Long-time/long-distance asymptotics of the two-point functions in the non-linear Schodinger model

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Finite-Size Technology in Low-Dimensional Quantum Systems - V

In collaboration with : *N. Kitanine, J.M. Maillet, N. A. Slavnov and V. Terras* .

"On the thermodynamic limit of form factors in the massless XXZ Heisenberg chain", JMP **50** 095209, (2009).
"Thermodynamic limit of particle-hole form factors in the massless XXZ Heisenberg chain", ArXiV: 1003.4557.
N. Kitanine, K. K. Kozlowski, J.M. Maillet, N. A. Slavnov and V. Terras

"Riemann–Hilbert approach to the time-dependent generalized sine kernel".
"Multidimensional Natte series for the time-dependent correlation functions in the non-linear Schrodinger model".
in preparation

Outline

Asymptotics of Correlation functions at $T = 0K$

★ 1D *gapless* models at $T = 0K$ are critical $\rightsquigarrow$ Algebraic in distance decay of correlators.

$\Rightarrow$ predictions of critical exponents by Luttinger liquid/CFT from finite size corrections (Batchelor, Destri, DeVega, Klumper, Pearce, Woynarowich, Wehner, Zittartz)

♦ XXZ spin $1/2$ chain at $-1 < \Delta < 1 \equiv$ quantum critical magnet:

Haldane, Peschel-Luther for XXZ $\quad \langle \sigma_1^z \sigma_m^z \rangle \simeq \langle \sigma^z \rangle^2 + \dfrac{C_1}{m^2} + C_2 \dfrac{\cos(2mp_F)}{m^{2\mathcal{Z}^2}} + \ldots$

no access to constants C_1, C_2, no methods for higher corrections in m

indirect conjecture for the constants at zero magnetic field '99 Lukyanov , '03 Lukyanov,Terras .

Very restricted predictions for the long-distance/long-time behavior

Correlation functions at the free fermion points $T = 0K$

- Correlators for free fermionic models $\equiv$ Fredholm determinants

Density-to-density correlators (Bose gaz): $\frac{1}{2}\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial \beta^2}\langle \mathrm{e}^{\beta Q_x}\rangle = \langle \rho(x)\rho(0)\rangle$

$$\langle \mathrm{e}^{\beta Q_x}\rangle = \det_{[-q;q]}[I+S] = \sum_{n\geq 0}\frac{1}{n!}\int_{-q}^{q}\mathrm{d}^n\lambda \det_n\begin{bmatrix} S(\lambda_1,\lambda_1)\dots S(\lambda_1,\lambda_n) \\ \dots \\ S(\lambda_n,\lambda_1)\dots S(\lambda_n,\lambda_n)\end{bmatrix} \qquad S(\lambda,\mu) = \frac{1-\mathrm{e}^{\beta}}{\pi}\,\frac{\sin\frac{x}{2}(\lambda-\mu)}{\lambda-\mu}$$

- Sine kernel satisfies PV: $\langle \psi^{\dagger}(x)\psi(0)\rangle$ for imp. bosons ('80 Jimbo, Miwa, Mori, Sato) ;
- transverse Ising, imp. bosons ('83-'86 McCoy, Perk, Schrock, Tang);
- T>0K, large x, large t by RHP ('90 Its, Izergin, Korepin, Varzugin, Slavnov,...) .

Today, well developped methods of asymptotic analysis :

★ Operator methods (separation theorem) Basor, Bottcher, Silbermann, Szego, Widom

★ Riemann–Hilbert problems Deift, Its, Izergin, Korepin, Krasovsky, Zhou .

Correlation functions define new classes of special functions

- Long history of obtaining algebraic expressions for correlators
  Izergin, Korepin, Kyoto group, Lyon group, Wuppertal group;
- Algebraic separation of integrals, hidden free fermionic structure
  '04-'08 Boos, Jimbo, Miwa, Smirnov, Takeyama ;
- Various series of multiple integrals from ABA Lyon Group ;
- First method of asymptotic analysis '08 Kitanine, K., Maillet, Slavonv, Terras .

