Boundary operators in the $O(n)$ and ADE matrix models

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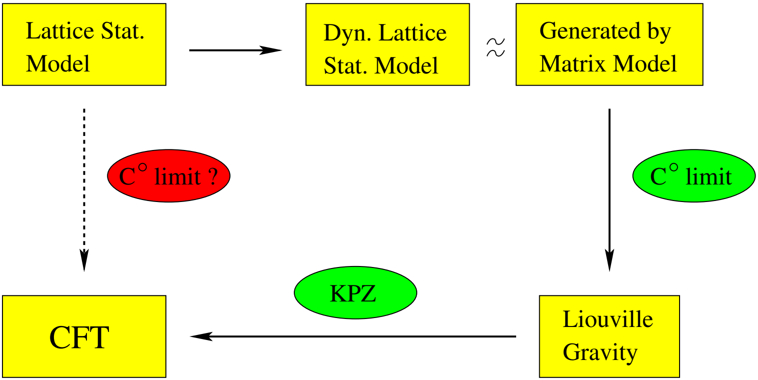
Strategy



Lattice Stat. Model
Dyn. Lattice Stat. Model
≈
Generated by Matrix Model
C° limit ?
C° limit
KPZ
CFT
Liouville Gravity

Outline

Outline

The $O(n)$ lattice model

Definition of the $O(n)$ model

- We consider a lattice Γ, to each point $r \in \Gamma$ we associate an $O(n)$ spin $S_a(r)$ with $a = 1 \cdots n$ and normalized such that $\operatorname{tr} S_a(r) S_b(r') = \delta_{ab}\delta_{rr'}$.
- The 'geometric' partition function reads

$$\mathcal{Z}_\Gamma(T) = \operatorname{tr} \prod_{<rr'>} \left(1 + \frac{1}{T} \sum_{a=1}^{n} S_a(r) S_a(r') \right).$$

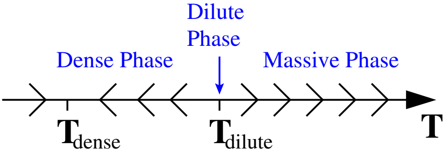
The $O(n)$ lattice model

Reformulation as a loop gas model

- The $O(n)$ model can be reformulated as a sum over configurations of self-avoiding, mutually avoiding loops of weight n,

$$\mathcal{Z}_{\Gamma}(T) = \sum_{\text{loops}} T^{-\text{length}} n^{\#\text{ loops}}.$$

- This formulation makes sense for arbitrary n. It exhibits a critical behavior when $|\, n \,| \leq 2$.
- The phase diagram has two fixed points:



The $O(n)$ lattice model

Boundary conditions

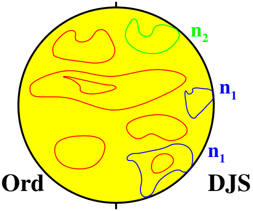
Ordinary boundary conditions:

Free spins, bulk behavior.

Loops avoid the boundary.

JS boundary conditions:

Loops with weight k on the boundary.

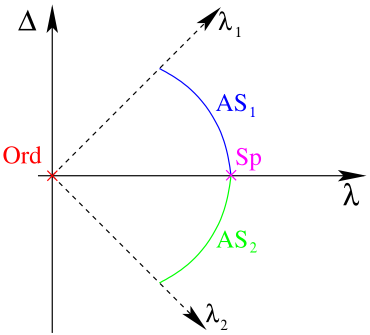


Dilute JS boundary conditions:

Split the spin components in two orthogonal sets $\vec{S} = \vec{S}_1 + \vec{S}_2$. Leads to two kinds of loops, with weights n_1 and n_2 ($n = n_1 + n_2$), and coupling constants λ_1 and λ_2.

Flat lattice results from DJS

DJS BC in the dilute phase



- **Ord**: Loops avoid the boundary.
- **AS**$_1$: Loops of weight n_1 critically enhanced.
- **AS**$_2$: Loops of weight n_2 critically enhanced.
- **Sp**: Both loops touch the boundary.

