Finite-Size Dyonic Giant Magnons in TsT-transformed $AdS_5 \times S^5$

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Outline

Introduction

Strings on $AdS_5 \times S^5 / \mathcal{N} = 4$ SYM duality

- Giant Magnons
- Finite-size corrections

Strings on $AdS_5 \times S^5_{\gamma_i}$/non-SUSY deformation of $\mathcal{N} = 4$ SYM

- γ-deformed Giant Magnons
- Finite-size corrections

Strings on $R_t \times S^3$ and the NR Integrable System

- Lagrangian and Virasoro constraints

$$\mathcal{L}_s = \frac{T}{2}\left(G_{00} - G_{11}\right), \qquad G_{ab} = g_{MN}\partial_a X^M \partial_b X^N,$$
$$G_{00} + G_{11} = 0, \qquad G_{01} = 0$$

- Embedding

$$Z_0 = Re^{it(\tau,\sigma)}, \qquad W_j = Rr_j(\tau,\sigma)e^{i\phi_j(\tau,\sigma)}, \qquad \sum_{j=1}^{2} W_j\bar{W}_j = R^2$$
$$G_{ab} = R^2\left[-\partial_a t\partial_b t + \sum_{j=1}^{2}\left(\partial_a r_j\partial_b r_j + r_j^2\partial_a\phi_j\partial_b\phi_j\right)\right]$$
$$\mathcal{L} = \mathcal{L}_s + \Lambda_s\left(\sum_{j=1}^{2} r_j^2 - 1\right)$$

Strings on $R_t \times S^3$ and the NR Integrable System

Conserved Quantities

The string energy E_s and two angular momenta J_j

$$E_s = -\int d\sigma \frac{\partial \mathcal{L}_s}{\partial(\partial_0 t)}, \qquad J_j = \int d\sigma \frac{\partial \mathcal{L}_s}{\partial(\partial_0 \phi_j)}$$

Strings on $R_t \times S^3$ and the NR Integrable System

NR Quantities

- Neumann-Rosochatius ansatz

$$t(\tau,\sigma) = \kappa\tau, \qquad r_j(\tau,\sigma) = r_j(\xi), \qquad \phi_j(\tau,\sigma) = \omega_j\tau + f_j(\xi),$$
$$\xi = \alpha\sigma + \beta\tau, \qquad \kappa, \omega_j, \alpha, \beta = constants$$

- NR Lagrangian

$$L_{NR} = (\alpha^2 - \beta^2)\sum_{j=1}^{2}\left[r_j'^2 - \frac{1}{(\alpha^2-\beta^2)^2}\left(\frac{C_j^2}{r_j^2} + \alpha^2\omega_j^2 r_j^2\right)\right] + \Lambda_s\left(\sum_{j=1}^{2} r_j^2 - 1\right)$$

C_j are integration constants coming from single time integration of the equations of motion for $f_j(\xi)$:

$$f_j' = \frac{1}{\alpha^2-\beta^2}\left(\frac{C_j}{r_j^2} + \beta\omega_j\right)$$

Strings on $R_t \times S^3$ and the NR Integrable System

NR Quantities

- NR Hamiltonian The Virasoro constraints give the conserved Hamiltonian H_{NR} and a relation between the parameters

$$H_{NR} = (\alpha^2 - \beta^2) \sum_{j=1}^{2} \left[r_j'^2 + \frac{1}{(\alpha^2 - \beta^2)^2} \left(\frac{C_j^2}{r_j^2} + \alpha^2 \omega_j^2 r_j^2 \right) \right] = \frac{\alpha^2 + \beta^2}{\alpha^2 - \beta^2} \kappa^2,$$

$$\sum_{j=1}^{2} C_j \omega_j + \beta \kappa^2 = 0$$

Strings on $R_t \times S^3$ and the NR Integrable System

The Solution

In order to identically satisfy the embedding condition

$$\sum_{j=1}^{2} r_j^2 - 1 = 0,$$

we introduce a new variable $\theta(\xi)$ by

$$r_1(\xi) = \sin\theta(\xi), \qquad r_2(\xi) = \cos\theta(\xi)$$

Then

$$\begin{aligned} \theta'(\xi) &= \pm\frac{1}{\alpha^2-\beta^2}\left[(\alpha^2+\beta^2)\kappa^2 - \frac{C_1^2}{\sin^2\theta} - \frac{C_2^2}{\cos^2\theta} - \alpha^2\left(\omega_1^2\sin^2\theta + \omega_2^2\cos^2\theta\right)\right]^{1/2} \\ &\equiv \pm\frac{1}{\alpha^2-\beta^2}\,\Theta(\theta), \end{aligned}$$

Strings on $R_t \times S^3$ and the NR Integrable System

The Solution

which can be integrated to give

$$\xi(\theta) = \pm(\alpha^2 - \beta^2) \int \frac{d\theta}{\Theta(\theta)},$$
$$f_1 = \frac{\beta\omega_1\xi}{\alpha^2 - \beta^2} \pm C_1 \int \frac{d\theta}{\sin^2\theta\, \Theta(\theta)},$$
$$f_2 = \frac{\beta\omega_2\xi}{\alpha^2 - \beta^2} \pm C_2 \int \frac{d\theta}{\cos^2\theta\, \Theta(\theta)}.$$

All these solve formally the NR system for the present case.