Density-to-density correlators (Bose gaz):

$$\langle e^{\beta Q_x} \rangle = \sum_{n \geq 0} \frac{1}{n!} \int_{-q}^{q} \frac{d^n \lambda}{(2i\pi)^n} \int_{\Gamma([-q;q])} \frac{d^n z}{(2i\pi)^n} \mathcal{F}_n \begin{pmatrix} \lambda_1, \dots, \lambda_n \\ z_1, \dots, z_n \end{pmatrix} \prod_{k=1}^{n} \frac{e^{ix(z_k - \lambda_k)}}{z_j - \lambda_j} \operatorname{det}_n \left[\frac{1}{z_j - \lambda_k} \right] .$$

Today's main message

★ Correlatiors in interacting models ≡ multidimensional deformations of Fredholm determinants.

★ Construction of multidimensional Natte series allowing to read off asymptotics.

The non-linear Schrodinger model

- NLSE $\equiv$ 1D limit of 3D Bose gas. Test of Bogoliubov theory.
- Eigenfunctions and spectrum ('63 Lieb, Liniger).
- Simplest possible *interacting* **massless** integrable model.

$$H = \int_0^L \left\{ \partial_y \Psi^\dagger(y) \partial_y \Psi(y) + c \Psi^\dagger(y) \Psi^\dagger(y) \Psi(y) \Psi(y) - h \Psi^\dagger(y) \Psi(y) \right\} \mathrm{d}y$$

L: length of circle, $c > 0$ coupling constant (repulsive regime), $h > 0$ chemical potential.

★ N-body eigenstates constructed by algebraic Bethe Ansatz $\quad | \{\lambda_j\} \rangle$

The parameters $\{\lambda_j\}$, $\lambda_j \in \mathbb{R}$, are the unique solutions to the Bethe Ansatz equations

$$\frac{L}{2\pi} p_0(\lambda_j) - \frac{1}{2\pi} \sum_{a=1}^{N} \theta(\lambda_j - \lambda_a) = n_j - \frac{N+1}{2} \quad \text{with} \quad \left\{ \begin{array}{c} p_0(\lambda) = \lambda \\ \theta(\lambda) = i \ln\left(\dfrac{ic + \lambda}{ic - \lambda} \right) \end{array} \right.$$

★ All choices of $n_1 < \cdots < n_N$ yield the complete set of eigenstates of H ('90 Dorlas).

The form factor approach

Form factor expansion for finite L of $j(x,t) \equiv \mathrm{e}^{iHt}\Psi^{\dagger}(x)\,\Psi(x)\,\mathrm{e}^{-iHt}$

$$\begin{aligned}
\langle\, G.S. | j(x,t)\, j(0,0) \,|\, G.S. \,\rangle &= \sum_{\{\mu\}_{\mathrm{ex}}} \langle\, G.S. | \mathrm{e}^{-ixP+itH} j(0,0)\, \mathrm{e}^{ixP-itH} |\, \{\mu\}_{\mathrm{ex}} \,\rangle\langle\, \{\mu\}_{\mathrm{ex}} \,| j(0,0) \,|\, G.S. \,\rangle \\
&= \sum_{\{\mu\}_{\mathrm{ex}}} \mathrm{e}^{ix(P_{\mathrm{G.S.}}-P_{\mathrm{ex}})-it(\mathcal{E}_{G.S.}-\mathcal{E}_{\mathrm{ex}})} \left|\langle\, G.S. | j(0,0) \,|\, \{\mu\}_{\mathrm{ex}} \,\rangle\right|^2
\end{aligned}$$

presumed steps of the computation

- Characterize the excitations above the ground state;
- Get a determinant representation for $\langle\, G.S. | j(0,0) \,|\, \{\mu\}_{\mathrm{ex}} \,\rangle$;
- Asymptotic in *size* L formula for $\langle\, G.S. | j(0,0) \,|\, \{\mu\}_{\mathrm{ex}} \,\rangle$;
- Perform the summation over $\{\mu\}_{\mathrm{ex}}$;
- Re-cast in a form fit for the asymptotic analysis.

