Lüscher corrections in integrable and string sigma models

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- Introduction
- Lüscher correction for the mass gap
- Lüscher's F-term and the Bajnok-Janik formula
- Example: XXX model (O(2) model)
- Example: Konishi operator

Benasque, 5 July 2010

AdS/CFT conjecture

String σ-model on $AdS_5 \times S_5$ $\Longleftrightarrow$ $\mathcal{N} = 4\ N_c \to \infty$ SYM 3+1 dim

Energy of string states $\Longleftrightarrow$ Anomalous dims of local operators

String model coupling g $\Longleftrightarrow$ 't Hooft coupling $\lambda = 4\pi^2 g^2$

Integrability (both sides)!

- asymptotic BA: all power corrections, neglecting “wrapping”

  Beisert, Staudacher '05

- generalized Lüscher approach: 4-loop Konishi (agree with PT)

  Janik, Lukowski '07 Bajnok, Janik '09

$$\mathcal{K} = \mathrm{Tr}\{D^2 Z^2 - (DZ)^2\}$$

- higher twist-two operators in sl_2 sector: agree with BFKL

  Bajnok, Janik, Lukowski '09

- 5-loop Konishi

  Bajnok, Hegedus, Janik, Lukowski '10

$$\Delta^{(10)} = \Delta^{(10)}_{\text{asympt}} + g^{10}\left\{ -\frac{81\zeta(3)^2}{16} + \frac{81\zeta(3)}{32} - \frac{45\zeta(5)}{4} + \frac{945\zeta(7)}{32} - \frac{2835}{256}\right\}$$

- 5-loop twist-two

  Lukowski, Rej, Velizhanin '10

- TBA for $AdS_5 \times S_5$ string σ-model

  Arutyunov, Frolov '09 Bombardelli, Fioravanti, Tateo '09

  Gromov, Kazakov, Kozak, Vieira '09 Arutyunov, Frolov, Suzuki '10

- Linearized TBA $\longrightarrow$ Lüscher numerically

  Arutyunov, Frolov, Suzuki '10

Large λ expansion

- classical string:

  Gubser, Plebanov, Polyakov '98

  Arutyunov, Frolov, Staudacher '04

$$E_K(\lambda) = 2\sqrt[4]{\lambda} + \ldots$$

- numerical fit from TBA:

  Gromov, Kazakov, Vieira '09

$$E_K(\lambda) = \sqrt[4]{\lambda}\left\{2.0004 + \frac{1.988}{\sqrt{\lambda}} - \frac{2.60}{\lambda} + \frac{6.2}{\sqrt{\lambda}^3} + \ldots\right\}$$

$$\lambda < 664 < \lambda_{\mathrm{crit}}^{(1)}$$

- semiclassical expansion:

Roiban, Tseytlin '09

$$E_K(\lambda) = \sqrt[4]{\lambda}\left\{2 + \frac{1}{\sqrt{\lambda}} + \dots\right\}$$

- numerical fit from TBA:

Frolov '10

$$E_K(\lambda) = \sqrt[4]{\lambda}\left\{2.00045 + \frac{1.98}{\sqrt{\lambda}} - \frac{2.55}{\lambda} + \frac{6.7}{\sqrt{\lambda}^3} + \dots\right\}$$

$$\lambda < 2046 < \lambda_{\mathrm{crit}}^{(1)}$$

- no $\lambda_{\mathrm{crit}}^{(1)}$? ($\lambda_{\mathrm{crit}}^{(1)} \approx 774$ from ABA)

Lüscher corrections to the mass gap

Lüscher '83 Klassen, Melzer '91

All order diagrammatic expansion: Lüscher's F-term

$\Delta m_a^{(F)}(L)$

$$= -\frac{dL}{4\pi m_a} \sum_b \int_{-\infty}^{\infty} \mathrm{d}\theta \left(\frac{m_b \cosh\theta}{2\pi L} \right)^{d/2} K_{d/2-1}(m_b L \cosh\theta)\, F_{ab}\left(\theta + \frac{i\pi}{2}\right)$$

1+1 dimensional case: $(m_a = m_b = m)$

$$\Delta m_a^{(F)}(L) = -\frac{m}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\theta \cosh\theta e^{-mL\cosh\theta}\, S_a\left(\theta + \frac{i\pi}{2}\right)$$

$$S_a(\theta) = -n + \sum_{b=1}^{n} S_{ab}^{ab}(\theta)$$

μ-term (1+1 dim)

