Finite-Size Technology in Low-Dimensional Quantum Systems

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UNIVERSAL PROPERTIES OF ISING CLUSTERS AND DROPLETS NEAR CRITICALITY

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Based on :

GD, Nucl.Phys.B 818 (2009) 196

GD, J. Viti, arXiv:1006.2301 [hep-th]

• Recent developments (SLE, log CFT's, ...) put new emphasis on the dualism between local (spin) and non-local (cluster) observables

• The debate goes back to the "droplet model" (60's): is the ferromagnetic transition a percolative transition of spin clusters? No (70's), but ... (80's)

• The universal critical properties of both local and non-local observables can be studied by field theory. In 2D the exact fractal dimensions of clusters are known in many cases (CFT)

• I'll use integrable field theory for the Ising case to obtain two results:

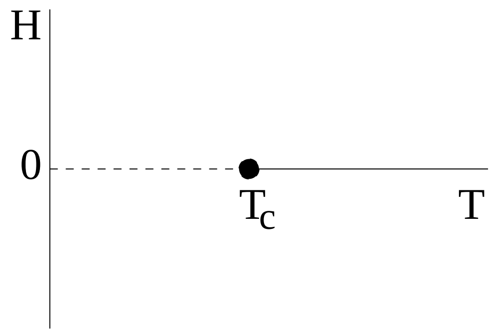
– exact non-perturbative mechanism explaining cluster criticality in absence of magnetic criticality

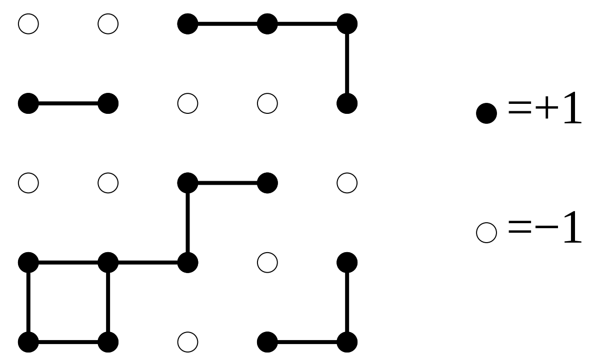
– quantitative characterization of universal cluster properties near criticality

Ising percolation

$$-\mathcal{H}_{Ising} = \frac{1}{T}\sum_{\langle ij\rangle}\sigma_i\sigma_j + H\sum_i \sigma_i\,, \qquad \sigma_i = \pm 1 \qquad (T \geq 0)$$

No magnetic transition away from $T \leq T_c$, $H = 0$





$P \equiv$ probability that a site belongs to an infinite cluster of $+$ spins (percolative order parameter)

$\forall\, T \geq 0\ \exists\ H_c(T)$ such that $P > 0$ for $H > H_c(T)$

$$H_c(0) = 0 \qquad\qquad \frac{e^{H_c(\infty)}}{2\cosh H_c(\infty)} = p_c^0$$

$p_c^0 =$ critical point of random percolation (non-universal)

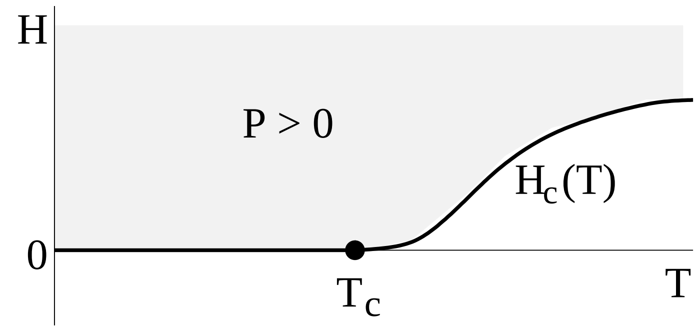
Three observations:

i) $H_c(T)$ monotonic

ii) spontaneous magnetization $\Longrightarrow$ infinite cluster (Coniglio et al '77)

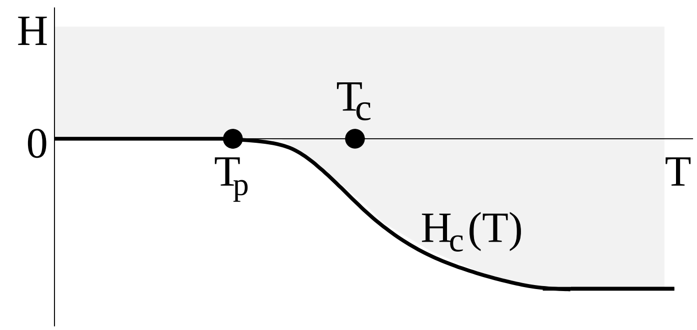
iii) $p_c \leq p_c^0$ at $H = 0$ (interaction makes percolation easier)