The thermodynamic limit I

Chemical potential $h \rightsquigarrow N$. *Thermodynamic limit* $N, L \to +\infty$, $N/L \to D$.

The ground state

$n_j = j \Rightarrow$ Bethe roots for GS *densify* on $[-q\,;\,q]$ with density $\rho(\lambda) \equiv \lim\limits_{\substack{L\to+\infty \\ \lambda_j \to \lambda}} \dfrac{1}{L\left(\lambda_j - \lambda_{j-1}\right)}$

$$\rho(\lambda) - \int_{-q}^{q} K(\lambda-\mu)\,\rho(\mu)\,\mathrm{d}\mu = \frac{p_0'(\lambda)}{2\pi} \qquad \text{and} \qquad D = \int_{-q}^{q} \rho(\lambda)\,\mathrm{d}\lambda$$

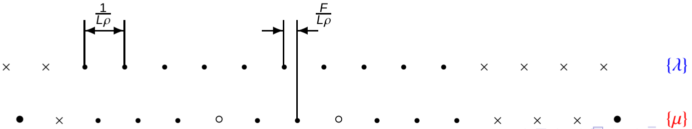
"particle-hole" excitations $\rightsquigarrow$ new choice of integers n_j:

$$n_j = j \;\text{ for }\; j \in \{1,\dots,N\} \setminus \{h_1,\dots,h_n\} \qquad \text{and} \qquad n_{h_a} = p_a \;\text{ for }\; a \in \{1,\dots,N\}$$

$\Rightarrow$ Excited state's roots μ shifted infinitesimally in respect to GS roots λ

$$\mu_j - \lambda_j = F\left(\lambda_j\right)/L\rho\left(\lambda_j\right) + \mathrm{O}\left(L^{-2}\right) \qquad j \in \{1,\dots,N\} \setminus \{h_1,\dots,h_n\}\ .$$

- "holes" in continuous distribution of rapidities at $\mu_{h_1}, \dots, \mu_{h_n}$
- new "particle" rapidities at $\mu_{p_1}, \dots, \mu_{p_n}$



The thermodynamic limit II

The *shift function* F solves a linear integral equation

$$F(\lambda) - \int_{-q}^{q} K(\lambda-\mu)\, F(\mu)\, \mathrm{d}\mu = -\frac{1}{2\pi} \sum_{p=1}^{n} \theta\left(\lambda - \mu_{p_a}\right) - \theta\left(\lambda - \mu_{h_a}\right) ,$$

With leading position of particles/holes $\xi\left(\mu_j\right) = \dfrac{j}{L}$ $\quad \xi(\lambda) = \dfrac{p}{2\pi} - \dfrac{N+1}{2L}$.
Additive structure in *"particles"* and *"holes"* rapidities.

$$F(\lambda) = \sum_{p=1}^{n} \phi\left(\lambda, \mu_{h_a}\right) - \phi\left(\lambda, \mu_{p_a}\right) \quad \text{where} \quad \phi(\lambda,\tau) - \int_{-q}^{q} K(\lambda-\mu)\, \phi(\mu,\tau)\, \mathrm{d}\mu = \frac{1}{2\pi}\theta(\lambda-\tau) .$$

$\rightsquigarrow$ Additive excitation spectrum.

$$\mathcal{P}_{\mathrm{ex}} - \mathcal{P}_{G.S.} = \sum_{a=1}^{n} p\left(\mu_{p_a}\right) - p\left(\mu_{h_a}\right) \qquad \mathcal{E}_{\mathrm{ex}} - \mathcal{E}_{G.S.} = \sum_{a=1}^{n} \epsilon\left(\mu_{p_a}\right) - \epsilon\left(\mu_{h_a}\right)$$

$$p(\lambda) = 2\pi \int_{0}^{\lambda} \rho(s)\, \mathrm{d}s \qquad \left(I - \frac{K}{2\pi}\right) \cdot \epsilon(\lambda) = \epsilon_0(\lambda) \quad \epsilon_0(\lambda) = \lambda^2 - h .$$