Finite-Size Effects

- Strings on $R_t \times S^2$
- Strings on $R_t \times S^3$

Strings on $R_t \times S^2$

The Giant Magnon

- String solution:

$$W_1 = R\sqrt{1-\left(1-\beta^2\kappa^2/\alpha^2\omega_1^2\right) dn^2\left(C\xi|m\right)}$$
$$\times \exp\left\{\frac{i\omega_1}{1-\beta^2/\alpha^2}\left[(1-\beta^2\kappa^2/\alpha^2\omega_1^2)\tau + (1-\kappa^2/\omega_1^2)\frac{\beta}{\alpha}\sigma\right]\right.$$
$$\left. + \frac{i\beta(1-\kappa^2/\omega_1^2)}{\alpha\omega_1\sqrt{1-\beta^2/\alpha^2}}\Pi\left(am(C\xi-\mathbf{K}), -\frac{m}{\kappa^2}\middle|m\right)\right\},$$
$$W_2 = R\sqrt{1-\beta^2\kappa^2/\alpha^2\omega_1^2}\, dn\left(C\xi|m\right), \qquad Z_0 = R\exp(i\kappa\tau),$$

where $\sigma \in [-\sigma_0, \sigma_0]$ and

$$C = \mp\frac{\omega_1\sqrt{1-\beta^2\kappa^2/\alpha^2\omega_1^2}}{\alpha(1-\beta^2/\alpha^2)}, \qquad m \equiv \frac{\kappa^2(1-\beta^2/\alpha^2)}{\omega_1^2(1-\beta^2\kappa^2/\alpha^2\omega_1^2)},$$

Strings on $R_t \times S^2$

The Giant Magnon

- The corresponding solution of the (C)SG:

$$\sin^2(\phi/2) = \frac{\kappa^2\left(1-\kappa^2/\omega_1^2\right) sn^2\left(C\xi - \mathbf{K}|m\right)}{M^2\left(1-\beta^2\kappa^2/\alpha^2\omega_1^2\right) dn^2\left(C\xi - \mathbf{K}|m\right)}.$$

This solution of the CSG system reduces to that of the SG equation for $M^2 = \kappa^2$.

Strings on $R_t \times S^2$

The Giant Magnon

- Conserved Quantities and Worldsheet Momentum:

$$\mathcal{E}_s \equiv \frac{2\pi}{\sqrt{\lambda}} E_s = 2\sqrt{(1-v^2)(1-\epsilon)}\mathbf{K}(1-\epsilon),$$

$$\mathcal{J} \equiv \frac{2\pi}{\sqrt{\lambda}} J_1 = 2\sqrt{\frac{1-v^2}{1-v^2\epsilon}}\left[\mathbf{K}(1-\epsilon) - \mathbf{E}(1-\epsilon)\right],$$

$$p = 2v\sqrt{\frac{1-v^2\epsilon}{1-v^2}}\left[\frac{1}{v^2}\Pi\left(1-\frac{1}{v^2}\middle|1-\epsilon\right) - \mathbf{K}(1-\epsilon)\right],$$

where

$$p \equiv \Delta\varphi_1 = \varphi_1(\tau,\sigma_0) - \varphi_1(\tau,-\sigma_0), \qquad \epsilon \equiv 1-m, \qquad v \equiv -\beta/\alpha.$$

Strings on $R_t \times S^2$

The Giant Magnon

We are interested in the behavior of these quantities in the limit $\epsilon \to 0$. Our approach is as follows. We introduce $v(\epsilon)$ according to the rule

$$v(\epsilon) = v_0(p) + v_1(p)\epsilon + v_2(p)\epsilon \log(\epsilon)$$

and expand $\mathcal{E}_s$, $\mathcal{J}$ and p about $\epsilon = 0$. For p to be finite, we find

$$v_0(p) = \cos(p/2), \qquad v_1(p) = \frac{1}{4}\sin^2(p/2)\cos(p/2)(1 - \log(16)),$$
$$v_2(p) = \frac{1}{4}\sin^2(p/2)\cos(p/2).$$

Strings on $R_t \times S^2$

The Giant Magnon

After that, from the expansion for $\mathcal{J}$, we obtain ϵ as a function of $\mathcal{J}$ and p

$$\epsilon = 16 \exp\left(-\frac{\mathcal{J}}{\sin(p/2)} - 2\right).$$

Finally, using all these in the expansion for $\mathcal{E}_s - \mathcal{J}$, we derive

$$\mathcal{E}_s - \mathcal{J} = 2\sin(p/2)\left[1 - 4\sin^2(p/2)\exp\left(-\frac{\mathcal{J}}{\sin(p/2)} - 2\right)\right],$$

which reproduces the leading finite-$\mathcal{J}$ correction to the GM energy-charge relation derived earlier.

