Exact solutions of p+ip-wave pairing models

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arXiv:1001.1591 (TCD,MI,JL,GS,S-YZ) and to appear

Exactly solvable many-body models

Bardeen, Cooper, Schreiffer (BCS) model of superconductivity (57)

$$H = \sum_{j=1,\sigma=\pm 1}^{L} \varepsilon_j c_{j\sigma}^\dagger c_{j\sigma} - \sum_{j,k}^{L} g_{jk} c_{j+}^\dagger c_{j-}^\dagger c_{k-} c_{k+}$$

L doubly degenerate energy levels ε_j, fermion creation operators $c_{j\pm}^\dagger$

Reduced model $g_{jk} = g$ has exact solution (Richardson (64))

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More recent interest due to

- BCS s-wave superfluidity seen in $^{40}\mathrm{K}$ and $^{6}\mathrm{Li}$
- Hidden integrability (Cambiaggio, Rivas, Saraceno (97))
- Experimental work on Al nanograins (Ralph, Black and Tinkham (96)

BCS model with unconventional superconductivity

BCS pairing with p-wave symmetry observed in $^{3}\mathrm{He}$ ($\mathrm{Sr_2Ru0_4}$?)

$$\sum_{j,k}^{L} g_{jk} c_{j+}^{\dagger} c_{j-}^{\dagger} c_{k-} c_{k+} \to \sum_{\mathbf{k}\neq\mathbf{k}'} (k_x - ik_y)(k_x + ik_y) c_{\mathbf{k}}^{\dagger} c_{-\mathbf{k}}^{\dagger} c_{-\mathbf{k}'} c_{\mathbf{k}'}$$

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Recent interest due to

- connection with the Moore-Read pfaffian state in FQHE
- topological quantum phase transitions in groundstate
- applications in topological quantum computation (Nayak, Simon,Stern, Freedman, Das Sarma (08))
- possible experimental realisations through Feshbach resonance induced p-wave pairing in ultracold Fermi gases

A $p+ip$-wave pairing model with resonant molecules

Our Hamiltonian

$$\begin{aligned} H &= \delta b^\dagger b + \sum_{\mathbf{k}} \frac{\mathbf{k}^2}{2m} c^\dagger{}_{\mathbf{k}} c_{\mathbf{k}} \\ &\quad - \frac{G}{4} \sum_{\mathbf{k}\neq\mathbf{k}'} (k_x - ik_y)(k_x + ik_y) c^\dagger_{\mathbf{k}} c^\dagger_{-\mathbf{k}} c_{-\mathbf{k}'} c_{\mathbf{k}'} \\ &\quad - \frac{K}{2} \sum_{\mathbf{k}} \left((k_x - ik_y) c^\dagger_{\mathbf{k}} c^\dagger_{-\mathbf{k}} b + \mathrm{h.c.} \right) \end{aligned}$$

where

$c^\dagger_{\mathbf{k}}, c_{\mathbf{k}}$ create/destroy 2D fermions with momentum $\mathbf{k}$

$b^\dagger, b$ create/destroy zero-momentum molecular condensate

3-parameter family of couplings $\{\delta, K, G\}$

BCS to BEC crossover

By varying couplings $\{\delta, K, G\}$ we can probe the crossover from the BCS groundstate to the BEC groundstate

Hard to do for generic couplings

But we find there is a submanifold in parameter space for which the model is integrable

Interacting → non-interacting model

Set $\delta = K = 0$

$$
\begin{aligned}
H &= \delta b^\dagger b + \sum_{\mathbf{k}} \frac{\mathbf{k}^2}{2m} c_{\mathbf{k}}^\dagger c_{\mathbf{k}} \\
&\quad - \frac{G}{4} \sum_{\mathbf{k} \neq \mathbf{k}'} (k_x - ik_y)(k_x + ik_y) c_{\mathbf{k}}^\dagger c_{-\mathbf{k}}^\dagger c_{-\mathbf{k}'} c_{\mathbf{k}'} \\
&\quad - \frac{K}{2} \sum_{\mathbf{k}} \left((k_x - ik_y) c_{\mathbf{k}}^\dagger c_{-\mathbf{k}}^\dagger b + \mathrm{h.c.} \right)
\end{aligned}
$$