The main result

T=0K long-time & distance asymptotics of currents: $j(x,t) \equiv \mathrm{e}^{iHt}\Psi^{\dagger}(x)\Psi(x)\mathrm{e}^{-iHt}$

x/t fixed $\rightarrow$ $p(\lambda) - t\epsilon(\lambda)/x$ has a unique simple saddle-point λ_0: $p'(\lambda_0) - t\epsilon'(\lambda_0)/x = 0$.

In the space-like regime ($\lambda_0 > q$):

$$\langle j(x,t)\, j(0,0)\rangle = \left(\frac{p_F}{\pi}\right)^2 - \frac{\mathcal{Z}^2}{2\pi^2}\frac{x^2+t^2v_F^2}{\left(x^2-t^2v_F^2\right)^2} + \frac{2\cos(2xp_F)}{[-i(x-v_Ft)]^{\mathcal{Z}^2}\,[i(x+v_Ft)]^{\mathcal{Z}^2}}\left|\mathcal{F}\begin{pmatrix} q \\ -q\end{pmatrix}\right|^2(1+\mathrm{o}(1))$$

$$+\frac{p'(\lambda_0)\sqrt{2\pi}}{[i(t\epsilon''(\lambda_0)-xp''(\lambda_0))]^{\frac{1}{2}}}\,\frac{\exp\{i(xp(\lambda_0)-p_F-t\epsilon(\lambda_0))\}}{[-i(x-v_Ft)]^{(F_+(q)-1)^2}\,[i(x+v_Ft)]^{(F_+(-q))^2}}\left|\mathcal{F}\begin{pmatrix} \lambda_0 \\ q\end{pmatrix}\right|^2(1+\mathrm{o}(1))$$

$$+\frac{p'(\lambda_0)\sqrt{2\pi}}{[i(t\epsilon''(\lambda_0)-xp''(\lambda_0))]^{\frac{1}{2}}}\,\frac{\exp\{i(xp(\lambda_0)+p_F-t\epsilon(\lambda_0))\}}{[-i(x-v_Ft)]^{(F_-(q))^2}\,[i(x+v_Ft)]^{(F_-(-q)+1)^2}}\left|\mathcal{F}\begin{pmatrix} \lambda_0 \\ -q\end{pmatrix}\right|^2(1+\mathrm{o}(1))$$

$$F_{\pm}(\lambda) = \phi(\lambda,\pm q) - \phi(\lambda,\lambda_0) \qquad \mathcal{Z} = 1+\phi(q,-q)-\phi(q,q)\,.$$

v_F: Fremi velocity.

Asymptotic behavior of form factors: The result

NLSE, **Slavnov** **'90**, *XX* **Arikawa,Kabrach,Mller,Wiele** '06
6-Vertex R matrix **Kitanine, K, Maillet, Slavnov, Terras** '09-'10

- $\{\mu\}$: excited state with particles $\mu_{p_1}, \dots, \mu_{p_n}$ and holes $\mu_{h_1}, \dots, \mu_{h_n}$.
- $\{\lambda\}$ GS roots.
- F shift function associated with $\{\mu\}$ and $\{\lambda\}$.
- Power-law behavior in L with exponent $\alpha_c[F]$.