Perturbation : Boundary thermal operator $B_{1,3}$

Boundary anisotropic operator $B_{3,3}$

Study of the DJS BC using the matrix model

In collaboration with K. Hosomichi and I. Kostov

1. Ord/JS bcc op. in the dense phase [JHEP 0901 (2009) 009]
   - Conformal weight of Ord/JS operators.
   - Relation JS / Alt b.c. in the RSOS matrix model.

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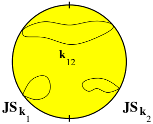

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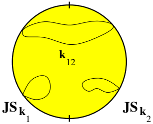


3. Ord/DJS bcc op. in the dilute phase [arXiv:0910.1581]
   - Phase diagram of DJS boundary conditions (Δ, λ).
   - Conformal weight of Ord/Sp and Ord/AS bcc operators.
   - Conformal weight of operators generating the flows.
   - Bulk thermal flow of conformal b.c. $\delta_{r,s} \to \delta_{s-1,r}$

Outline

The $O(n)$ model on a dynamical lattice

Introduction of the matrix model I

- The partition function on the dynamical lattice is obtained as a sum over random lattices

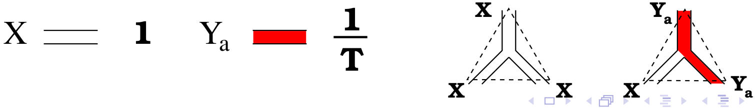
$$\mathcal{Z}_{\text{dyn}}(\kappa, T) = \sum_{\Gamma} \kappa^{-A(\Gamma)} \mathcal{Z}_{\Gamma}(T)$$

- It can be generated as an expansion of the $O(n)$ matrix model

$$\mathcal{Z} = \int dX \prod_{a=1}^{n} dY_a e^{\beta \text{tr} \left(-\frac{1}{2}X^2 + \frac{1}{3}X^3 - \frac{T}{2}\sum_{a=1}^{n} Y_a^2 + \sum_{a=1}^{n} XY_a^2 \right)}.$$

where X and Y_a are $N \times N$ Hermitian matrices.

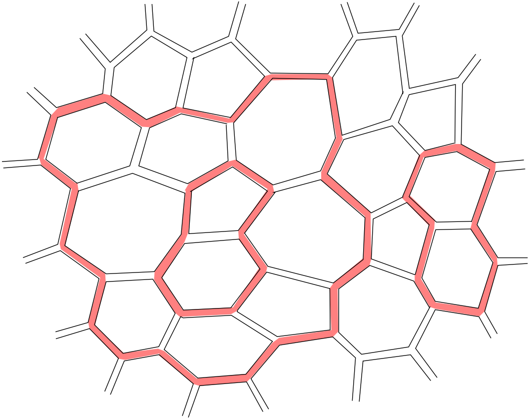
- Propagators and vertices:



The $O(n)$ model on a dynamical lattice

Introduction of the matrix model II

- A loop configuration:



- Planar limit (disc corr): $(\beta, N) \to \infty$, $\beta/N = \kappa^2$.

The $O(n)$ model on a dynamical lattice

The disc partition function

- The partition function on the disc is

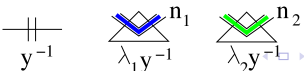
$$\mathcal{Z}_{\text{dyn}}(\kappa, x, T) = \sum_{\Gamma:\ \text{disc}} \frac{1}{L(\Gamma)} \kappa^{-A(\Gamma)} x^{-L(\Gamma)} Z_\Gamma(T),$$

- It is generated by correlators of the matrix model

$$\text{Ord}: \quad W(x) = \frac{1}{\beta} \left\langle \operatorname{tr} \frac{1}{x - X} \right\rangle$$

$$\text{JS}: \quad \tilde{W}(y) = \frac{1}{\beta} \left\langle \operatorname{tr} \frac{1}{y - Y_k^2} \right\rangle, \qquad Y_k^2 = \sum_{a=1}^{k} Y_a^2$$

$$\text{DJS}: \quad H(y|\lambda_1, \lambda_2) = \frac{1}{\beta} \left\langle \operatorname{tr} \frac{1}{y - X - \lambda_1 Y_{n_1}^2 - \lambda_2 Y_{n_2}^2} \right\rangle$$