- Remark: The length of the GM string is proportional to $\mathcal{J}$

Strings on $R_t \times S^3$

String solution

$$Z_0 = R \exp(i\kappa\tau),$$

$$W_1 = R\sqrt{1 - z_+^2 dn^2\left(C\xi|m\right)} \exp\left\{ i\omega_1\tau + \frac{2i\beta/\alpha}{z_+\sqrt{1-\omega_2^2/\omega_1^2}} \right.$$

$$\left. \times \left[F\left(am(C\xi)|m\right) - \frac{\kappa^2/\omega_1^2}{1-z_+^2}\Pi\left(am(C\xi), -\frac{z_+^2 - z_-^2}{1-z_+^2}\middle| m\right)\right]\right\},$$

$$W_2 = Rz_+ dn\left(C\xi|m\right) \exp\left\{ i\omega_2\tau + \frac{2i\beta\omega_2/\alpha\omega_1}{z_+\sqrt{1-\omega_2^2/\omega_1^2}} F\left(am(C\xi)|m\right)\right\},$$

Strings on $R_t \times S^3$

String solution

where

$$z_\pm^2 = \frac{1}{2(1-\frac{\omega_2^2}{\omega_1^2})}\left\{y_1 + y_2 - \frac{\omega_2^2}{\omega_1^2} \pm \sqrt{(y_1-y_2)^2 - \left[2\left(y_1+y_2-2y_1y_2\right) - \frac{\omega_2^2}{\omega_1^2}\right]\frac{\omega_2^2}{\omega_1^2}}\right\},$$

$$y_1 = 1-\kappa^2/\omega_1^2, \qquad y_2 = 1-\beta^2\kappa^2/\alpha^2\omega_1^2,$$

$$C = \mp\frac{\alpha\sqrt{\omega_1^2-\omega_2^2}}{\alpha^2-\beta^2}z_+, \qquad m \equiv 1 - z_-^2/z_+^2.$$

- Note This string solution contains both cases: $\alpha^2 > \beta^2$ for the GM and $\alpha^2 < \beta^2$ for the SS.

Strings on $R_t \times S^3$

The “dual” CSG solution

For the present case, the CSG field $\psi = \sin(\phi/2)\exp(i\chi/2)$ is defined by

$$\sin^2(\phi/2) = \frac{\omega_1^2/M^2}{\beta^2/\alpha^2 - 1}\left[\left(1 - \kappa^2/\omega_1^2\right) - \left(1 - \omega_2^2/\omega_1^2\right)\left(z_+^2 cn^2(C\xi|m) + z_-^2 sn^2(C\xi|m)\right)\right]$$

$$\chi = \frac{A}{\beta}(\beta\sigma + \alpha\tau) - C_\chi(\alpha\sigma + \beta\tau) + \frac{C_\chi}{CD}\Pi\left(am(C\xi), n|m\right),$$

where

$$D = \frac{\omega_1^2/M^2}{\beta^2/\alpha^2 - 1}\left[\left(1 - \kappa^2/\omega_1^2\right) - \left(1 - \omega_2^2/\omega_1^2\right) z_+^2\right], \qquad n = \frac{\left(1 - \omega_2^2/\omega_1^2\right)\left(z_+^2 - z_-^2\right)}{\left(1 - \kappa^2/\omega_1^2\right) - \left(1 - \omega_2^2/\omega_1^2\right) z_+^2}.$$

Strings on $R_t \times S^3$

The Giant Magnon

- Conserved Quantities and Worldsheet Momentum:

$$\mathcal{E}_s = \frac{2\kappa(1-\beta^2/\alpha^2)}{\omega_1 z_+ \sqrt{1-\omega_2^2/\omega_1^2}} \mathbf{K}\left(1 - z_-^2/z_+^2\right),$$

$$\mathcal{J}_1 = \frac{2z_+}{\sqrt{1-\omega_2^2/\omega_1^2}} \left[\frac{1-\beta^2\kappa^2/\alpha^2\omega_1^2}{z_+^2} \mathbf{K}\left(1 - z_-^2/z_+^2\right) - \mathbf{E}\left(1 - z_-^2/z_+^2\right)\right],$$

$$\mathcal{J}_2 = \frac{2z_+\omega_2/\omega_1}{\sqrt{1-\omega_2^2/\omega_1^2}} \mathbf{E}\left(1 - z_-^2/z_+^2\right),$$

$$p = -\frac{2\beta/\alpha}{z_+\sqrt{1-\omega_2^2/\omega_1^2}} \left[\frac{\kappa^2/\omega_1^2}{1-z_+^2} \Pi\left(-\frac{z_+^2 - z_-^2}{1-z_+^2} \middle| 1 - z_-^2/z_+^2\right) - \mathbf{K}\left(1 - z_-^2/z_+^2\right)\right].$$