$$\left| \frac{\langle \{\mu\} | j(0,0) | \{\lambda\} \rangle}{\|\{\mu\}\| \cdot \|\{\lambda\}\|} \right|^2 = \left(\frac{2\pi}{L}\right)^{\alpha_c[F]} \cdot \mathcal{F}_n\left(\begin{array}{c} \{\mu_{p_a}\} \\ \{\mu_{h_a}\} \end{array} \right) \cdot \left(1 + \mathrm{O}\left(\frac{\ln L}{L}\right)\right) .$$

$$\mathcal{F}_n\left(\begin{array}{c} \{\mu_{p_a}\} \\ \{\mu_{h_a}\} \end{array} \right) = \mathcal{R}_n\left(\begin{array}{c} \{p_a\};\{\mu_{p_a}\} \\ \{h_a\};\{\mu_{h_a}\} \end{array} \right)[F] \cdot \mathcal{S}_n\left(\begin{array}{c} \{\mu_{p_a}\} \\ \{\mu_{h_a}\} \end{array} \right)[F]$$

- Particle-hole excitations **on** Fermi boundary $\quad \alpha_c[F] \equiv$ **critical exponent**

eg Particle at q / hole at $-q$ $\rightsquigarrow$ $\Delta E = 0 \;\; \Delta P = 2p_F$- excitation, $\quad \alpha_c[F] = 2\mathcal{Z}^2$

Asymptotic behavior of form factors: the problems

Form factors ≡ derivatives of scalar products $\left|\langle\, \{\mu\} \,| j\,(0,0) \,|\, \{\lambda\}\,\rangle\right|^2 = \partial_\kappa^2 \,|\langle \{\mu\}_\kappa \mid \{\lambda\}\rangle|^2$

scalar products **Slavnov** '89 ;

form factors **Izergin, Korepin** '84 , **Korepin, Slavnov** '99 , **Oota** '04 ;

$$\langle \{\mu\}_\kappa \mid \{\lambda\}\rangle = \frac{\det_N \left[\Omega_\kappa\left(\{\mu\}, \{\lambda\}\right)\right]}{\prod\limits_{j<k}^{N} \sinh\left(\mu_k - \mu_j\right) \sinh\left(\lambda_j - \lambda_k\right)}$$

Ω_κ has singularities at $\mu_j = \lambda_k$: $\left[\Omega_\kappa\left(\{\mu\}, \{\lambda\}\right)\right]_{jk} \sim \dfrac{1}{\mu_j - \lambda_k}$ $\quad L \to +\infty$??

⊛ **factorization of singularities**:

$$\left|\frac{\langle \{\mu\}_\kappa \mid \{\lambda\}\rangle}{\|\{\mu\}_\kappa\| \cdot \|\{\lambda\}\|}\right|^2 = \underbrace{\mathcal{G}_\kappa\begin{pmatrix} \{\mu\}_1^N \\ \{\lambda\}_1^N \end{pmatrix} \cdot \det\nolimits^2_{\Gamma([-\Lambda_h;\Lambda_h])}\left[I + \frac{\widehat{U}_\kappa^{(\lambda)}}{2i\pi}\right]}_{\text{Smooth limit } L\to+\infty} \cdot \underbrace{\det\nolimits_N^2\left[\frac{1}{\lambda_j - \mu_k}\right]}_{\text{Free fermionic like term}}$$

The generating function

$$\frac{1}{2}\frac{\partial^2}{\partial\kappa^2}\frac{\partial^2}{\partial x^2}\mathcal{L}(\kappa) = \langle j(x,t)\, j(0,0)\rangle \qquad \text{with} \qquad \mathcal{L}_N(\kappa) = \sum_{\substack{n_1<\dots<n_N \\ n_i\in\mathbb{Z}}} \mathrm{e}^{ix\mathcal{P}_{\mathrm{ex}}-it\mathcal{E}_{\mathrm{ex}}} \cdot \left| \frac{\langle \{\mu\}_\kappa \mid \{\lambda\}\rangle}{\|\{\mu\}_\kappa\| \cdot \|\{\lambda\}\|} \right|^2$$

$$\mathcal{P}_{\mathrm{ex}} = \sum_{a=1}^{n} p\left(\mu_{p_a}\right) - p\left(\mu_{h_a}\right) + \mathrm{O}\left(L^{-1}\right) \qquad \mathcal{E}_{\mathrm{ex}} = \sum_{a=1}^{n} \epsilon\left(\mu_{p_a}\right) - \epsilon\left(\mu_{h_a}\right) + \mathrm{O}\left(L^{-1}\right)$$