Consequences:



a) $p_c^0 > 1/2$, i.e. $H_c(\infty) > 0$

i), ii) $\Longrightarrow H_c(T) = 0$ for $T \leq T_c$



b) $p_c^0 < 1/2$, i.e. $H_c(\infty) < 0$

$p = 1/2$ for $T \geq T_c$, $H = 0$

iii) $\Longrightarrow T_p < T_c$

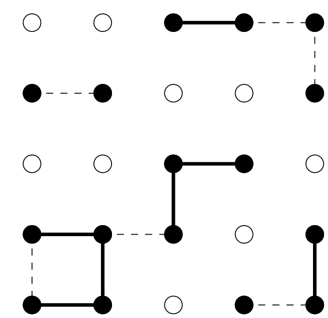
Ordinary lattices in $d = 2$ have $p_c^0 \geq 1/2 \longrightarrow$ a)

$d = 3$ have $p_c^0 < 1/2 \longrightarrow$ b)

Kasteleyn-Fortuin representation

$$-\mathcal{H}_q = -\mathcal{H}_{Ising} + J \sum_{\langle ij \rangle} t_i t_j \left(\delta_{s_i,s_j} - 1\right), \qquad t_i = \frac{1}{2}(\sigma_i + 1) = 0, 1, \qquad s_i = 1, \ldots, q$$

$$\begin{aligned} Z_q &= \sum_{\{t_i\}} \sum_{\{s_i\}} e^{-\mathcal{H}_q} = \quad \text{(Murata '79)} \\ &= \sum_{\{t_i\}} e^{-\mathcal{H}_{Ising}} \, q^{N_v} \sum_G p_B^b (1 - p_B)^{\bar{b}} \, q^{N_c} \end{aligned}$$



$N_v = \#$ of empty sites

$G =$ graph made of bonds between occupied sites

$b = \#$ of bonds in G; $\quad \bar{b} = \#$ of absent bonds (dashed)

$N_c = \#$ of connected components in G (KF clusters)

$p_B \equiv 1 - e^{-J}$ prob that a bond is present (KF cl=Ising cl for $p_B = 1$)

q continuous parameter

$$\langle X \rangle_{q \to 1} = Z_{Ising}^{-1} \sum_{\{t_i\}} e^{-\mathcal{H}_{Ising}} \sum_G X \, p_B^b (1 - p_B)^{\bar{b}}$$ dilute percolation average

pure percolation for $H = +\infty$

Example of percolative observable: mean cluster number per site

$$\frac{\langle N_c \rangle}{N} = -\,\partial_q f_q|_{q=1} - \frac{1}{2}(1-M) \qquad\qquad M \text{ Ising magnetization per site}$$

$$f_q \equiv -\frac{1}{N} \ln Z_q = f_{Ising} + (q-1)F + O((q-1)^2)$$

Dilute Potts model yields

– Ising magnetic properties at $q = 1$

– KF cluster properties at $q = 1 + \epsilon$

– Ising cluster properties at $q = 1 + \epsilon$, $J \to +\infty$

RG analysis in $d = 2$ (Coniglio-Klein '80 + CFT)

Look for fixed points of $\mathcal{H}_{q\to 1}(T,H,J) = \mathcal{H}_{Ising}(T,H) - J \sum_{\langle ij \rangle} t_i t_j \,(\delta_{s_i,s_j} - 1)$

Need a magnetic fixed point to start with $\Longrightarrow$ $T = T_c,\ H = 0.$ $\quad J\,?$

d=2 fixed points characterized by

– central charge $\quad c = 1 - 6/[m(m+1)]$

– primary fields $\varphi_{r,s}$ with scaling dimensions $\quad X_{r,s} = \frac{[(m+1)r - ms]^2 - 1}{2m(m+1)}$

$\mathcal{H}_{Ising}(T_c, 0)$: $\quad m = 3, \quad c = 1/2, \quad X_\sigma = X_{1,2} = 1/8, \quad X_\varepsilon = X_{1,3} = 1$