Strings on $R_t \times S^3$

The Giant Magnon

- The Technology I : We introduce the new parameters

$$u \equiv \omega_2^2/\omega_1^2, \qquad v \equiv -\beta/\alpha, \qquad \epsilon \equiv z_-^2/z_+^2.$$

This will allow us to eliminate κ/ω_1 and $z_\pm$ from the coefficients in the previous expressions and rewrite them as functions of u, v and ϵ only:

$$\begin{aligned}
\mathcal{E}_s &= 2K_e \mathbf{K}\left(1-\epsilon\right), \\
\mathcal{J}_1 &= 2K_{11}\left[K_{12}\mathbf{K}\left(1-\epsilon\right) - \mathbf{E}\left(1-\epsilon\right)\right], \\
\mathcal{J}_2 &= 2K_2 \mathbf{E}\left(1-\epsilon\right), \\
p &= 2K_{\varphi 1}\left[K_{\varphi 2}\Pi\left(K_{\varphi 3}|1-\epsilon\right) - \mathbf{K}\left(1-\epsilon\right)\right].
\end{aligned}$$

Strings on $R_t \times S^3$

The Giant Magnon

- The Technology II : We introduce $u(\epsilon)$ and $v(\epsilon)$ according to the rule

$$u(\epsilon) = u_0 + u_1\epsilon + u_2\epsilon\log(\epsilon), \qquad v(\epsilon) = v_0 + v_1\epsilon + v_2\epsilon\log(\epsilon),$$

and expand $\mathcal{E}$, $\mathcal{J}_1$, $\mathcal{J}_2$ and p about $\epsilon = 0$. Requiring $\mathcal{J}_2$ and p to be finite, we find

$$u_0 = \frac{\mathcal{J}_2^2}{\mathcal{J}_2^2 + 4\sin^2(p/2)}, \qquad v_0 = \frac{\sin(p)}{\sqrt{\mathcal{J}_2^2 + 4\sin^2(p/2)}}$$

$$u_1 = u_1(u_0, v_0), \qquad v_1 = v_1(u_0, v_0), \qquad u_2 = u_2(u_0, v_0), \qquad v_2 = v_2(u_0, v_0).$$

Strings on $R_t \times S^3$

The Giant Magnon

- The Technology III : The parameter ϵ can be obtained from the expansion for $\mathcal{J}_1$

$$\epsilon = 16 \exp\left[-\left(\sqrt{1-u_0-v_0^2}\,\mathcal{J}_1 + 2\left(1-v_0^2/\left(1-u_0\right)\right)\right)/(1-v_0^2)\right].$$

Using all of the above in the expansion for $\mathcal{E}_s - \mathcal{J}_1$, one arrives at

$$\mathcal{E}_s - \mathcal{J}_1 = \sqrt{\mathcal{J}_2^2 + 4\sin^2(p/2)} - \frac{16\sin^4(p/2)}{\sqrt{\mathcal{J}_2^2 + 4\sin^2(p/2)}}$$
$$\exp\left[-\frac{2\left(\mathcal{J}_1 + \sqrt{\mathcal{J}_2^2 + 4\sin^2(p/2)}\right)\sqrt{\mathcal{J}_2^2 + 4\sin^2(p/2)}\,\sin^2(p/2)}{\mathcal{J}_2^2 + 4\sin^4(p/2)}\right].$$

- Conclusion : This energy-charge relation coincides with the one found recently, describing the finite-size effects for dyonic GM.

Finite-Size Effects- Deformed Case

An interesting example of correspondence between gauge and string theory models with reduced supersymmetry is provided by an exactly marginal deformation of $\mathcal{N} = 4$ SYM and string theory on a β-deformed $AdS_5 \times S^5$ background. When $\beta \equiv \gamma$ is real, the deformed background can be obtained from $AdS_5 \times S^5$ by the so-called TsT transformation. It includes T-duality on one angle variable, a shift of another isometry variable, then a second T-duality on the first angle. Taking into account that the five-sphere has three isometric coordinates, one can consider generalization of the above procedure, consisting of chain of three TsT transformations. The result is a regular three-parameter deformation of $AdS_5 \times S^5$ string background, dual to a non-supersymmetric deformation of $\mathcal{N} = 4$ super Yang-Mills, which is conformal in the planar limit to any order of perturbation theory. The action for this γ_i-deformed $(i = 1, 2, 3)$ gauge theory can be obtained from the initial one after replacement of the usual product with associative $*$-product.