- Only implicit description of Bethe roots;
- each form factor introduce high couplings in the summation variables;
- Exact resummation possible with use of multidimensional residues;

reasonable simplifications

- Only states having the same per-site energy as GS contribute in $L \to +\infty$
- Only the leading in L behavior of individual form factors contributes
- The roots of excited states can be approximated by their leading position $\xi\left(\mu_{p_a}\right) = p_a/L$ and $\xi\left(\mu_{h_a}\right) = h_a/L$.

The effective form factor series I

$$\mathcal{L}_N(\kappa) = \sum_{n=0}^{N} \sum_{p_1<\cdots<p_n} \sum_{h_1<\cdots<h_n} \left(\frac{2\pi}{L}\right)^{\alpha_c[F]} \prod_{a=1}^{n} \left\{ \frac{e^{ixp(\mu_{p_a})-it\epsilon(\mu_{p_a})}}{e^{ixp(\mu_{h_a})-it\epsilon(\mu_{h_a})}} \right\} \cdot \mathcal{R}_n \begin{pmatrix} \{p_a\}\,;\{\mu_{p_a}\} \\ \{h_a\}\,;\{\mu_{h_a}\} \end{pmatrix} [F] \cdot \mathcal{S}_n \begin{pmatrix} \{\mu_{p_a}\} \\ \{\mu_{h_a}\} \end{pmatrix} [F]$$

- Summation dependent shift function $F(\lambda) = \dfrac{\ln \kappa}{2i\pi} Z + \sum_{a=1}^{n} \phi(\lambda; \mu_{h_a}) - \phi(\lambda; \mu_{p_a})$.
- Divergent sums not replaceable by integrals because of discreet part .
- Smooth part highly coupled, symmetric function with reduction properties

$$\mathcal{S}_n \begin{pmatrix} \{\mu_{p_a}\}_{a=1}^{n} \\ \{\mu_{h_a}\}_{a=1}^{n} \end{pmatrix} [G] \Bigg|_{\mu_{p_k}=\mu_{h_\ell}} = \mathcal{S}_{n-1} \begin{pmatrix} \{\mu_{p_a}\}_{\substack{a=1 \\ \neq k}}^{n} \\ \{\mu_{h_a}\}_{\substack{a=1 \\ \neq \ell}}^{n} \end{pmatrix} [G] \quad .$$

$$\mathcal{G}[F] = \prod_{a=1}^{n} \exp\left\{ \int_{\mathbb{R}} \phi(\lambda, \mu_{h_a}) - \phi(\lambda, \mu_{p_a}) \frac{\delta}{\delta\rho(\lambda)} \right\} \cdot \mathcal{G}[\nu_\rho] \quad \text{with} \quad \nu_\rho = \frac{\beta Z}{2i\pi} + \rho \,.$$

The effective form factor series II

Archeologist theorem

Every symmetric holomorphic function of n variables $\{\lambda_j\}_1^n$ anf n variables $\{\mu_j\}_1^n$ that has the reduction property admits the decomposition with holomorphic functions g_r

$$\mathcal{S}_n\begin{pmatrix} \{\lambda_j\}_{j=1}^n \\ \{\mu_j\}_{j=1}^n \end{pmatrix} = \sum_{r\geq 0} \alpha_r \prod_{a=1}^{n} \mathrm{e}^{g_r(\lambda_a)-g_r(\mu_a)} \, .$$

$$\mathcal{L}_N(\kappa) =: \sum_{r\geq 0} \alpha_r \exp\left\{-\int_{-q}^{q} \ln'\left[\widehat{E}_-^2(\lambda)\right]\nu_\rho(\lambda)\right\} \cdot X_N\left[\nu_\rho, \widehat{E}_-^2\right] : \quad \widehat{E}_-^2(\lambda) = \mathrm{e}^{ixp(\lambda)-it\epsilon(\lambda)+g(\lambda)}$$

$$g(\lambda) = g_r(\lambda) + \int_{\mathbb{R}} \mathrm{d}\mu \phi(\mu,\lambda) \frac{\delta}{\delta\rho(\mu)}$$