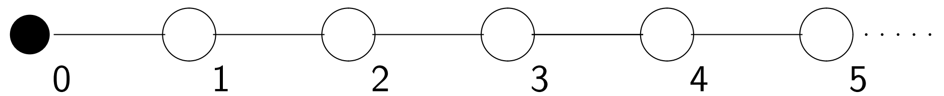
$$\Delta m_a^{(\mu)}(L) = -\frac{1}{8m_a^2}\sum_{bc}\theta(m_a^2 - |m_b^2 - m_c^2|)\,\frac{\lambda_{abc}^2}{\mu_{abc}}\,\mathrm{e}^{-\mu_{abc}L}$$

λ_{abc}: 3-point coupling

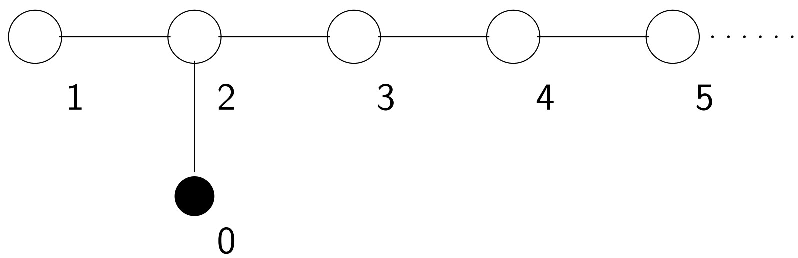
μ_{abc}: height of mass triangle $(m_a,\, m_b,\, m_c)$

- LWW coupling and "step scaling" function Lüscher, Weisz, Wolff '91
- Lattice MC measurements
- Sigma model perturbation theory

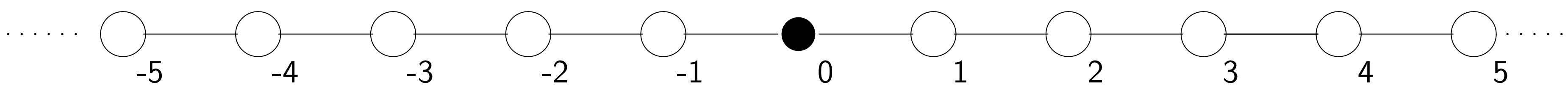
Generalized Lüscher formulae



O(2) model TBA diagram



O(3) model TBA diagram



O(4) model TBA diagram

SIGMA MODELS ($n = 2, 3, 4$)

Bethe-Yang limit energy:

$$E^{(0)} = \sum_{j=1}^{r} \mu \cosh \theta_j$$

Bethe-Yang equations (QC)

$$QC_k^{(0)}(\theta_1, \ldots, \theta_r) = \mathrm{e}^{i\mu L \sinh \theta_k} \, \mathrm{e}^{i\mathcal{R}_k} = -1 \,, \qquad k = 1, \ldots, r \,,$$

$$\mathrm{e}^{i\mathcal{R}_k} = \sigma(\theta_k | \theta_1, \ldots, \theta_r)$$

$\sigma(\theta|\theta_1, \ldots, \theta_r)$: eigenvalue of the transfer matrix

(unitary and crossing symmetric S-matrix)

Lüscher's F-term:

$$E^{(1)} = E^{(0)} + \delta E =$$

$$\delta E = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\theta\, \mu \cosh\theta\, \mathrm{e}^{-\mu L \cosh\theta}\, \sigma\left(\theta + \tfrac{i\pi}{2} \middle| \theta_1, \ldots, \theta_r\right) .$$

Bajnok-Janik formula:

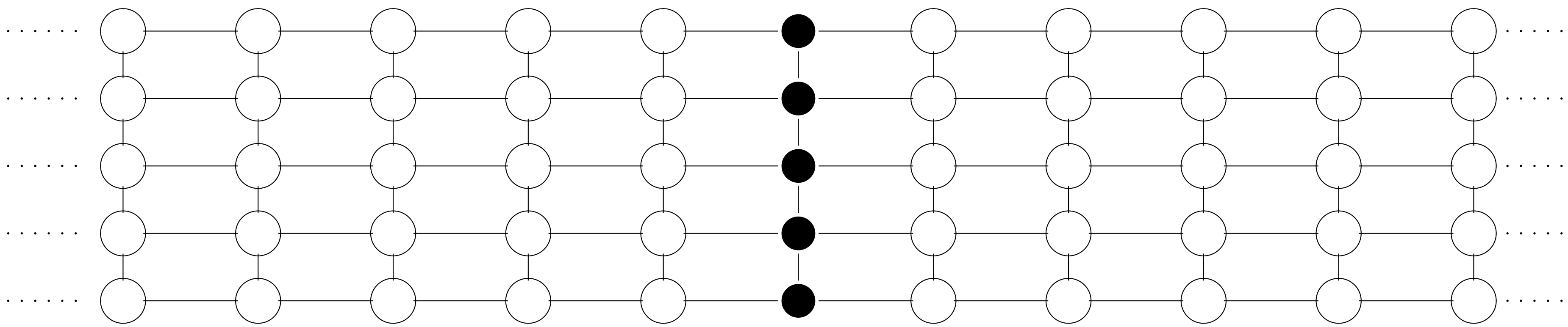
$$QC_k^{(1)}(\theta_1, \ldots, \theta_r) = QC_k^{(0)}(\theta_1, \ldots, \theta_r)\,\{1 + i\,\delta\mathcal{R}_k\} = -1\,,$$