Finite-Size Effects- Deformed Case

An essential property of the TsT transformation is that it preserves the classical integrability of string theory on $AdS_5 \times S^5$. The γ-dependence enters only through the *twisted* boundary conditions and the *level-matching* condition. The last one is modified since a closed string in the deformed background corresponds to an open string on $AdS_5 \times S^5$ in general.
The finite-size correction to the giant magnon energy-charge relation, in the γ-deformed background, has been found by Bykov and Frolov, by using conformal gauge and the string sigma model reduced to $R_t \times S^3$. For the deformed case, this is the smallest consistent reduction due to the *twisted* boundary conditions. It turns out that even for the three-parameter deformation, the reduced model depends only on one of them - γ_3. As far as there are two isometry angles ϕ_1, ϕ_2 on S^3, the solution can carry two non-vanishing angular momenta J_1, J_2. Then, the giant magnon is an open string solution with only one charge $J_1 \neq 0$. The momentum p of the magnon excitation in the corresponding spin chain is identified with the angular difference $\Delta\phi_1$ between the end-points of the string, since in the light-cone gauge $t = \tau$, $p_{\phi_1} = 1$, it is equal to the worldsheet momentum p_{ws} of a soliton . The other angle satisfies the following *twisted* boundary conditions $\Delta\phi_2 = 2\pi(n_2 - \gamma_3 J_1)$, where n_2 is an integer winding number of the string in the second isometry direction of the deformed sphere S^3_γ.

Finite-Size Effects- Deformed Case

An interesting extension of this study is the dyonic giant magnon. This state corresponds to bound states of the fundamental magnons and stable even in the deformed theory. Understanding its string theory analog in the strong coupling limit can be helpful to extend the AdS/CFT duality to the deformed theories.

Finite-Size Effects- Deformed Case

The bosonic part of the Green-Schwarz action for strings on the γ-deformed $AdS_5 \times S^5_\gamma$ reduced to $R_t \times S^5_\gamma$

$$\begin{aligned} S &= -\frac{T}{2}\int d\tau d\sigma \Big\{ \sqrt{-\gamma}\gamma^{ab}\Big[-\phi_a t\phi_b t + \phi_a r_i \phi_b r_i + G r_i^2 \phi_a\varphi_i\phi_b\varphi_i \\ &+ G r_1^2 r_2^2 r_3^2 \left(\hat\gamma_i \phi_a\varphi_i\right)\left(\hat\gamma_j\phi_b\varphi_j\right)\Big] \\ &- 2G\,\epsilon^{ab}\left(\hat\gamma_3 r_1^2 r_2^2 \phi_a\varphi_1\phi_b\varphi_2 + \hat\gamma_1 r_2^2 r_3^2 \phi_a\varphi_2\phi_b\varphi_3 + \hat\gamma_2 r_3^2 r_1^2\phi_a\varphi_3\phi_b\varphi_1\right)\Big\}, \end{aligned} \tag{1}$$

where T is the string tension, γ^{ab} is the worldsheet metric, φ_i are the three isometry angles of the deformed S^5_γ, and

$$\sum_{i=1}^{3} r_i^2 = 1, \qquad G^{-1} = 1 + \hat\gamma_3 r_1^2 r_2^2 + \hat\gamma_1 r_2^2 r_3^2 + \hat\gamma_2 r_1^2 r_3^2. \tag{2}$$

Finite-Size Effects- Deformed Case

The deformation parameters $\hat{\gamma}_i$ are related to γ_i which appear in the dual gauge theory as

$$\hat{\gamma}_i = 2\pi T \gamma_i = \sqrt{\lambda}\gamma_i.$$

When $\hat{\gamma}_i = \hat{\gamma}$ this becomes a supersymmetric background, and the deformation parameter γ enters the $\mathcal{N} = 1$ SYM superpotential in the following way

$$W \propto tr\left(e^{i\pi\gamma}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\gamma}\Phi_1\Phi_3\Phi_2\right).$$

Finite-Size Effects- Deformed Case

By using the TsT transformations which map the string theory on $AdS_5 \times S^5$ to the γ_i-deformed theory, one can relate the angle variables ϕ_i on S^5 to the angles φ_i of the γ_i-deformed geometry :

$$p_i = \pi_i, \qquad r_i^2 \phi_i' = r_i^2 \left(\varphi_i' - 2\pi\epsilon_{ijk}\gamma_j p_k\right), \qquad i = 1, 2, 3, \tag{3}$$

where p_i, π_i are the momenta conjugated to ϕ_i, φ_i respectively, and the summation is over j, k. The equality $p_i = \pi_i$ implies that the charges

$$J_i = \int d\sigma p_i$$

are invariant under the TsT transformation.