⊛ $: \cdot :$ operator order "all functional derivative to the left"

$$X_N\left[\nu, E_-^2\right] = \sum_{n=0}^{N} \sum_{p_1<\dots<p_n} \sum_{h_1<\dots<h_n} \left(\frac{2\pi}{L}\right)^{\alpha_c[\nu]} \prod_{a=1}^{n} \frac{E_-^2(\mu_{h_a})}{E_-^2(\mu_{p_a})} \cdot \mathcal{R}_n\begin{pmatrix} \{p_a\};\{\mu_{p_a}\} \\ \{h_a\};\{\mu_{h_a}\} \end{pmatrix}[\nu]$$

where $\xi\left(\mu_{p_a}\right) = p_a/L$.

Fredholm determinant representation for $X_N\left[\nu, E_-^2\right]$

Introduce two sets of parameters $\widehat{\xi}_\lambda\left(\lambda_j\right)=j/L$ and $\widehat{\xi}_\mu\left(\mu_j\right)=j/L$:

$$\widehat{\xi}_\mu=\frac{p}{2\pi}+\frac{N+1}{2L}\,,\qquad \text{and}\quad \widehat{\xi}_\lambda=\frac{p}{2\pi}+\frac{\nu}{L}+\frac{N+1}{2L}\,.$$

$$X_N\left[\nu, E_-^2\right]=\prod_{j=1}^{N}\left\{2\sin\left[\pi\nu\left(\lambda_j\right)\right]\right\}^2\sum_{n_1<\cdots<n_N}\prod_{j=1}^{N}\frac{E_-^2\left(\lambda_j\right)/E_-^2\left(\mu_{n_j}\right)}{4\pi^2\widehat{\xi}_\mu'\left(\mu_{n_j}\right)\widehat{\xi}_\lambda'\left(\lambda_j\right)}\cdot\det{}_N^2\left[\frac{1}{L\left(\mu_{n_j}-\lambda_k\right)}\right].$$

- separation of sums is possible

$$X_N\left[\nu, E_-^2\right]\underset{N,L\to+\infty}{\longrightarrow}\det{}_{[-q;q]}\left[I+V\right]$$

- $I+V$ is an integrable integral operator with kernel

$$V\left(\lambda,\mu\right)=4\frac{\sin\left[\pi\nu\left(\lambda\right)\right]\sin\left[\pi\nu\left(\mu\right)\right]}{2i\pi\left(\lambda-\mu\right)}\left\{E_+\left(\lambda\right)E_-\left(\mu\right)-E_+\left(\mu\right)E_-\left(\lambda\right)\right\}$$

$$E_+\left(\lambda\right)=iE_-\left(\lambda\right)\left\{\fint_{\mathbb{R}}\frac{\mathrm{d}\mu}{2\pi}\,\frac{E_-^{-2}\left(\mu\right)}{\mu-\lambda}+\frac{E_-^{-2}\left(\lambda\right)}{2}\cot\pi\nu\left(\lambda\right)\right\}.$$

Natte series for Fredholm determinants

RHP **Its, Izergin, Korepin, Slavnov** '90

A.E. **Deift, Zhou** '92 **Its, Izergin, Korepin, Varzugin** '92

$$\det\left[I+V\right]=\det\left[I+V\right]^{(0)}\left\{1+\sum_{n\geq 1}\sum_{\Sigma s\ell_s=n}\sum_{\substack{\Sigma_{t_3}\epsilon_t=0\\ \epsilon_t\in\{\pm1,0\}}}\int_{\mathscr{C}}\frac{\mathrm{d}^n z_t}{(2i\pi)^n}H_n\left(\{\ell\},\{\epsilon_t\},\{z_t\}\right)[\nu]\prod_{t\in J_{\{\ell\}}}\mathrm{e}^{\epsilon_t g(z_t)}\right\}.$$