$$k = 1, \ldots, r\,,$$

$$\delta\mathcal{R}_k(\theta_1, \ldots, \theta_r) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\theta\, \mathrm{e}^{-\mu L \cosh\theta}\, \partial_k\, \sigma\left(\theta + \tfrac{i\pi}{2} \middle| \theta_1, \ldots, \theta_r\right)$$

$$\partial_k = \partial/\partial\theta_k.$$

PRINCIPAL MODELS (SU(n))



SU(6) model TBA diagram

several types of particles: masses μ_a

transfer matrix: $\sigma^a(\theta|\theta_1,\ldots,\theta_r)$

(all particles in $a=n-1$ (anti-vector) representation)

Lüscher corrections to the energy:

$$E^{(1)}=E^{(0)}+\delta E$$

$$E^{(0)} = \sum_{j=1}^{r} \mu_{n-1} \cosh \theta_j$$

$$\delta E = i \sum_{a=1}^{n-1} \mu_a \int_{-\infty}^{\infty} \frac{\mathrm{d}\theta}{2\pi} \sinh\left(\theta + \tfrac{i\pi}{n}\right) \mathrm{e}^{i\mu_a L \sinh\left(\theta + \frac{i\pi}{n}\right)} \sigma^a \left(\theta + \tfrac{i\pi}{n} \middle| \theta_1, \ldots, \theta_r\right)$$

quantization conditions: $(k = 1, \ldots, r)$

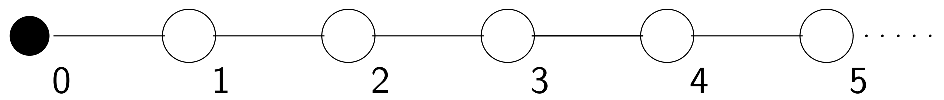
$$QC_k^{(1)} = QC_k^{(0)} \left\{1 + i\,\delta\mathcal{R}_k\right\}$$

$$= \mathrm{e}^{i\mu_{n-1} L \sinh\theta_k}\, \sigma^{n-1}(\theta_k|\theta_1, \ldots, \theta_r)\ \left\{1 + i\,\delta\mathcal{R}_k\right\} = -1\,,$$

Bajnok-Janik formula:

$$\delta\mathcal{R}_k(\theta_1, \ldots, \theta_r) = \sum_{a=1}^{n-1} \int_{-\infty}^{\infty} \frac{\mathrm{d}\theta}{2\pi} \mathrm{e}^{i\mu_a L \sinh\left(\theta + \frac{i\pi}{n}\right)} \partial_k\, \sigma^a \left(\theta + \tfrac{i\pi}{n} \middle| \theta_1, \ldots, \theta_r\right)$$

Example: O(2) model



O(2) model TBA diagram

BETHE-YANG LIMIT SOLUTION (BETHE ANSATZ):

$$T_m(\theta+\tfrac{i\pi}{2})T_m(\theta-\tfrac{i\pi}{2}) = T_{m-1}(\theta)T_{m+1}(\theta)+T_0(\theta+i\tfrac{m+1}{2}\pi)T_0(\theta-i\tfrac{m+1}{2}\pi)$$

$m = 1, 2, \ldots$ $\qquad$ ($T_{-1}(\theta) \equiv 0$ boundary cond.)

$$T_0(\theta) = \prod_k (\theta - \theta_k) \qquad\qquad T_m(\theta) : \text{BA sol'n}$$

Bethe Ansatz

O(2) S-matrix: $S^{ab}_{cd}(\theta) = U(\theta)\left\{\theta\,\delta^{ab}\delta_{cd} + (i\pi - \theta)\delta^a_d\,\delta^b_c\right\}$

$U(\theta)$: unitarity, crossing

Transfer matrix (3-particles)

$$T^{b_1 b_2 b_3}_{a_1 a_2 a_3}(\theta|\theta_1,\theta_2,\theta_3) = S^{u_1 b_1}_{u_3 a_1}(\theta-\theta_1)\,S^{u_2 b_2}_{u_1 a_2}(\theta-\theta_2)\,S^{u_3 b_3}_{u_2 a_3}(\theta-\theta_3)$$