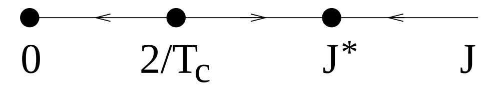
$\mathcal{H}_q$ possesses two critical lines as functions of q:

$$\sqrt{q} = 2 \sin \frac{\pi(t-1)}{2(t+1)}, \qquad m = \begin{cases} t, \ \text{critical } (H = +\infty): & X_s = X_{\frac{m-1}{2},\frac{m+1}{2}}, \quad X_{t_1} = X_{2,1} \\ t+1, \ \text{tricritical:} & X_s = X_{\frac{m}{2},\frac{m}{2}}, \ X_{t_1} = X_{1,2}, \ X_{t_2} = X_{1,3} \end{cases}$$

$$q \to 1: \qquad m = \begin{cases} 2, \ \text{pure perc:} & c = 0, \quad X_s = 5/48, \quad X_{t_1} = 5/4 \\ 3, \ \text{dilute perc:} & c = 1/2, \quad X_s = 5/96, \quad X_{t_1} = 1/8, \quad X_{t_2} = 1 \end{cases}$$

We found a fixed point of dilute percolation for $J = J^*$

A trivial (purely magnetic) fixed point is at $J = 0$



J irrelevant at these two fixed points $\Longrightarrow$ $\exists$ a third one

$$-\mathcal{H}_q\big|_{J=2/T} = \frac{2}{T}\sum_{\langle ij\rangle}(\delta_{\nu_i,\nu_j} - 1) + (\ln q - 2H)\sum_i \delta_{\nu_i,0}\,, \qquad \nu_i = 0, 1, \ldots, q$$

$\Longrightarrow$ fixed point at $J = 2/T_c$ as $q \to 1$, with $X_s = X_\sigma = 1/8$

(KF clusters with $J = 2/T$) $=$ Ising droplets

RG flows among fixed points with $c = 1/2$ (c-theorem does not apply)

Critical behavior of Ising clusters is ruled by J^* (agrees with numerics)

cluster size $\sim$ (linear extension)^D $\qquad D = d - X_s$ fractal dimension

$$D = \begin{cases} 91/48 = 1.89.. & \text{pure percolation} \\ 187/96 = 1.94.. & \text{Ising clusters} \\ 15/8 = 1.87.. & \text{Ising droplets} \end{cases}$$

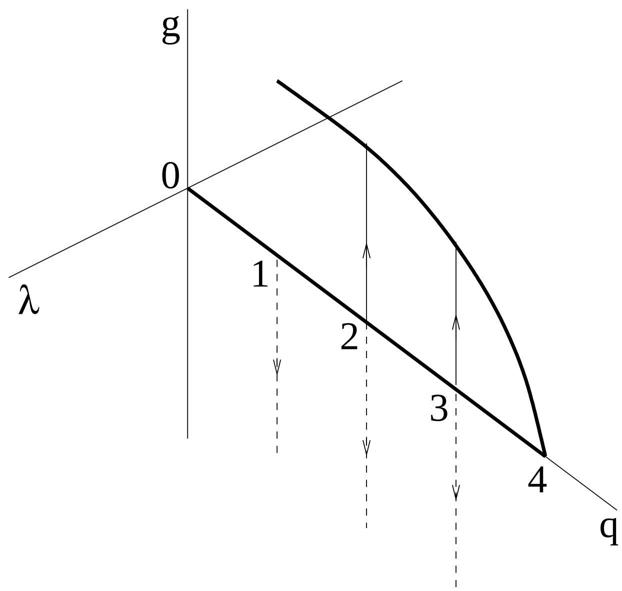
Field theory of Ising clusters

Ising field theory:

$$\mathcal{A}_{Ising} = \mathcal{A}_{CFT}^{Ising} - \tau \int d^2x\, \varepsilon(x) - h \int d^2x\, \sigma(x), \qquad \tau \sim T - T_c,\ \ h \sim H$$

Dilute Potts field theory:

$$\mathcal{A}_q = \mathcal{A}_{CFT}^{tricr} - g \int d^2x\, \varphi_{1,3}(x) - \lambda \int d^2x\, \varphi_{1,2}(x)$$