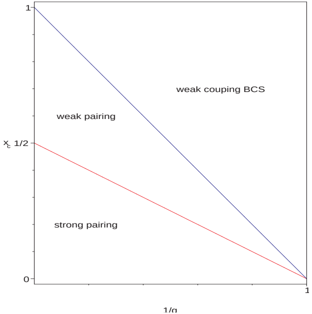
BCS model with $p + ip$-symmetry

$$
\begin{aligned}
H_{K=0,\delta=0} &= \sum_{\mathbf{k}} \frac{\mathbf{k}^2}{2m} c_{\mathbf{k}}^{\dagger} c_{\mathbf{k}} \\
&\quad - \frac{G}{4} \sum_{\mathbf{k} \neq \mathbf{k}'} (k_x - i k_y)(k_x + i k_y) c_{\mathbf{k}}^{\dagger} c_{-\mathbf{k}}^{\dagger} c_{-\mathbf{k}'} c_{\mathbf{k}'}
\end{aligned}
$$

Mean field theory (Read, Green (00))

Exact solution (MI,JL,GS,S-YZ (09) and TCD,MI,JL,GS,S-YZ (10))

Groundstate of $p + ip$-wave BCS model



Phase	Filling fraction	GS Energy
Weak coupling	$x_c > 1 - 1/g$	$E > 0$
Moore-Read line	$x_c = 1 - 1/g$	$E = 0$
Weak pairing	$(1 - 1/g)/2 < x_c < 1 - 1/g$	$E < 0$
Read-Green line	$x_c = (1 - 1/g)/2$	$E < 0$
Strong pairing	$x_c < (1 - 1/g)/2$	$E < 0$

($2L$ fermion momentum states, N_c Cooper pairs, $x_C = N_c/L$ filling fraction, $g = GL$)

Groundstate of fermion and boson model without interaction

When $\delta = K = 0$

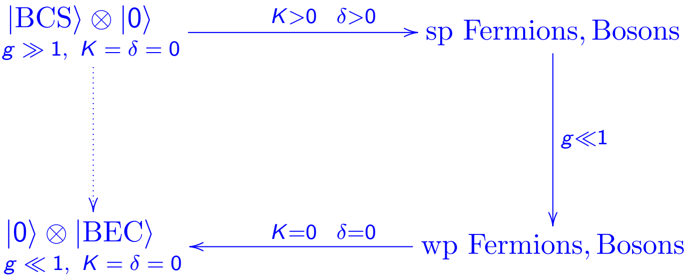
$$|\Psi\rangle = |\Psi_{\mathrm{BCS}}\rangle \otimes |N_b\rangle$$

with N_b chosen to minimise energy

Phase	$g = 1/G$	Filling fraction	$\lvert\Psi_{\mathrm{BCS}}\rangle$	N_b
BEC	$g < 1$	all	vacuum	N
Mixed	$g > 1$	$x > (1 - 1/g)/2$	Read-Green	$0 < N_b < N$
BCS	$g > 1$	$x < (1 - 1/g)/2$	Strong pairing	0

($x = x_b + x_C$, $x_b = N_b/L$, $N = N_b + N_c$)

BCS to BEC


|BCS⟩ ⊗ |0⟩
g ≫ 1, K = δ = 0
K>0 δ>0
sp Fermions, Bosons
g≪1
wp Fermions, Bosons
K=0 δ=0
|0⟩ ⊗ |BEC⟩
g ≪ 1, K = δ = 0

Exact solution via QISM

Trigonometric solution of R-matrix	$R_{\mathrm{XXZ}}(u)$
L-operator for boson	$\tilde{L}(u)$
L-operator for Cooper pair $c_{j_-} c_{j+}$	$L(u)$