Bethe roots: $\{u_j\}_{j=1}^{M}$

Baxter's Q operator: $Q(\theta) = \prod_{j=1}^{M}(\theta - u_j)$

Bethe equations:

$$\frac{Q(u_j+i\pi)}{Q(u_j-i\pi)} = -\frac{T_0(u_j+\frac{i\pi}{2})}{T_0(u_j-\frac{i\pi}{2})} \qquad j = 1, \ldots, M$$

$T_k(\theta)$ polynomials:

$$\xi(\theta) = \frac{T_0(\theta)}{Q(\theta+\frac{i\pi}{2})\,Q(\theta-\frac{i\pi}{2})}$$

$$T_k(\theta) = Q(\theta + i\,\tfrac{k+1}{2}\,\pi)\,Q(\theta - i\,\tfrac{k+1}{2}\,\pi)\,\sum_{j=0}^{k} \xi(\theta + i\,\tfrac{k-2j}{2}\,\pi)$$

$$\mathrm{Y-system} \Longleftrightarrow \mathrm{T-system} \quad \mathrm{(gauge)}$$

$$y_m(\theta) = \frac{T_{m-1}(\theta)\, T_{m+1}(\theta)}{T_0(\theta{+}i\frac{m+1}{2}\pi)\, T_0(\theta{-}i\frac{m+1}{2}\pi)}$$

$$Y_m(\theta) = 1 + y_m(\theta) = \frac{T_m(\theta{+}\frac{i\pi}{2})\, T_m(\theta{-}\frac{i\pi}{2})}{T_0(\theta{+}i\frac{m+1}{2}\pi)\, T_0(\theta{-}i\frac{m+1}{2}\pi)}$$

Y-system equations:

$$y_m(\theta + \tfrac{i\pi}{2})\, y_m(\theta - \tfrac{i\pi}{2}) = Y_{m-1}(\theta)\, Y_{m+1}(\theta) \quad m = 1, 2 \ldots$$

y_m roots: T_{m+1} roots $+$ T_{m-1} roots

only T_0 has roots: $\Longrightarrow$ only y_1 has roots: $\{\theta_k\}$

T_m: transfer matrix in spin $\frac{m}{2}$ representation (up to scale factor)

T_1: transfer matrix in defining representation

$$\frac{T_1(\theta+\frac{i\pi}{2})\,T_1(\theta-\frac{i\pi}{2})}{T_0(\theta+i\pi)\,T_0(\theta-i\pi)} = Y_1(\theta)$$

physical transfer matrix (unitary + crossing symmetric):

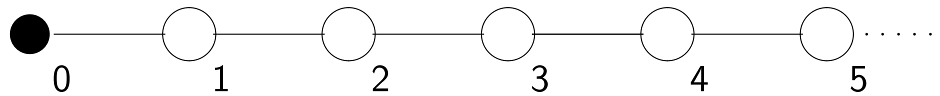
$$\Lambda(\theta) = \sigma(\theta + \tfrac{i\pi}{2})$$

$$\Lambda(\theta + \tfrac{i\pi}{2})\,\Lambda(\theta - \tfrac{i\pi}{2}) = Y_1(\theta)$$

Bethe-Yang equations:

$$\mathrm{e}^{i\ell\sinh\theta_k}\sigma(\theta_k) = -1 = \mathrm{e}^{i\ell\sinh\theta_k}\Lambda(\theta_k - \tfrac{i\pi}{2})$$

EXACT TBA EQUATIONS:



$$y^e_m(\theta); \quad Y^e_m(\theta) = 1 + y^e_m(\theta); \quad L^e_m(\theta) = \ln Y^e_m(\theta); \quad m = 0, 1, \ldots$$

exact Y-system equations:

$$y^e_m(\theta + \tfrac{i\pi}{2})\, y^e_m(\theta - \tfrac{i\pi}{2}) = Y^e_{m-1}(\theta)\, Y^e_{m+1}(\theta) \quad m = 0, 1 \ldots$$

($Y^e_{-1}(\theta) \equiv 1$ by convention)

Asymptotic condition: $\quad y^e_0(\theta) = \mathrm{e}^{-\ell \cosh\theta} \Lambda^e(\theta)$

$$\Lambda^e(\theta + \tfrac{i\pi}{2})\, \Lambda^e(\theta - \tfrac{i\pi}{2}) = Y^e_1(\theta)$$

Quantization conditions (QC): $y^e_1(\theta_k) = 0 \quad \Rightarrow \quad Y^e_0(\theta_k \pm \tfrac{i\pi}{2}) = 0$