Finite-Size Effects- Deformed Case

If none of the variables r_i is vanishing on a given string solution, one gets

$$\phi_i' = \varphi_i' - 2\pi\epsilon_{ijk}\gamma_j p_k .$$

Integrating the above equations and taking into account that for a closed string in the γ-deformed background

$$\Delta\varphi_i = \varphi_i(r) - \varphi_i(-r) = 2\pi n_i, \qquad n_i \in \mathbb{Z},$$

one finds the *twisted* boundary conditions for the angles ϕ_i on the original S^5 space

$$\Delta\phi_i = \phi_i(r) - \phi_i(-r) = 2\pi\left(n_i - \nu_i\right), \qquad \nu_i = \varepsilon_{ijk}\gamma_j J_k, \qquad J_i = \int_{-r}^{r} d\sigma p_i.$$

If the *twists* ν_i are not integer, then a closed string on the deformed background is mapped to an open string on $AdS_5 \times S^5$.

Finite-Size Effects- Deformed Case

The particular case considered by Bykov and Frolov corresponds to $J_2 = J_3 = 0$, $\nu_1 = 0$, and as a result the angles $\phi_{1,2}$ of the undeformed S^3 satisfy the following *twisted* boundary conditions

$$p = \Delta\phi_1 = \phi_1(r) - \phi_1(-r), \qquad \delta = \Delta\phi_2 = \phi_2(r) - \phi_2(-r) = 2\pi\left(n_2 - \gamma_3 J_1\right),$$

where in fact δ plays the role of the deformation parameter. By using the ansatz

$$\phi_1 = \omega\tau + \frac{p}{2r}(\sigma - v\tau) + \phi(\sigma - v\tau),$$
$$\phi_2 = \nu\tau + \frac{\delta}{2r}(\sigma - v\tau) + \alpha(\sigma - v\tau),$$
$$\chi = \chi(\sigma - v\tau),$$

where ϕ, α and χ satisfy periodic boundary conditions, they found that the giant magnon string solution can be completely determined from the equations

Finite-Size Effects- Deformed Case

$$\begin{aligned}
&\mathcal{E} \equiv \frac{E_s}{\frac{\sqrt{\lambda}}{2\pi}} = 2\int_{-r}^{0} d\sigma = 2r, \\
&\mathcal{J}_1 = \frac{J_1}{\frac{\sqrt{\lambda}}{2\pi}} = \frac{2}{1-v^2}\left(rv^2 A_1 + \omega \int_{\chi_{min}}^{\chi_{max}} d\chi \frac{1-\chi}{|\chi'|}\right), \\
&\mathcal{J}_2 = \frac{J_1}{\frac{\sqrt{\lambda}}{2\pi}} \propto rv^2 A_2 + \nu \int_{\chi_{min}}^{\chi_{max}} d\chi \frac{\chi}{|\chi'|} = 0, \\
&\frac{p}{2} + \frac{rv\omega}{1-v^2} = -\frac{vA_1}{1-v^2}\int_{\chi_{min}}^{\chi_{max}} \frac{d\chi}{(1-\chi)|\chi'|}, \\
&\delta + \frac{rv\nu}{1-v^2} = -\frac{vA_2}{1-v^2}\int_{\chi_{min}}^{\chi_{max}} \frac{d\chi}{\chi|\chi'|},
\end{aligned} \tag{4}$$

Finite-Size Effects- Deformed Case

A_1 and A_2 are parameters related by $\omega A_1 + \nu A_2 + 1 = 0$, $\chi = 1 - r_1^2 = r_2^2$, and

$$|\chi'| = \frac{2\sqrt{\omega^2 - \nu^2}}{1 - v^2}\sqrt{(\chi_{max} - \chi)(\chi - \chi_{min})(\chi - \chi_n)},$$
$$0 < \chi_{min} < \chi < \chi_{max} < 1, \qquad \chi_n < 0.$$

Finite-Size Effects- Deformed Case

The dispersion relation in the large $\mathcal{J}_1$ limit can be found from (4) as an expansion in

$$\exp\left(-\frac{\mathcal{J}_1}{\sin(p/2)}\right),$$

and up to the leading order it is

$$E - J_1 = \frac{\sqrt{\lambda}}{\pi}\sin(p/2)\left[1 - \frac{4}{e^2}\sin^2(p/2)\cos(\Phi)\exp\left(-\frac{\mathcal{J}_1}{\sin(p/2)}\right)\right], \qquad (5)$$

where

$$\Phi = \frac{\delta}{2^{3/2}\cos^3(p/4)}, \qquad -\pi \le \delta \le \pi, \qquad -\pi \le p \le \pi.$$

In the limit $\Phi \to 0$ the formula (5) gives the result for the undeformed case.