Monodromy matrix

$$
\begin{aligned}
T(u) &= U \tilde{L}_{0a}(u z_a^{-1}) \, L_{0L}(u z_L^{-1}) \ldots L_{01}(u z_1^{-1}) \\
&= \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}
\end{aligned}
$$

where

$$
U = \begin{pmatrix} e^{-i\alpha} & 0 \\ 0 & e^{i\alpha} \end{pmatrix}
$$

An integrable Hamiltonian

Transfer matrix

$$t(u) = \left(z_a \prod_{i=1}^{L} z_i \right) \mathrm{tr}_0 \, T(u) = \sum_{j=-L-1}^{L+1} t_j \, u^j$$

Commuting transfer matrices

$$[t_j, t_k] = 0 \qquad j, k = -L - 1 \ldots L + 1$$

Hamiltonian arises from t_{L-1} $\qquad (t_{L+1} = \mathrm{const}, t_L = 0)$

Quasi-classical limit

R matrix has quasi-classical property

$$\lim_{\eta\to 0} R(u) = I \otimes I$$

$$\begin{aligned} R(u) &\sim I \otimes I + \eta\,\mathcal{R}(u) + \ldots \\ T(u) &\sim I \otimes I + \eta\,\mathcal{T}(u) + \ldots \\ t(u) &\sim 1 + \eta\,\tilde{t}(u) + \ldots \end{aligned}$$

$$\Uparrow$$

Integrable submanifold in coupling space $\{\delta, K, G\}$

In quasi-classical limit $\eta \to 0$ we find

$$
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H &= \delta b^\dagger b + \sum_{\mathbf{k}} \frac{\mathbf{k}^2}{2m} c^\dagger_{\mathbf{k}} c_{\mathbf{k}} \\
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with $\delta = -K^2 G$

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Pairing Hamiltonian is integrable for $\{\delta, K, G\} = \{-K^2 G, K, G\}$

Algebraic Bethe ansatz

Eigenstates

$$\left|\Psi(y_j)\right\rangle = \prod_{j=1}^{N=N_b+N_c} C(y_j)\left|0\right\rangle$$

Bethe ansatz equations

$$\frac{F^2}{y_j^2} + \frac{G^{-1} - L + 2N - 1}{y_j} + \sum_{k=1}^{L} \frac{1}{y_j - \mathbf{k}^2} - \sum_{l \neq j}^{N} \frac{2}{y_j - y_l} = 0 \quad , \quad F = K/G$$

Energy

$$E = (1+G)\sum_{j=1}^{N} y_j$$

Bethe ansatz solutions

Analytical solution - hard but Perron-Frobenius theorem shows that groundstate is unique and has all Bethe roots real and negative

Numerical solution - easier (if L, x not too large) since iterative scheme converges quickly

Correlation functions

Write $c_j^\dagger, c_j, b_a^\dagger, b_a, N_j, b_a^\dagger b_a$ in terms of the matrix elements

$$\mathcal{A}(u), \mathcal{B}(u), \mathcal{C}(u) \text{ and } \mathcal{D}(u)$$

of monodromy matrix $\mathcal{T}(u)$

Compute scalar products $\langle \Psi(\{y_j\}) | \Psi(\{\tilde{y}_j\}) \rangle$

Obtain explicit expressions for

$$\begin{array}{cc} \langle c_j^\dagger \rangle & \langle N_j \rangle \\ \langle b_a^\dagger \rangle & \langle b_a^\dagger \, b_a \rangle \\ \langle b_a \, c_j^\dagger \rangle & \langle c_j^\dagger \, c_k \rangle \\ \langle N_j N_k \rangle & \end{array}$$

One-point function

Boson fraction expectation value

$$\langle N_b \rangle = \frac{\langle \Psi | b^\dagger b | \Psi \rangle}{\langle \Psi | \Psi \rangle} = 1 - \frac{\det(\mathcal{G} - \mathcal{Q})}{\det \mathcal{G}}$$

where

$$\begin{aligned}
\mathcal{G}_{ii} &= \frac{F}{y_i^2} + \sum_{\mathbf{k} \in \mathbf{K}_+} \frac{\mathbf{k}^2}{(y_i - \mathbf{k}^2)^2} - 2 \sum_{m \neq i} \frac{y_m}{(y_i - y_m)^2} \\
\mathcal{G}_{ij} &= \frac{2 y_j}{(y_j - y_i)^2} \qquad\qquad i \neq j \\
\mathcal{Q}_{ij} &= \frac{F^2}{y_i^2}
\end{aligned}$$