$\mathcal{A}_{q=1} = \mathcal{A}_{Ising} \qquad (g = \tau,\ \lambda = h)$

$\mathcal{A}_q$ integrable for g and/or λ equal zero

S_q symmetry breaks spontaneously at $\lambda = 0$

q degenerate vacua for $\lambda < 0$

The $q \to 1$ limit of the Potts critical surface is the Ising percolation transition:
1st order (massive) at $T < T_c$, 2nd order (massless) at $T > T_c$

$\mathcal{A}_{q=1} = \mathcal{A}_{Ising}$, however, is purely massive: no transition above T_c

We can take the limit analytically

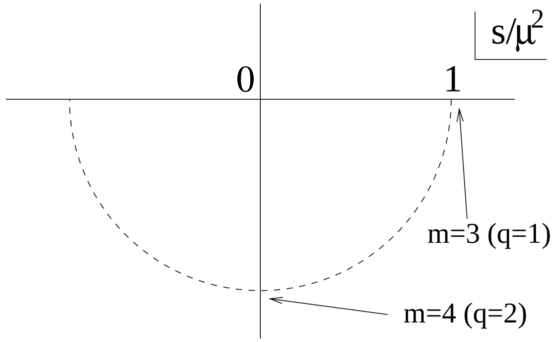
The massless surface of $\mathcal{A}_q$ is an integrable field theory (Fendley, Saleur, Zamolodchikov '93, in RSOS basis)

Fundamental particles: right/left movers A_k, $k = 1, \dots, q-1$ with $p^1 = \pm p^0$

Poles of two-particle amplitudes contained in $S_{-1/2}(\theta)/(S_{1/2}(\theta)\cosh\rho(i\pi - \theta))$

$$S_\gamma(\theta) = \prod_{n=0}^{\infty} \frac{\Gamma\left(\frac{1}{2} + \left(2n + \frac{3}{2} - \gamma\right)\rho - \frac{\rho\theta}{i\pi}\right)\Gamma\left(\frac{1}{2} + \left(2n + \frac{1}{2} - \gamma\right)\rho + \frac{\rho\theta}{i\pi}\right)}{\Gamma\left(\frac{1}{2} + \left(2n + \frac{3}{2} - \gamma\right)\rho + \frac{\rho\theta}{i\pi}\right)\Gamma\left(\frac{1}{2} + \left(2n + \frac{1}{2} - \gamma\right)\rho - \frac{\rho\theta}{i\pi}\right)}, \qquad \rho = 1/(m-1)$$

$$s = \mu^2 e^\theta \; : \qquad \begin{cases} \mathrm{Im}\,\theta \in (0, \pi) & \text{physical sheet} \\ \mathrm{Im}\,\theta \in (0, -\pi) & \text{second sheet} \end{cases}$$



No poles on physical sheet

Pole at $\theta_0 = -i\pi(m-3)/2$, on 2nd sheet for $m \in (3, 5)$

$q \to 1^+$: resonance B with $\mathrm{Im}\, s_0 \propto (q-1)\mu^2$

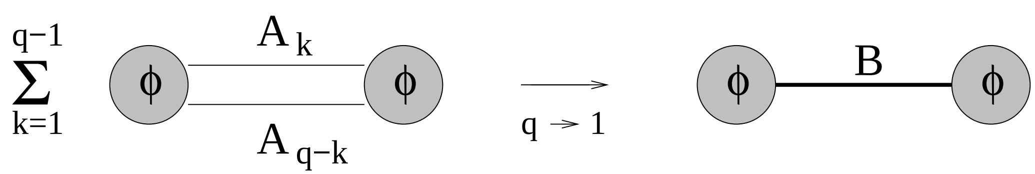
- $q = 1 + \epsilon$: ϵ massless particles A_k, one resonance B with lifetime $\propto 1/\epsilon$
- $q = 1$: 0 massless particles, one stable particle B with mass μ

$\Longrightarrow$ percolative transition in absence of magnetic singularities above T_c

B S_q-singlet (survives at $q=1$) $\Longrightarrow$ only S_q-invariant fields ϕ have non-zero correlations at $q=1$:

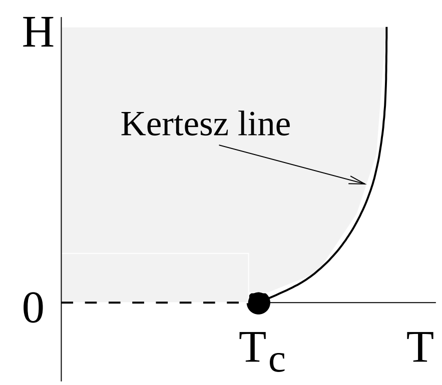
$$\begin{aligned}
\lim_{q\to 1}\langle\phi(r)\phi(0)\rangle &= \lim_{q\to 1}(q-1)\int d\theta_1 d\theta_2\,|\langle 0|\phi(0)|A_k(\theta_1)A_{q-k}(\theta_2)\rangle|^2\,e^{-rE_{2,0}(\theta_1,\theta_2)}+\dots\\
&= \lim_{q\to 1}(q-1)\int d\beta d\theta\,\frac{|R_\phi|^2}{(\theta-\theta_0)(\theta+\theta_0)}\,e^{-rE_{2,0}((\beta+\theta)/2,(\beta-\theta)/2)}+\dots\\
&\propto \int d\beta\,e^{-rE_{1,\mu}(\beta)}+\dots
\end{aligned}$$

$$E_{2,0}(\theta_1,\theta_2)\equiv\mu(e^{\theta_1}+e^{-\theta_2})/2,\qquad \theta_0\propto -i(q-1),\qquad \mathrm{E}_{1,\mu}(\beta)\equiv\mu\cosh(\beta/2)$$



This is how the canonical space of fields $[I]\oplus[\sigma]\oplus[\epsilon]$ of Ising field theory is recovered at $q=1$

Field theory of Ising droplets



Droplets are KF clusters with $J = 2/T$, i.e. $p_B = 1 - e^{-2/T}$

$T = \infty$: $p_B = 0$, no percolation

$H = +\infty$: transition at $T = -2/\ln(1 - p_c^0)$

$p_c^0 =$ threshold of random bond percolation

Scaling limit :

$$\tilde{\mathcal{A}}_q = \mathcal{A}_{CFT}^{(q+1)} - \tau_q \int d^2x\, \varphi_{2,1}(x) + 2h_q \int d^2x\, \delta_{\nu(x),0}\,, \qquad \nu(x) = 0, 1, \ldots, q$$

RG invariant : $\eta_q = \tau_q / h_q^{(2-X_{2,1})/(2-X_s)}$ $\qquad (\tilde{\mathcal{A}}_{q=1} = \mathcal{A}_{Ising}, \quad \eta_1 \equiv \eta = \tau/h^{8/15})$

h_q breaks S_{q+1} into S_q ; for $q > 1$ S_q breaks spontaneously at $\eta_q = \eta_q^c$

The Kertész line is the limit $q \to 1$ of the flow from S_{q+1} fixed point at $h_q = 0$ to S_q fixed point at $h_q = +\infty$

Again, no transition at $q = 1$; resonance mechanism most likely, but no integrability in this case

• Universal limit of the Kertész line: lattice data (Fortunato, Satz '01) + lattice-continuum relations (Caselle, Grinza, Rago '04) give $\eta_K \equiv \eta_{q \to 1}^c \simeq 0.12$

Cluster observables

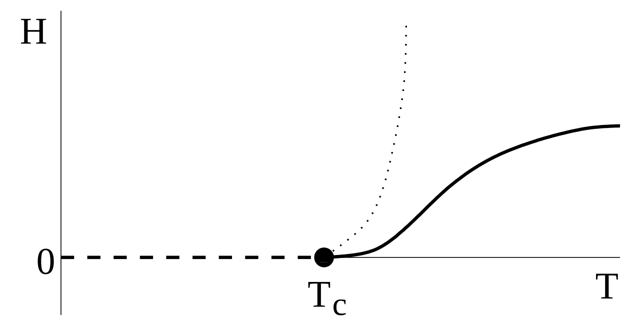
$P = \lim_{q\to 1} \frac{\langle\sigma_1(x)\rangle}{q-1}$ percolative order parameter

$S = \lim_{q\to 1} \frac{1}{q-1} \sum_x \langle\sigma_1(x)\sigma_1(0)\rangle_c$ mean cluster size

$\lim_{q\to 1} \frac{1}{q-1}\langle\sigma_1(x)\sigma_1(0)\rangle_c \sim e^{-|x|/\xi_t}, \quad |x| \to \infty$ ξ_t (true) connectivity length