TBA Lemma

$$f(\theta + \tfrac{i\pi}{2})\, f(\theta - \tfrac{i\pi}{2}) = F(\theta)$$

$\{t_k\}$ roots in physical strip: $f(t_k) = 0$

$$f(\theta) = \left\{ \prod_k \tau(\theta - t_k) \right\} \exp\{(K \star \ln F)(\theta)\}$$

$$\tau(\theta) = \tanh\tfrac{\theta}{2} \qquad \frac{\tau'(\theta)}{\tau(\theta)} = \frac{1}{\sinh\theta}$$

$$K(\theta) = \frac{1}{\cosh\theta} \qquad (A \star B)(x) = \int_{-\infty}^{\infty} \mathrm{d}y\, A(x-y)B(y)$$

Exact TBA equations:

$$\Lambda^e(\theta) = \exp\left\{(K \star L_1^e)(\theta)\right\}$$

$$y_1^e(\theta) = \left\{\prod_k \tau(\theta - \theta_k)\right\} \exp\left\{(K \star L_0^e)(\theta) + (K \star L_2^e)(\theta)\right\}$$

$$y_m^e(\theta) = \exp\left\{(K \star L_{m+1}^e)(\theta) + (K \star L_{m-1}^e)(\theta)\right\} \qquad m = 2, 3, \dots$$

Exact Bethe-Yang equations (QC):

$$\mathrm{e}^{i\ell \sinh \theta_k} \Lambda^e(\theta_k - \tfrac{i\pi}{2}) = -1$$

$$\Lambda^e(\theta_k - \tfrac{i\pi}{2}) = \mathrm{e}^{i\mathcal{B}^e(\theta_k)}$$

$$\mathcal{B}^e(x) = -\tfrac{\mathcal{P}}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}y \, \tfrac{L_1^e(y)}{\sinh(y-x)} \qquad \mathcal{P} : \text{superfluous}$$

QC (exact Bethe-Yang equation):

$$\mathcal{B}^e(\theta_k) + \ell \sinh\theta_k = 2\pi I_k \quad \text{(half integer)}$$

Energy:

$$\epsilon = \sum_k \cosh\theta_k - \tfrac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\theta \cosh\theta \, L_0^e(\theta)$$

Back to B-Y limit:

$$\Lambda(\theta) = \exp\{(K \star L_1)(\theta)\}$$

$$y_1(\theta) = \left\{\prod_k \tau(\theta - \theta_k)\right\} \exp\{(K \star L_2)(\theta)\}$$

$$y_m(\theta) = \exp\{(K \star L_{m+1})(\theta) + (K \star L_{m-1})(\theta)\} \quad m = 2, 3, \ldots$$

$$\Lambda(\theta - \tfrac{i\pi}{2}) = \sigma(\theta) = \mathrm{e}^{i\mathcal{B}(\theta)}$$

B-Y equations:

$$\mathcal{B}(\theta_k) + \ell \sinh \theta_k = 2\pi I_k$$

LINEARIZED TBA EQUATIONS:

$$f^e = f + \delta f$$

$$\frac{\delta y_1}{y_1} = K \star L_0^e + K \star \delta L_2$$

$$\frac{\delta y_m}{y_m} = K \star \delta L_{m+1} + K \star \delta L_{m-1} \quad m = 2, 3, \ldots$$

$$L_0^e(\theta) = \ln\left[1 + \mathrm{e}^{-\ell \cosh\theta}\,\Lambda^e(\theta)\right] \approx \mathrm{e}^{-\ell\cosh\theta}\,\Lambda(\theta) = \lambda(\theta)$$

Lüscher's F-term:

$$-\tfrac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\theta \cosh\theta\, L_0^e(\theta) \approx -\tfrac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\theta \cosh\theta\, \mathrm{e}^{-\ell\cosh\theta}\Lambda(\theta)$$

$$= -\tfrac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\theta \cosh\theta\, \mathrm{e}^{-\ell\cosh\theta}\sigma(\theta + \tfrac{i\pi}{2})$$

Structure of linear system: $\mathbf{M}\,\xi = J$

$$\mathbf{M} = \begin{pmatrix} D_1 & -K\star & 0 & 0 & \dots \\ -K\star & D_2 & -K\star & 0 & \dots \\ 0 & -K\star & D_3 & -K\star & \dots \\ 0 & 0 & -K\star & D_4 & \dots \\ & & \vdots & & \end{pmatrix}$$

$$\xi = \begin{pmatrix} \delta L_1 \\ \delta L_2 \\ \delta L_3 \\ \vdots \end{pmatrix} \qquad J = \begin{pmatrix} j(\theta) \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

$$j(\theta) = K \star \lambda$$

Bethe-Yang equation:

$$\mathcal{B}(\theta_k) + \delta\mathcal{B}(\theta_k) + \ell \sinh\theta_k = 2\pi I_k$$

Bajnok-Janik:

$$\delta\mathcal{B}_k = \delta\mathcal{B}(\theta_k) = -\tfrac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}y \, \tfrac{\delta L_1(y)}{\sinh(y-\theta_k)}$$

$$\delta\mathcal{B}_k = -\tfrac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}y \, \int_{-\infty}^{\infty} \mathrm{d}x \, \tfrac{R_{11}(y,x)\, j(x)}{\sinh(y-\theta_k)}$$