Finite-Size Effects- Deformed Case

Dyonic Giant Magnons

As explained already, instead of considering strings on the γ-deformed background $AdS_5 \times S^5_\gamma$, we can consider strings on the original $AdS_5 \times S^5$ space, but with *twisted* boundary conditions. Actually, here we are interested in string configurations living in the $R_t \times S^3$ subspace, which can be described by the NR integrable system.

Finite-Size Effects- Deformed Case

Dyonic Giant Magnons

If we introduce the variable

$$\chi = 1 - r_1^2 = r_2^2,$$

the NR Hamiltonian can be rewritten as

$$\chi'^2 = \frac{4\omega_1^2(1-u^2)}{\alpha^2(1-v^2)^2}\left\{-\chi^3 + \frac{(1-w^2)+(1-v^2w^2)-u^2}{1-u^2}\chi^2 - \frac{1-(1+v^2)w^2+v^2[(w^2-u^2j)^2-j^2]}{1-u^2}\chi - \frac{v^2u^2j^2}{1-u^2}\right\}$$
$$= \frac{4\omega_1^2(1-u^2)}{\alpha^2(1-v^2)^2}(\chi_{max}-\chi)(\chi-\chi_{min})(\chi-\chi_n), \tag{6}$$

$$v = -\frac{\beta}{\alpha}, \qquad u = \frac{\omega_2}{\omega_1}, \qquad w = \frac{\kappa}{\omega_1}, \qquad j = -\frac{C_2}{\beta\omega_2}.$$

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Correspondingly, the conserved quantities transform to

$$\begin{aligned} \mathcal{E} &= \frac{\kappa}{\alpha}\int_{-r}^{r} d\xi = \frac{(1-v^2)w}{\sqrt{1-u^2}}\int_{\chi_{min}}^{\chi_{max}} \frac{d\chi}{\sqrt{(\chi_{max}-\chi)(\chi-\chi_{min})(\chi-\chi_n)}}, \\ \mathcal{J}_1 &= \frac{1}{\sqrt{1-u^2}}\int_{\chi_{min}}^{\chi_{max}} \frac{\left[1-v^2\left(w^2-u^2 j\right)-\chi\right]d\chi}{\sqrt{(\chi_{max}-\chi)(\chi-\chi_{min})(\chi-\chi_n)}}, \\ \mathcal{J}_2 &= \frac{u}{\sqrt{1-u^2}}\int_{\chi_{min}}^{\chi_{max}} \frac{\left(\chi-v^2 j\right)d\chi}{\sqrt{(\chi_{max}-\chi)(\chi-\chi_{min})(\chi-\chi_n)}}. \end{aligned} \tag{7}$$

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The angular differences

$$p = \Delta\phi_1 = \phi_1(r) - \phi_1(-r), \qquad \delta = \Delta\phi_2 = \phi_2(r) - \phi_2(-r) = 2\pi\left(n_2 - \gamma_3 J_1\right).$$

$$\begin{aligned} p &= \int_{-r}^{r} d\xi f_1' = \frac{\beta\omega_1}{\alpha^2(1-v^2)} \int_{-r}^{r} \left(1 - \frac{w^2 - u^2 j}{r_1^2}\right) d\xi \\ &= \frac{v}{\sqrt{1-u^2}} \int_{\chi_{min}}^{\chi_{max}} \left(\frac{w^2-u^2 j}{1-\chi} - 1\right) \frac{d\chi}{\sqrt{(\chi_{max}-\chi)(\chi-\chi_{min})(\chi-\chi_n)}}, \end{aligned} \tag{8}$$

$$\begin{aligned} \delta &= \int_{-r}^{r} d\xi f_2' = \frac{\beta\omega_2}{\alpha^2(1-v^2)} \int_{-r}^{r} \left(1 - \frac{j}{r_2^2}\right) d\xi \\ &= \frac{uv}{\sqrt{1-u^2}} \int_{\chi_{min}}^{\chi_{max}} \left(\frac{j}{\chi} - 1\right) \frac{d\chi}{\sqrt{(\chi_{max}-\chi)(\chi-\chi_{min})(\chi-\chi_n)}}. \end{aligned} \tag{9}$$

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For correspondence with the Bykov, Frolov notations, we fix $\kappa = \alpha = 1$, rename $\omega_1 \to \omega$, $\omega_2 \to \nu$, introduce the parameters A_1, A_2, and the functions $\phi(\xi)$, $\alpha(\xi)$ as follows

$$C_1 = -vA_1, \qquad C_2 = -vA_2,$$
$$f_1(\xi) = \frac{p}{2r}\xi + \phi(\xi), \qquad f_2(\xi) = \frac{\delta}{2r}\xi + \alpha(\xi).$$