Construction of the correlators

Disc with two boundaries

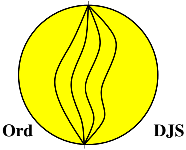
We consider a disc with mixed Ord-DJS boundary conditions,

$$D_L^{(i)}(x,y) = \frac{1}{\beta}\left\langle \mathrm{tr}\left(\frac{1}{x-X}\mathbb{S}_L^{(i)}\frac{1}{y-X-\lambda_1 Y_{n_1}^2-\lambda_2 Y_{n_2}^2}\mathbb{S}_L^{(i)\dagger}\right)\right\rangle$$

between both boundaries L open lines are inserted by the operators

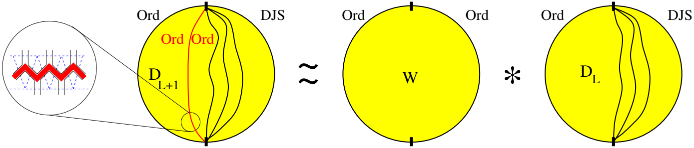
$$\mathbb{S}_L^{(1)} = \sum_{\{a_1,\cdots,a_L\}\subset\{1,\cdots,n_1\}} Y_{a_1}\cdots Y_{a_L}$$

$$\mathbb{S}_L^{(2)} = \sum_{\{a_1,\cdots,a_L\}\subset\{n_1+1,\cdots,n\}} Y_{a_1}\cdots Y_{a_L}$$



Derivation of the loop equations I

- The loop equations are obtained using the invariance of the matrix measure.
- They describe the removing of loops.
- Correlators satisfy a recursion relation $D_{L+1} = W * D_L$



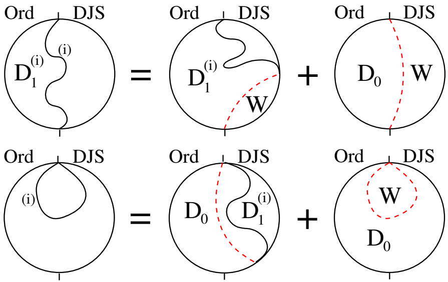
- This 'cutting' of correlators involve the star product

$$(F * G)(x) = \oint \frac{dx'}{2i\pi} \frac{F(x') - F(x)}{x - x'} G(-x')$$

where the countour circles the branch cut of the Ord boundary cosmological constant.

Derivation of the loop equations II

All the physics is contained in the 0th order equation which couples D_0 and $D_1^{(i)}$:



$$\mathcal{A}^{(i)}(x)\mathcal{B}^{(i)}(-x) = \mathcal{C}^{(i)}(x)$$

with

$$\mathcal{A}^{(i)}(x) = \lambda_i D_0(x) - 1$$

$$\mathcal{B}^{(i)}(x) \propto D_1^{(i)}(x) + \text{n.u.}$$

$$\mathcal{C}^{(i)}(x) \propto W(x) + n_i W(-x) + \text{poly}$$

$\Longrightarrow$ This equation will be studied in the continuum limit.

Take the continuum limit

Reminder

- Taking the continuum limit leads to:

Statistical model at the critical point on a flat lattice.	$\longrightarrow$	CFT, operators $V_{r,s}$, $B_{r,s}$, conformal dimension $\delta_{r,s}$
Statistical model at the critical point on a dynamical lattice.	$\longrightarrow$	CFT$\otimes$Liouville$\otimes$ghost, dressed operators $e^{2\alpha_{r,s}\phi}V_{r,s}$, gravitational dimension $\Delta_{r,s}$

- The KPZ formulas relate the central charges and the dimensions $\delta_{r,s}$ and $\Delta_{r,s}$.