Inverse operator:

$$\mathbf{M}^{-1} = \mathbf{R} \qquad \xi_m(x) = \sum_s \int_{-\infty}^{\infty} \mathrm{d}y \, R_{ms}(x,y)\, J_s(y)$$

$\mathbf{M} = \mathbf{M}^T$ symmetric: $\mathbf{R} = \mathbf{R}^T$

$$R_{ms}(x,y) = R_{sm}(y,x)$$

DERIVATIVE OF B-Y LIMIT:

$$\partial_k = \frac{\partial}{\partial \theta_k}$$

$$\frac{\partial_k \Lambda}{\Lambda} = K \star \partial_k L_1$$

$$\frac{\partial_k y_1}{y_1} = -\frac{1}{\sinh(\theta - \theta_k)} + K \star \partial_k L_2$$

$$\frac{\partial_k y_m}{y_m} = K \star \partial_k L_{m+1} + K \star \partial_k L_{m-1} \quad m = 2, 3, \ldots$$

same linear system!

$$\xi = \begin{pmatrix} \partial_k L_1 \\ \partial_k L_2 \\ \vdots \end{pmatrix} \qquad J = \begin{pmatrix} j_k(\theta) \\ 0 \\ \vdots \end{pmatrix} \qquad j_k(\theta) = -\frac{1}{\sinh(\theta - \theta_k)}$$

Relation from the derivative case:

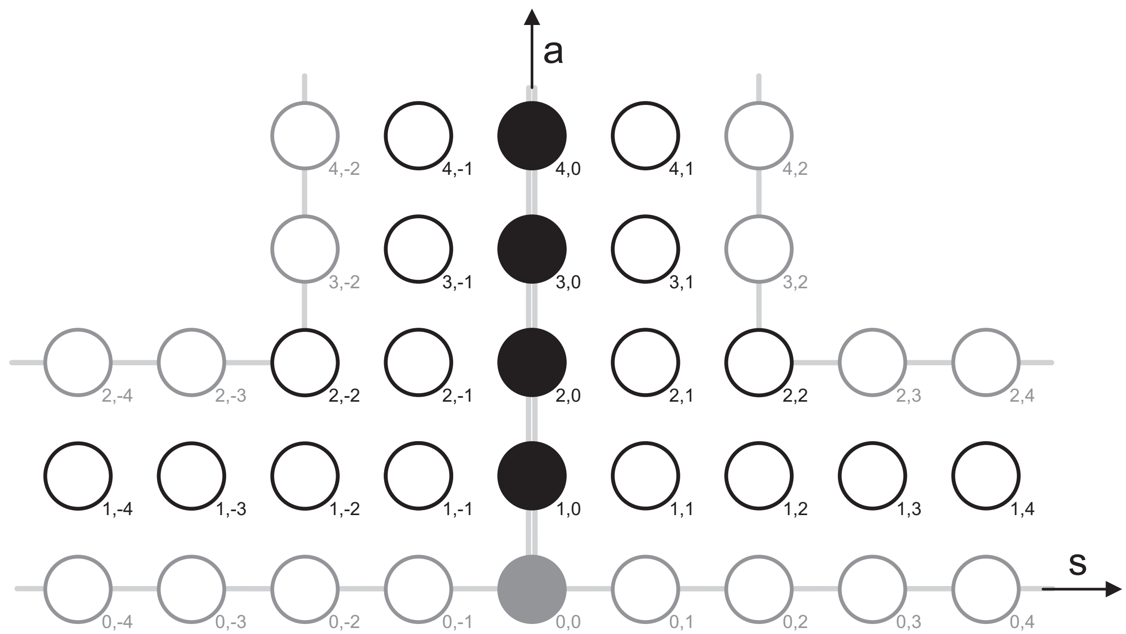
$$\partial_k L_m(x) = -\int_{-\infty}^{\infty} \mathrm{d}y\, R_{m1}(x,y)\, \tfrac{1}{\sinh(y-\theta_k)}$$

Huge piece of luck: just what we need!

$$\begin{aligned}
\delta\mathcal{B}_k &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}x\, \partial_k L_1(x) j(x) \\
&= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}x\, \partial_k L_1(x)\, (K\star\lambda)(x) \\
&= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}x\, \lambda(x)\, (K\star\partial_k L_1)(x)
\end{aligned}$$

$$= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}x\, \lambda(x)\, \tfrac{\partial_k\Lambda(x)}{\Lambda(x)} = \tfrac{1}{2\pi}\,\partial_k \int_{-\infty}^{\infty} \mathrm{d}x\ \mathrm{e}^{-\ell\cosh x}\,\Lambda(x)$$