$\langle N_c\rangle/N = -\,\partial_q f_q\big|_{q=1} - \frac{1}{2}(1-M)$ mean cluster number per site

$\sigma_k(x) \equiv \left(\delta_{s(x),k} - \frac{1}{q}\right) t(x), \qquad k = 1,\ldots,q$ Potts spin



Critical behavior :

$$\begin{aligned} P &= B\, g^{\beta} \\ S &= \Gamma\, g^{-\gamma} \\ \xi &= f\, g^{-\nu} \\ \left(\frac{\langle N_c\rangle}{N} - \frac{M}{2}\right)_{sing} &= \left(\frac{A_{-1}}{2}\ln^2 g - A_0 \ln g + \delta_{A_{-1},0}\, A_1\right) g^{\mu} \end{aligned}$$

$$g = \begin{cases} |T - Tc| & \text{if } H = 0 \\ |H| & \text{if } T = T_c \end{cases} \quad \to 0$$

Universal amplitude ratios

- provide the canonical way of characterizing universal critical behavior *around* critical points

- two universality classes can have the same exponents but different amplitude ratios (e.g. droplet size vs magnetic susceptibility)

- allow to test methods of non-conformal field theory (e.g. calculation of correlation functions from the S-matrix)

- in the percolative case we test results of non-conventional field theory (non-unitary, non-rational, ...)

- challenge for any approach alternative to field theory (e.g. massive SLE)

Universal amplitude ratios for Ising percolation (GD, J. Viti, 2010) :

	clusters	droplets
Γ_a/Γ_b^+	non-universal	40.3
$f_{2nd,a}/f_{t,a}$	"	0.99959..
$f_{t,a}/f_{t,b}^+$	"	2
$f_{t,a}/\widehat{f}_{t,a}$	"	1
$A_{k,a}/A_{k,b}^+; k=0,-1$	"	1
Γ_b^+/Γ_b^-	-	1
$f_{t,b}^+/f_{t,b}^-$	1/2	1
$f_{2nd,b}^-/f_{t,b}^-$	0.6799	0.61
$f_{2nd,b}^+/f_{2nd,b}^-$	-	1
$f_{t,b}^+/\widehat{f}_{t,b}^\pm$	1/2	1
U_b	24.72	15.2
$A_{k,b}^+/A_{k,b}^-; k=0,-1$	1	1
$A_{0,b}^\pm/A_{-1,b}^\pm$	$-\gamma-\ln\pi=-1.7219..$	$-\gamma-\ln\pi=-1.7219..$
R_b	$\frac{3\sqrt{3}(\gamma+\ln\pi)}{64\pi^2}=0.014165..$	$-\frac{\gamma+\ln\pi}{12\pi^2}=-0.014539..$
$f_{t,c}^+/f_{t,c}^-$	1/2	-
$f_{2nd,c}^-/f_{t,c}^-$	1.002	-
$f_{t,c}^+/\widehat{f}_{t,c}^\pm$	$\sin\frac{\pi}{5}=0.58778..$	-
$A_{k,c}^+/A_{k,c}^-; k=0,1$	1	-
$A_{0,c}^\pm/A_{1,c}^\pm$	$-0.42883..$	-
R_c	$-3.7624..\times 10^{-3}$	-

$\gamma=0.5772..$ Euler-Mascheroni constant

Random percolation case (GD, J. Viti, J. Cardy, J.Phys.A43 (2010) 152001):

	Field Theory	Lattice
A^+/A^-	1	1^a
f_t^+/f_t^-	2	-
f_{2nd}^+/f_t^+	1.001	-
f_{2nd}^+/f_{2nd}^-	3.73	4.0 ± 0.5^c
U	2.22	2.23 ± 0.10^d
$R_{\xi_{2nd}}^+$	0.926	$\approx 0.93^{a+b}$
Γ^+/Γ^-	160.2	162.5 ± 2^e

a) Domb, Pearce, J. Phys. A 9 (1976) L137

b) Aharony, Stauffer, J. Phys. A 30 (1997) L301

c) Corsten, Jan, Jerrard, Physica A 156 (1989) 781

d) Daboul, Aharony, Stauffer, J. Phys. A 33 (2000) 1113

e) Jensen, Ziff, Phys. Rev. E 74 (2006) 20101

Conclusion

- There is more physics in the Ising model than we are used to think

- This extends to more general spin models

- Field theory is able to describe this physics, through analytic continuation from unitary cases