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Then one finds

$$\mathcal{E} = \frac{4\tilde{\kappa}}{\sqrt{(1-\chi_n)(1-\tilde{v}^2)}}\mathbf{K}(1-\epsilon),$$

$$\mathcal{J}_1 = \frac{4\tilde{\kappa}}{(1-v^2)\sqrt{(1-\chi_n)(1-\tilde{v}^2)}}\left[\left(\omega(1-\chi_n) - \frac{v^2}{\omega}(1+vA_2)\right)\mathbf{K}(1-\epsilon)\right.$$
$$\left. - \omega(1-\chi_n)(1-\tilde{v}^2)\mathbf{E}(1-\epsilon)\right],$$

$$\mathcal{J}_2 = \frac{4\tilde{\kappa}}{(1-v^2)\sqrt{(1-\chi_n)(1-\tilde{v}^2)}}\left[\left(v^2 A_2 + v\chi_n\right)\mathbf{K}(1-\epsilon)\right.$$
$$\left. + v(1-\chi_n)(1-\tilde{v}^2)\mathbf{E}(1-\epsilon)\right],$$

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$$p = \frac{4\tilde{\kappa} v}{(1-v^2)\sqrt{(1-\chi_n)(1-\tilde{v}^2)}} \left[\frac{1+\nu A_2}{\omega(1-\chi_n)\tilde{v}^2} \Pi\left(\frac{\tilde{v}^2-1}{\tilde{v}^2}(1-\epsilon) | 1-\epsilon \right) - \omega \mathbf{K}(1-\epsilon) \right],$$

$$\delta = -\frac{2\tilde{\kappa} v}{(1-v^2)\sqrt{(1-\chi_n)(1-\tilde{v}^2)}} \left[\frac{A_2}{(1-\tilde{v}^2)\left(1+\chi_n \frac{\tilde{v}^2}{1-\tilde{v}^2}\right)} \Pi\left(\frac{1-\chi_n}{1+\chi_n \frac{\tilde{v}^2}{1-\tilde{v}^2}}(1-\epsilon) | 1-\epsilon \right) \right.$$
$$\left. + \nu \mathbf{K}(1-\epsilon) \right].$$

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Notations

$$\tilde{v}^2 = \frac{1-\chi_{max}}{1-\chi_n}, \qquad \epsilon = \frac{\chi_{min}-\chi_n}{\chi_{max}-\chi_n}, \qquad \tilde{\kappa} = \frac{1-v^2}{2\sqrt{\omega^2-\nu^2}}$$

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In order to obtain the finite-size correction to the energy-charge relation, we have to consider the limit $\epsilon \to 0$. We make the following ansatz for the parameters

$$v = v_0 + v_1\epsilon + v_2\epsilon\log(\epsilon), \qquad \tilde{v} = \tilde{v}_0 + \tilde{v}_1\epsilon + \tilde{v}_2\epsilon\log(\epsilon), \qquad \omega = 1 + \omega_1\epsilon,$$

$$\nu = \nu_0 + \nu_1\epsilon + \nu_2\epsilon\log(\epsilon), \qquad A_2 = A_{21}\epsilon, \qquad \chi_n = \chi_{n1}\epsilon.$$

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Now, we impose the conditions:

1. p - finite
2. $\mathcal{J}_2$ - finite
3. $\mathcal{E} - \mathcal{J}_1 = \frac{2\sqrt{1-v_0^2-\nu_0^2}}{1-\nu_0^2} - \frac{(1-v_0^2-\nu_0^2)^{3/2}}{2(1-\nu_0^2)}\cos(\Phi)\epsilon$

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Finally, the dispersion relation, including the leading finite-size correction, takes the form

$$\mathcal{E} - \mathcal{J}_1 = \sqrt{\mathcal{J}_2^2 + 4\sin^2(p/2)} - \frac{16\sin^4(p/2)}{\sqrt{\mathcal{J}_2^2 + 4\sin^2(p/2)}} \cos(\Phi)$$
$$\exp\left[-\frac{2\left(\mathcal{J}_1 + \sqrt{\mathcal{J}_2^2 + 4\sin^2(p/2)}\right)\sqrt{\mathcal{J}_2^2 + 4\sin^2(p/2)}\,\sin^2(p/2)}{\mathcal{J}_2^2 + 4\sin^4(p/2)}\right].$$

For $\mathcal{J}_2 = 0$, this reduces to the result found by Bykov and Frolov.

Concluding Remarks

Possible extensions to:

- Finite-Size Strings on $R_t \times S^5_\gamma$
- Finite-Size Strings on $R_t \times CP^3_\gamma$

An interesting open problem:

To reproduce the energy-charge relation by using the Lüscher's approach . To this end, we need a generalization of the Lüscher's formulas for the case of nontrivial *twisted* boundary conditions.