Take the continuum limit

Continuum limit in matrix models I

- We adjust the parameters to their critical values where the mean length and area of loops diverge.

$$\epsilon^2\mu = \kappa - \kappa^*, \quad \epsilon^{1/g}\xi = x - x^*, \cdots$$

- The phase diagram is obtained from criticality conditions.
- Critical correlators correspond to boundary 2pt functions of

$$\begin{aligned}
d_L^{(i)}(\xi,\zeta) &\to (Ord|S_L^{(i)}|AS_{(1)}), && \Delta > 0 \\
d_L^{(i)}(\xi,\zeta) &\to (Ord|S_L^{(i)}|AS_{(2)}), && \Delta < 0 \\
d_L^{(i)}(\xi,\zeta) &\to (Ord|S_L^{(i)}|Sp), && \Delta = 0,\ \lambda = \lambda^*
\end{aligned}$$

- The dimension of the perturbations gives the operators that generate the flows (e.g. $\epsilon^{\theta/g} t_B = \lambda - \lambda_c$).

Take the continuum limit

Continuum limit in matrix models II

Loop equations are shift equations on boundary parameters $\xi(\tau)$, $\mu_B(\sigma)$, e.g. on the $\mathrm{AS}_{(1)}$ branch (for any ξ, μ_B, t, t_B):

$$d_0^{(2)}(\tau,\sigma)d_1^{(1)}(\tau \pm i\pi, \sigma) + w(\tau) + n_1 w(\tau \pm i\pi) = \mu_B - t_B \xi$$

$$w(\tau) = \cosh{(1+\theta)\tau} + t\cosh{(1-\theta)\tau}, \quad n = 2\cos\pi\theta$$

- Perturbed FZZ equation for the Liouville boundary 2pt function.
- Depends on bulk t and boundary t_B temperature.
  $\Rightarrow$ The evolution of b.c. under thermal flows can be tracked down.

Outline

Results

Results in the Liouville context

- Dense phase, Ord/JS bcc operators:
    I. The solution of the loop equation is the Liouville boundary 2pt function.

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Results in the Liouville context

- Dense phase, Ord/JS bcc operators:
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- Dense phase, JS/JS bcc operators :
  I. The loop equations can be mapped on shift relations for the Liouville boundary 3pt functions.
- Dilute phase, Ord/DJS bcc operators :
  I. Loop equation for QFT coupled to 2D gravity (t, t_B).
  II. Solution on the critical curves AS (Liouville boundary 2pt functions).
  III. Solution at $\mu = \mu_B = 0$, perturbed Liouville gravity

$$\delta\mathcal{S} = t\int_{\text{bulk}} \mathcal{O}_{1,3} + \xi\int_{\text{Ord.}} \mathcal{O}^B_{1,1} + t_B\int_{\text{DJS}} \mathcal{O}^B_{1,3}$$

Perspectives and open problems

Work in progress...

1. Open problems:
   - Calculation of the DJS disc partition function $H(y)$.
     - $\Rightarrow$ Explicit expression for the AS curve.
     - $\Rightarrow$ Dimension of the DJS boundary.

Perspectives and open problems

Work in progress...

1. Open problems:
   - Calculation of the DJS disc partition function $H(y)$.
     - $\Rightarrow$ Explicit expression for the AS curve.
     - $\Rightarrow$ Dimension of the DJS boundary.
2. Perspectives:
   - Bulk anisotropy

$$\mathcal{S}[X, Y_a] = \mathrm{tr}\left(-\frac{1}{2}X^2 + \frac{1}{3}X^3 - \frac{T_1}{2}Y_{n_1}^2 - \frac{T_2}{2}Y_{n_2}^2 + XY_n^2\right)$$

   - Other models with boundaries.
   - SLE, Liouville gravity and Matrix Models...

Outline

ADE models

Two different realisations of unitary minimal models:

- $O(n)$ matrix model with $n = 2\cos\pi/h,\ h \in \mathbb{Z}$.
- ADE matrix model.

Both models can be mapped on a loop gas $\Longrightarrow$ Similar loop equations

Interest ???

- Unitary minimal models $(h, h \pm 1)$
- Different interpretation (ex: n_i parametrisation)
- Operators mixing,...

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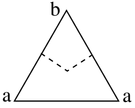
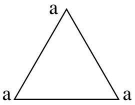
!!! CONCENTRATE HERE ON RSOS !!!