Example: Konishi operator



$AdS_5 \times S_5$ string model TBA diagram

Gromov, Kazakov, Kozak, Vieira '09

Arutyunov, Frolov '09

$AdS_5 \times S_5$ STRING MODEL

String energies:

$$E = J + \sum_{i=1}^{N} \mathcal{E}(p_i) - \frac{1}{2\pi} \sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}u \frac{d\widetilde{p}^Q}{du} \log(1 + Y_Q)$$

Dispersion relation:

$$\mathcal{E}(p) = \sqrt{1 + 4g^2 \sin^2 \frac{p}{2}}$$

Leading order "wrapping" corrections:

$$\pi(2n_k + 1) = J\, p_k + i \sum_{j=1}^{N} \log S^{1_* 1_*}_{\mathfrak{sl}(2)}(u_j, u_k) + \delta\mathcal{R}_k^{(\mathrm{BJ})} + O(g^9)$$

Bajnok-Janik formula:

$$\delta\mathcal{R}_k^{(\mathrm{BJ})} = \frac{1}{2\pi} \sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}u \; \frac{\partial}{\partial u_k} Y_Q^o(u)\Big|_{\{u_j\}=\{u_j^o\}}$$

The exact Bethe equations

$$\pi i(2n_k+1) = \log Y_{1_*}(u_k) = iL\,p_k - \sum_{j=1}^{N} \log S_{\mathfrak{sl}(2)}^{1_*1_*}(u_j,u_k)$$

$$+\,2\sum_{j=1}^{N} \log \mathrm{Res}(S) \star K_{vwx}^{11_*}(u_j^-,u_k) - 2\sum_{j=1}^{N} \log\left(u_j - u_k - \frac{2i}{g}\right)\frac{x_j^- - \frac{1}{x_k^-}}{x_j^- - x_k^+}$$

$$-\,2\sum_{j=1}^{n_1}\left(\log S \,\hat{\star}\, K_{y1_*}(r_j^{(1)-},u_k) - \log S(r_j^{(1)} - u_k)\right)$$

$$+\log\left(1+Y_Q\right)\star\left(K_{\mathfrak{sl}(2)}^{Q1_*} + 2\,s\star K_{vwx}^{Q-1,1_*}\right)$$

$$+\,2\log\left(1+Y_{1|vw}\right)\star\left(s\,\hat{\star}\,K_{y1_*}+\tilde{s}\right) - 2\log\frac{1-Y_-}{1-Y_+}\,\hat{\star}\,s\star K_{vwx}^{11_*}$$

$$+\log\frac{1-\frac{1}{Y_-}}{1-\frac{1}{Y_+}}\,\hat{\star}\,K_1 + \log\left(1-\tfrac{1}{Y_-}\right)\left(1-\tfrac{1}{Y_+}\right)\hat{\star}\,K_{y1_*}$$

LINEARIZED STRING TBA EQUATIONS

Arutyunov, Frolov, Suzuki '10

$$Y = Y^o(1+\mathcal{Y}).$$

- $m|w$-strings: $m \geq 1$, $\mathcal{Y}_{0|w} = 0$

$$\mathcal{Y}_{m|w} = (A_{m-1|w}\mathcal{Y}_{m-1|w} + A_{m+1|w}\mathcal{Y}_{m+1|w}) \star s + \delta_{m1}\Big(\frac{\mathcal{Y}_+}{1-Y_+^o} - \frac{\mathcal{Y}_-}{1-Y_-^o}\Big)\hat{\star}s$$

coefficients: $A_{m|w} = \frac{Y^o_{m|w}}{1+Y^o_{m|w}}$

- $m|vw$-strings: $m \geq 1$, $\mathcal{Y}_{0|vw} = 0$

$$\mathcal{Y}_{m|vw} = (A_{m-1|vw}\mathcal{Y}_{m-1|vw} + A_{m+1|vw}\mathcal{Y}_{m+1|vw}) \star s - Y^o_{m+1} \star s + \delta_{m1}\left(\frac{\mathcal{Y}_-}{1-\frac{1}{Y_-^o}} - \frac{\mathcal{Y}_+}{1-\frac{1}{Y_+^o}}\right)\hat{\star}s$$

coefficients: $A_{m|vw} = \frac{Y^o_{m|vw}}{1+Y^o_{m|vw}}$

• y-particles:

$$\mathcal{Y}_+ - \mathcal{Y}_- = Y_Q^o \star K_{Qy}$$

Perturbative orders:

$$Y_Q^o(u) = O(g^8)$$

$$\mathcal{Y}_+ - \mathcal{Y}_- = O(g^9)$$

Y_+^o and Y_-^o coincide at leading order:

$$\frac{Y_+^o(u)}{Y_-^o(u)} = 1 + O(g^2).$$

Decoupling:

$$\mathcal{Y}_{m|w} = O(g^9) \qquad\qquad \mathcal{Y}_{m|vw} = O(g^8)$$

Leading order (rescaled):

$$\frac{\delta y_m}{y_m} - K \star (\delta L_{m-1} + \delta L_{m+1}) = -K \star Y^o_{m+1}, \quad m = 1, 2, ...$$