Boson fraction $\langle N_b \rangle / N$

Number of momentum states $2L = 900$
Filling fraction $x = 1/3$
Number of boson and Cooper pairs $N = 300$
Full Hilbert space has dimension 10^{123}

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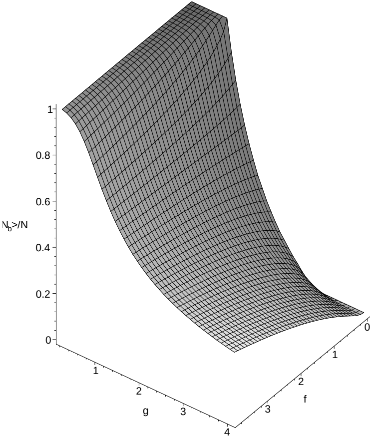
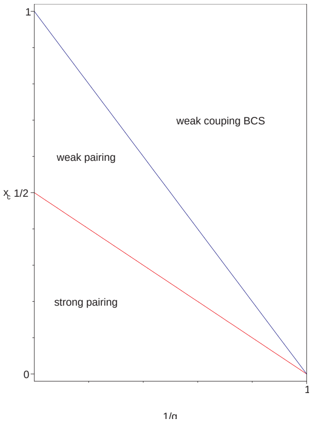


The $p + ip$ model: topological structure



Groundstate of weak pairing phase has a non-trivial topology in momentum space (Read, Green (00))

The $p + ip$ model: topological structure

Wavefunction $\phi(\mathbf{k}) = \phi_x(\mathbf{k}) + i\phi_y(\mathbf{k})$ induces map $\tilde{\phi}$ from (k_x, k_y) to (ϕ_x, ϕ_x)

$$\tilde{\phi} : S^2 \to S^2$$

Define winding number (Volovik (88))

$$\omega = \frac{1}{\pi} \int_{\mathbb{R}^2} dk_x dk_y \frac{\partial_{k_x}\phi_x \partial_{k_y}\phi_y - \partial_{k_y}\phi_x \partial_{k_x}\phi_y}{(1 + \phi_x^2 + \phi_y^2)^2}$$

Can generalise to $\phi(\mathbf{k}_1, \ldots \mathbf{k}_N)$

The $p + ip$ model: topological nature

Mean field theory (Read,Green (00))

$\omega = 0$ for strong pairing phase
$\omega = +1$ for weak pairing phase

Exact solution (I,JL,GS,S-YZ (09))

$\omega = 0$ for strong pairing and weak coupling phase
$\omega = 1, 2, 3, \ldots, N$ for weak pairing phase
ω counts the number of Moore-Read pairs

The $p + ip$ model with bosons

Wavefunction

$$|\Psi\rangle = \sum_{j=0}^{n} |\psi_j\rangle \otimes |N_b\rangle$$

where

$$|\psi_j\rangle = \sum_{\mathbf{k}_1 \dots \mathbf{k}_j} \psi_j(\mathbf{k}_1, \dots, \mathbf{k}_j) c^\dagger_{\mathbf{k}_1} c^\dagger_{-\mathbf{k}_1} \dots c^\dagger_{\mathbf{k}_L} c^\dagger_{-\mathbf{k}_L} |0\rangle$$

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Result

$$\omega = 0$$

No topological phase transition

Smooth crossover from BCS to BEC

Summary and outlook

- Obtained exact Bethe ansatz solution of $p + ip$-wave pairing model coupled to a bosonic mode
- Computed explicit formulae for some simple correlators
- Can explore crossover from BCS state to BEC along integrable manifold
- Many open questions concerning the model without interaction especially the nature of the transition from weak pairing to weak coupling regimes