RSOS model

- The RSOS model assign an integer height $a \in [1, h-1]$ at each sites of the lattice, requiring $|a-b| = 1$ for two adjacent heights a and b.
- The lattice is made of two types of triangles,



$T^{-1}(S_b/S_a)^{1/6}$



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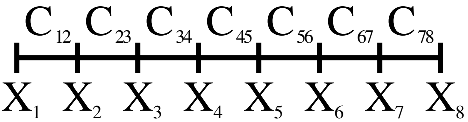
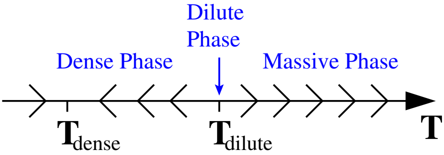
$$S_a = \sqrt{\frac{2}{h}} \sin\left(\frac{\pi a}{h}\right)$$

- Heights can be seen as taking values on a Dynkin graph of the A series,



Formulation as a loop gas

- The model can be reformulated as a loop gas, loops being domain wall surrounding domains of constant height, $n = 2\cosh \pi/h$.
- It exhibits the same phase behavior than the $O(n)$ model with a dense and a dilute phase,



- The continuum limit is described by a unitary minimal model, respectively $(h, h-1)$ and $(h+1, h)$ in the dense and dilute phase.

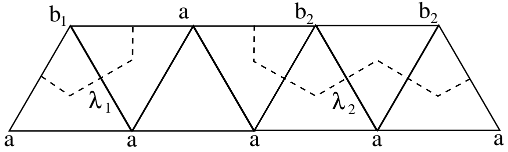
Boundary conditions

In the dense phase: [JEB, K. Hosomichi '08]

- **Fix**$_a$: Fixed heights on the boundary.
- **Alt**$_{<ab>}$: Alternating heights between two adjacent values $ababab\cdots$ [M. Bauer, H. Saleur '89]

In the dilute phase:

- **Alt**$_a$: Give a fugacity λ_b to nodes $b \sim a$ on the second row,



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- Map them to the $O(n)$ model loop equations.
- Re-interpret the results in this context.

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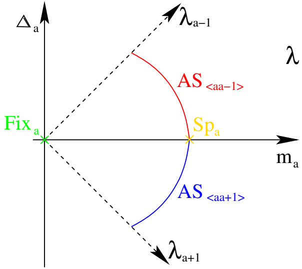


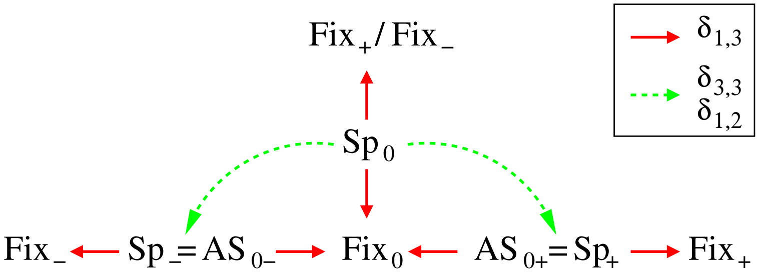
Critical Alt$_a$ boundary

Dilute Bound. State	Dense Bound. State
Fix$_a$ $\|a, 1 >$	Fix$_a$ $\|1, a >$
AS$_{<aa+1>}$ $\|1, a+1 >$	Alt$_{<aa+1>}$ $\|a, 1 >$
AS$_{<aa-1>}$ $\|1, a >$	Alt$_{<aa-1>}$ $\|a-1, 1 >$
Sp$_a$ $\|a, 2 >$	Fix$_a$ $\|1, a >$

Ising Tricritical

[P. Dorey, C. Rim, R. Tateo '09]

Boundary flows for the $(4,5)$ minimal model,



One matrix model

(advertisement)

- 1MM realises $(2, 2h+1)$ minimal models coupled to gravity.
- Recently, the boundary operators have been identified. [G. Ishiki, C. Rim '10]

To do:

- Perturbation ? Boundary phase diagram ? Flows ?
- 2MM which gives (p, q) minimal models.

Thank you !

