E}} = \rho^{\mathcal{SE}}$$

$$\Pi_j = |j\rangle\langle j|$$

Lazy states a "generalization": $\left[\rho^{\mathcal{SE}}, \rho^{\mathcal{S}} \otimes \mathbf{I}^{\mathcal{E}}\right] = 0$

$$\rho^{\mathcal{S}} = \sum_j p_j \Pi_j^{\mathcal{S}}$$

Orthonormal projectors,
don't have to be one-dimensional

Lazy states include:

- uncorrelated states
- classically correlated states
- maximally entangled states
- others

Purity Rate as a Discord Witness

- If purity rate is non-zero on S, there are quantum correlations to E, from: $\forall\, H_{tot},\ \left[\frac{d}{dt}\mathbf{P}_t^{\mathcal{S}}\right]_{t=\tau} = 0 \quad \Leftrightarrow \quad \mathfrak{C}\left(\rho^{\mathcal{S}\mathcal{E}}\right) = 0$

If: $\dot{\mathbf{P}}^{\mathcal{X}} \neq 0$, Then: Quantum Correlations between $\mathcal{X}$ and $\mathcal{Y}$

$$\left[\rho^{\mathcal{X}} \otimes \mathbb{I}^{\mathcal{Y}}, \rho^{\mathcal{X}\mathcal{Y}}\right] \neq 0$$

Yes, you can detect bipartite correlations from local ***dynamical*** info

Lazy States: $\mathfrak{C}\left(\rho^{\mathcal{SE}}\right) = \left[\rho^{\mathcal{S}} \otimes \mathbf{I}^{\mathcal{E}}, \rho^{\mathcal{SE}}\right] = 0$

<u>Almost all quantum states have non-classical correlations</u>

[Ferraro, *Leandro* Aolita, Cavalcanti, Cucchietti, Acín (2010)]

Lazy states are the same as C_0

- Sparse in the space of density matrices, in both volume and topology
- Measure zero in the whole Hilbert space
- Nowhere dense

Lazy States are hard to find

Decoherence is common (duh!)

- Rarity of lazy states imply:

$$\rho^{\mathcal{SE}}_{t=t_0} \approx \rho^{\mathcal{S}}_{t=t_0} \otimes \rho^{\mathcal{E}} \text{ is not reasonable!}$$

- Commonality of non-lazy states imply:

$$\text{For the most part: } \dot{\mathbf{P}}^{\mathcal{S}} \neq 0 \quad \forall\, H_{tot}$$

except for...

Measurements and Pointer States

Quantum state $\mathcal{Q}$ measured by apparatus $\mathcal{M}$ with outcomes $\{\ |\mu_i\rangle\ \}$

$$\rho^{\mathcal{M}\mathcal{Q}} = \sum_i p_i |\mu_i\rangle\langle\mu_i|^{\mathcal{M}} \otimes \rho_i^{\mathcal{Q}}$$

$\rho^{\mathcal{M}\mathcal{Q}}$ is a Lazy State

- Purity Stability for Measurement Apparatus?

Bound on Decoherence Rate

$$\left[\frac{d}{dt}\mathbf{P}_t^{\mathcal{S}}\right]_{t=\tau} = 2i\mathrm{tr}_{\mathcal{SE}}\left\{H_{int}\left[\rho_\tau^{\mathcal{S}}\otimes\mathbb{I}^{\mathcal{E}},\rho_\tau^{\mathcal{SE}}\right]\right\}$$

$$|\mathrm{tr}\left[A\sigma\right]| \leq \| A\sigma\|_1 \leq \|A\|\|\sigma\|_1 \qquad \|A\|_1 := \mathrm{Tr}\sqrt{A^\dagger A}$$

$$\left|\frac{d}{dt}\mathbf{P}_t^{\mathcal{S}}\right|_{t=\tau} \leq 2\left\|H_{int}\right\| \left\|\mathfrak{C}\left(\rho^{\mathcal{SE}}\right)\right\|_1$$

$\|\mathfrak{C}\left(\rho^{\mathcal{SE}}\right)\|_1$ System eigenbasis vs. system-environment eigenbasis

Intuition for: $\|\mathfrak{E}\left(\rho^{\mathcal{SE}}\right)\|_1$

On the spirit of the Church of the Larger Hilbert space, purify the total state!

$$\rho^{\mathcal{SE}} = |\Psi\rangle\langle\Psi|^{\mathcal{SE}}, \qquad |\Psi\rangle := \sum_j \sqrt{p_j}|\Psi_j\rangle^{\mathcal{SE}} = \sum_j \sqrt{p_j}\,|\mathcal{S}_j\rangle \otimes |\mathcal{E}_j\rangle$$

bound by the total system-environment correlations

$$\left\| \left[\rho^{\mathcal{SE}},\, \rho^{\mathcal{S}} \otimes \mathbf{I}^{\mathcal{E}}\right] \right\|_1 \leq 4\sqrt{\mathfrak{E}\left(|\Psi\rangle^{\mathcal{SE}}\right)}$$

Entropy of entanglement: $\mathfrak{E}\left(|\Psi\rangle\right)$

- Rate of decoherence bound by correlations to environment

Contributions

System-Environment Quantum Discord for general dynamics

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If: $\rho^{\mathcal{SE}} = \sum_j p_j \Pi_j^{\mathcal{S}} \otimes \rho_j^{\mathcal{E}}$,

Then: $\mathfrak{B}\left(\rho^{\mathcal{S}}\right) = \sum_\alpha \lambda_\alpha C_\alpha \rho^{\mathcal{S}} C_\alpha^\dagger$ $\quad \lambda_\alpha \geq 0$

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$$\mathfrak{A}_{NB}\left[\sum_i q_i \mathbf{P}_i\right] = \sum_i q_i \mathbf{P}_i \otimes \mathbf{P}_i$$

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$$\forall\, H_{tot}, \quad \left[\frac{d}{dt}\mathbf{P}_t^{\mathcal{S}}\right]_{t=\tau} = 0 \quad \Leftrightarrow \quad \mathfrak{C}\left(\rho^{\mathcal{SE}}\right) = 0$$

$$\left|\frac{d}{dt}\mathbf{P}_t^{\mathcal{S}}\right|_{t=\tau} \leq 2\left\|H_{int}\right\| \left\|\mathfrak{C}\left(\rho^{\mathcal{SE}}\right)\right\|_1$$

Entropy fluctuations

Church of the larger Hilbert space: $\rho^{\mathcal{SE}} = |\Psi\rangle\langle\Psi|^{\mathcal{SE}}, \qquad |\Psi\rangle = \sum_{j=1}^{d} \sqrt{p_j}|\Psi_j\rangle$

$$\dot{\mathbb{S}} = -i\frac{k_B}{\hbar}\sum_{m,n}^{d}\sqrt{p_m\,p_n}\ln\frac{p_m}{p_n}\ \langle\Psi_m|H_{int}|\Psi_n\rangle$$

Theorem: Entropy fluctuations cannot be too large

Magnitude: $|\dot{\mathbb{S}}| \leq \frac{k_B}{\hbar}\|H_{int}\|\left|\sum_{m\neq n}^{d}\sqrt{p_m\,p_n}\ln\frac{p_m}{p_n}\right|$

Bounded: $\left|\sum_{m\neq n}^{d}\sqrt{p_m\,p_n}\ln\frac{p_m}{p_n}\right| \leq C(d-1)^2, \qquad C \approx 0.735\ldots$

Bound: $|\dot{\mathbb{S}}| \leq \frac{k_B C}{\hbar}(d-1)^2\|H_{int}\|$