$Y^o_{m|vw} = y_m$: XXX model $Q(u) = u$ solution $N = 2$

More general states in the sl_2 sector: $N > 2$ (roots for all y_m)

wrapping corrected (leading order) Bethe equations

$$\pi(2n_k + 1) \;\; = \;\; J\, p_k + i \sum_{j=1}^{N} \log S^{1_* 1_*}_{\mathfrak{sl}(2)}(\tfrac{u_j}{g}, \tfrac{u_k}{g}) + \delta\mathcal{R}_k + O(g^9)$$

$$\delta\mathcal{R}_k = \delta\mathcal{R}^{(1)}_k + \delta\mathcal{R}^{(2)}_k + \delta\mathcal{R}^{(3)}_k$$

$\delta\mathcal{R}_k^{(1)}$ & $\delta\mathcal{R}_k^{(3)}$: small g expansion of the Y_Q convolution terms

$\delta\mathcal{R}_k^{(2)}$: convolution terms containing $Y_{1|vw}$

$$\delta\mathcal{R}_k^{(1)} = \tfrac{1}{\pi}\sum_{m=1}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}u\, Y_m^o(u)\, \tfrac{u-u_k}{(m+1)^2+(u-u_k)^2}$$

$$\delta\mathcal{R}_k^{(2)} = \int_{-\infty}^{\infty} \mathrm{d}u\, \frac{\delta L_1(u)}{2\sinh\frac{\pi}{2}(u-u_k)}$$

$$\delta\mathcal{R}_k^{(3)} = \tfrac{1}{\pi}\sum_{m=1}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}u\, Y_{m+1}^o(u)\, \left\{\mathcal{F}_m(u-u_k) - \tfrac{u-u_k}{m^2+(u-u_k)^2}\right\}$$

$$\mathcal{F}_m(u) = \tfrac{-i}{4}\left\{\psi\left(\tfrac{m+iu}{4}\right) - \psi\left(\tfrac{m-iu}{4}\right) - \psi\left(\tfrac{m+2+iu}{4}\right) + \psi\left(\tfrac{m+2-iu}{4}\right)\right\}$$

$$\psi(z) = \Gamma'(z)/\Gamma(z)$$

Nontrivial part: $\delta\mathcal{R}_k^{(2)}$

Same problem solved for O(2) model !

$$\delta\mathcal{R}_k^{(2)} = \tfrac{1}{\pi}\sum_{m=1}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}u\, Y_{m+1}^{o}(u)\,\left\{\partial_k \log \hat{t}_m(u)\right\}$$

$$= \tfrac{1}{\pi}\sum_{m=1}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}u\, Y_{m+1}^{o}(u)\,\left\{\partial_k \log t_m(u) - \tfrac{r_m'(u-u_k)}{r_m(u-u_k)}\right\}.$$

$$\tfrac{r_m'(x)}{r_m(x)} = \mathcal{F}_m(x) - \tfrac{2x}{m^2+x^2}$$

transcendental parts cancel

result: rational functions

$$\delta\mathcal{R}_k = \tfrac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}u\, Y_1^o(u)\, \tfrac{u-u_k}{4+(u-u_k)^2}$$

$$+\tfrac{1}{\pi}\sum_{m=1}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}u\, Y_{m+1}^o(u)\left\{\partial_k \log t_m(u) + \tfrac{u-u_k}{m^2+(u-u_k)^2} + \tfrac{u-u_k}{(m+2)^2+(u-u_k)^2}\right\}$$

can be written compactly

$$\delta\mathcal{R}_k = \tfrac{1}{2\pi}\sum_{m=1}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}u\, Y_m^o(u)\,\partial_k \log j_m(u)$$

$$j_m(u) = \tfrac{64g^8}{(u^2+m^2)^4}\, \tfrac{t_{m-1}^2(u)}{t_0(u-im-i)\, t_0(u-im+i)\, t_0(u+im-i)\, t_0(u+im+i)}$$

important observation:

$$Y_m^o(u) = j_m(u)$$

Bajnok-Janik formula reproduced:

$$\delta\mathcal{R}_k = \delta\mathcal{R}_k^{(\mathrm{BJ})} = \tfrac{1}{2\pi} \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}u \, \partial_k \, Y_m^o(u)$$