$E_-^2=\mathrm{e}^{ixu+g}$ and $J_{\{\ell\}}$ set of triplets $\{(s,p,j)\ ,\ s\in[\![1\,;\,n]\!]\,,\ p\in[\![1\,;\,\ell_s]\!]\,,\ j\in[\![1\,;\,s]\!]\}$.

$$\det\left[I+V\right]^{(0)}=\mathcal{B}_x\left[\nu,u\right]\exp\left\{\int_{-q}^{q}\ln'\left[E_-^2\left(\lambda\right)\right]\nu\left(\lambda\right)\right\}\quad\text{with}\quad\mathcal{B}_x\left[\nu,u\right]=\frac{C\left[\nu,u\right]}{x^{\nu_+^2+\nu_-^2}}$$

- Functionals H_n are subdominant $\left|H_n\left(\{\ell\},\{\epsilon_t\},\{z_t\}\right)[\nu]\right|=\mathrm{o}\left(1\right)$ in L^1 sense
- Natte series obtained through *algebraic* manipulations on Fredholm series.
- perfectly fit for computing the asymptotic $x\to+\infty$ expansion of the determinant.

multidimensional Natte series for $\mathcal{L}(\kappa)$

It is allowed to send $N, L \to +\infty$ *directly* in the expansion for $\mathcal{L}_N(\kappa)$:

$$\mathcal{L}_N(\kappa) \to \mathcal{L}_N(\kappa) =: \sum_{r \geq 0} \alpha_r \exp\left\{ - \int_{-q}^{q} \ln'\left[\widehat{E}_-^2(\lambda)\right] \nu(\lambda) \right\} \cdot \det_{[-q;q]}[I+V]\left[\nu_\rho, \widehat{E}_-^2\right] :$$

⊛ One can then compute the sums over r and the effect of functional translations:

- Functional translation $\nu_\rho(\lambda) \mapsto \nu_{\epsilon_t}(\lambda; \{z_t\}) = \dfrac{\ln \kappa}{2i\pi} Z(\lambda) - \sum_{t \in J_{\{\ell\}}} \phi(\lambda, z_t)$;
- Sum over r reconstructs the smooth part $\mathcal{S}$

$$\mathcal{L}(\kappa)\kappa^{-\frac{xp_F}{\pi}} = \mathcal{B}_x[\nu_0, p - t\epsilon/x]\mathcal{S}_0[\nu_0] + \sum_{n \geq 1} \sum_{\Sigma s\ell_s = n} \sum_{\substack{\Sigma_{t_3} \epsilon_t = 0 \\ \epsilon_t \in \{\pm 1, 0\}}} \int_{\mathscr{C}} \frac{d^n z_t}{(2i\pi)^n}$$

$$\times H_n(\{\ell\}, \{\epsilon_t\} \mid \{z_t\}) \left[\nu_{\epsilon_t}(\cdot, \{z_t\})\right] \cdot \mathcal{S}_{a_{\{\epsilon_t\}}} \begin{pmatrix} \{z_t\}_{\epsilon_t > 0} \\ \{z_t\}_{\epsilon_t < 0} \end{pmatrix} \left[\nu_{\epsilon_t}(\cdot; \{z_t\})\right] \mathcal{B}_x\left[\nu_{\epsilon_t}(\cdot, \{z_t\}), p - t\epsilon/x\right].$$

Conslusion and perspectives

Review of the results

- ✔ Method for computing size-asymptotics of form factors ;
- ✔ Construction of Natte series for Fredholm determinant ;
- ✔ Application of Natte series for asymptoic analysis of correlation functions .

Next possible extensions

- ⊛ Field-conjugated field correlator in NLSE ;
- ⊛ Asymptotic behavior for $T > 0K$ and for massive models ;
- ⊛ Applying the method to models with a more complex description of Bethe roots (XXZ) .