Appendix

FZZ equation for the boundary 2pt function

In the dilute phase, the resolvant is given by ($\mathfrak{b}^{-2} = g = 1 + \theta$)

$$\omega(\tau) = \cosh(\mathfrak{b}^{-2}\tau), \quad \xi = \cosh\tau, \quad \mu_B = \cosh(\mathfrak{b}^{-2}\sigma)$$

The boundary 2pt function obeys

$$D(P + 1/\mathfrak{b}|\tau, \sigma) = \frac{1}{2}\left[\cosh(\mathfrak{b}^{-2}\tau \mp i\pi P) + \cosh(\mathfrak{b}^{-2}\sigma)\right] D(P|\tau \pm i\pi, \sigma)$$

with

$$D(P|\tau, \sigma) = \exp\left(-\int_{-\infty}^{\infty} \frac{dt}{t}\left[\frac{\sinh(\pi P t \mathfrak{b}^2)\cos(\tau t)\cos(\sigma t)}{\sinh(\pi t)\sinh(\pi t \mathfrak{b}^2)} - \frac{P}{\pi t}\right]\right)$$

Dimension of boundary operators

Central charges ($n = 2\cos\pi\theta$ with $0 \le \theta \le 1$),

$$c_{\text{dense}} = 1 - 6\frac{\theta^2}{1-\theta}, \quad c_{\text{dilute}} = 1 - 6\frac{\theta^2}{1+\theta}$$

Parameterisation ($g = 1+\theta$),

$$n_1 = \frac{\sin\pi(r-1)\theta}{\sin\pi r\theta}, \quad n_2 = \frac{\sin\pi(r+1)\theta}{\sin\pi r\theta}, \quad \delta_{r,s} = \frac{(rg-s)^2 - (g-1)^2}{4g}$$

We found

$$\begin{aligned} (Ord|S_L^{(2)}|AS_{(2)}) &\to \delta_{r+L,r}, & (Ord|S_L^{(1)}|AS_{(2)}) &\to \delta_{r-L,r}, \\ (Ord|S_L^{(2)}|AS_{(1)}) &\to \delta_{r-L,r+1}, & (Ord|S_L^{(1)}|AS_{(1)}) &\to \delta_{r+L,r+1}. \end{aligned}$$

Solution with $\mu = \mu_B = 0$

Parameterisation ($\omega(x) = x^{1+\theta} + tx^{1-\theta}$):

$$x = e^{\tau}, \quad t = -e^{2\gamma\theta}, \quad t_B = -2\omega_0 e^{\gamma\theta} \sinh\tilde{\gamma}\theta, \quad \omega_0 = \frac{\sin(\pi\theta)}{\sin(\pi r\theta)}.$$

Solution on AS_1:

$$\begin{aligned} d_0^{(2)}(\tau) &= \tfrac{1}{\omega_0} e^{-\frac{\tau}{2}-\gamma(r\theta-\frac{1}{2})+\frac{\tilde{\gamma}}{2}} V_{-r}(\tau-\gamma+\tilde{\gamma}) V_{\frac{1}{\theta}-r}(\tau-\gamma-\tilde{\gamma}), \\ d_1^{(2)}(\tau) &= -e^{\frac{\tau}{2}+\gamma(\frac{1}{2}+\theta-r\theta)+\frac{\tilde{\gamma}}{2}} V_{1-r}(\tau-\gamma+\tilde{\gamma}) V_{1+\frac{1}{\theta}-r}(\tau-\gamma-\tilde{\gamma}), \end{aligned}$$

with the function

$$\log V_r(\tau) = -\frac{1}{2}\int_{-\infty}^{\infty} \frac{d\omega}{\omega} \left[\frac{e^{-i\omega\tau}\sinh(\pi r\omega)}{\sinh(\pi\omega)\sinh\frac{\pi\omega}{\theta}} - \frac{r\theta}{\pi\omega} \right]$$

$\mathrm{SLE}_{\kappa,\rho}$

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - W_t}$$
$$dW_t = \sqrt{\kappa} dB_t - \rho \frac{dt}{X_t}$$
$$dX_t = \frac{2dt}{X_t} - dW_t$$

with the conformal map $g_t(z) \sim z + 2t/z + O(z^{-2})$ at infinity, W_t is the image of the tip of the growing curve (Brownian motion), B_t is the standard Brownian motion, κ the diffusion constant. $X_t = g_t(X_0)$ are auxiliary variables.