The exact g-function: Solution of a puzzle

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Periodic bc. vs. integrable boundary

Periodic boundary conditions:

$$F^P_{\min} = -LT \int_{-\infty}^{\infty} \frac{d\theta}{2\pi}\; p'(\theta) \log\left(1 + e^{-\varepsilon(\theta)}\right)$$

$$\varepsilon(\theta) = \frac{e(\theta)}{T} - \int \frac{d\theta'}{2\pi}\varphi(\theta-\theta') \log\left(1 + e^{-\varepsilon(\theta')}\right)$$

Boundaries:

$$F^B_{\min} = F^P_{\min} - T \int_{-\infty}^{\infty} \frac{d\theta}{4\pi}\Theta_{ab}(\theta) \log\left(1 + e^{-\varepsilon(\theta)}\right)$$

$$\Theta_{ab}(\theta) = -i\frac{d}{d\theta}\log\left(R_a(\theta)R_b(\theta)S(-2\theta)\right) - 2\pi\delta(\theta)$$

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A puzzle

- This result is not complete! The missing piece: boundary independent!

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- Using a cluster expansion of the partition function:

$$\log(g_a) = \frac{1}{2}\int_{-\infty}^{\infty}\frac{d\theta}{4\pi}\Theta_{aa}(\theta)\log\left(1+e^{-\varepsilon(\theta)}\right)+$$
$$+\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\left(\prod_{i=1}^{n}\frac{d\theta_i}{2\pi}\frac{1}{1+e^{\varepsilon(\theta_i)}}\right)\varphi(\theta_1+\theta_2)\varphi(\theta_2-\theta_3)\ldots\varphi(\theta_n-\theta_1)$$

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Z and Quadratic fluctuations

- The partition function:

$$Z = \sum_n e^{-E_n/T}$$

- In the thermodynamic limit:
  densities $\rho_o(\theta)$ $\rho_h(\theta)$ for occupied roots and holes

$$Z = \int \mathcal{D}\rho_o(\theta)\mathcal{D}\rho_h(\theta)\ e^{-F[\rho_o,\rho_h]}$$

- Saddle point contribution $\equiv$ TBA
- Let us calculate the quadratic fluctuations too!

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Woynarovich's result

- Periodic b.c.:

$$Z = \frac{1}{\det\left(\hat{1} - \hat{P}\right)} e^{-F^{P}_{min}/T}$$

$$\left(\hat{P}(f)\right)(\theta) = \int_{\infty}^{\infty} \frac{d\theta'}{2\pi} \frac{1}{1 + e^{\varepsilon(\theta')}} \varphi(\theta - \theta') f(\theta')$$

- Integrable boundaries:

$$Z = \frac{1}{\det\left(\hat{1} - \hat{Q}^{+}\right)} e^{-F^{B}_{min}/T}$$

$$\left(\hat{Q}^{+}(f)\right)(\theta) = \int_{0}^{\infty} \frac{d\theta'}{2\pi} \frac{1}{1 + e^{\varepsilon(\theta')}} (\varphi(\theta - \theta') + \varphi(\theta - \theta')) f(\theta')$$

? ? ?

$$Z = \sum_{n} e^{-E_{n}/T}$$

$$\Downarrow$$

$$Z = \int \mathcal{D}\rho_{o}(\theta)\mathcal{D}\rho_{h}(\theta)\ e^{-F[\rho_{o},\rho_{h}]}$$

The Jacobian

The Bethe-Yang equations:

$$Q_j(\theta_1, \ldots, \theta_N) = p_j L + \sum_{k \neq j} \delta(\theta_k - \theta_j) = 2\pi I_j$$

Define the denstiy of states as

$$\rho_N(\theta_1, \ldots, \theta_N) = \det \frac{\partial Q_j}{\partial \theta_k}$$

Alternatively: Gaudin-determinant, norm of state

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The Jacobian – examples

1 particle:

$$\rho_1(\theta) = p'(\theta)L$$

2 particles:

$$\rho_2(\theta_1, \theta_2) = p'(\theta_1)p'(\theta_2)L^2 + \varphi(\theta_1 - \theta_2)(p'(\theta_1) + p'(\theta_2))L$$

In general:

$$\rho_N = \prod_{j=1}^{N} \rho_1(\theta_j) \times \left(1 + \mathcal{O}\left(\frac{1}{L}\right)\right)$$

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The thermodynamic limit

Using the methods of Korepin and others:

$$\rho_N \quad \rightarrow \quad \mathcal{N} \times \prod_{j=1}^{N} 2\pi L(\rho_o(\theta_j) + \rho_h(\theta_j))$$

Proposal:

$$Z = \mathcal{N} \times \int \mathcal{D}\rho_o(\theta)\mathcal{D}\rho_h(\theta)\ e^{-F[\rho_o,\rho_h]}$$

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The exact g-function

The results of P. Dorey et al. reproduced!

$$\log\left(\frac{\det\left(\hat{1}-\hat{Q}^{-}\right)}{\det\left(\hat{1}-\hat{Q}^{+}\right)}\right)=$$

$$\sum_{n=1}^{\infty}\frac{1}{n}\int_{-\infty}^{\infty}\frac{d\theta_1}{2\pi}\cdots\int_{-\infty}^{\infty}\frac{d\theta_n}{2\pi}\left(\prod_{i=1}^{n}\frac{1}{1+e^{\varepsilon(\theta_i)}}\right)\varphi(\theta_1+\theta_2)\varphi(\theta_2-\theta_3)\ldots\varphi(\theta_n-\theta_1)$$

where

$$(\hat{Q}^{\pm}(f))(x)=\int_{0}^{\infty}\frac{dy}{2\pi}\left(\varphi(x-y)\pm\varphi(x+y)\right)\frac{1}{1+e^{\varepsilon(y)}}f(y)$$

General form of the BY equations

$$e^{i\alpha p_j L}\mathcal{R}_{BY}(\theta_j)\prod_{k\neq j}\mathcal{S}_{BY}(\theta_k,\theta_j)=1 \qquad \theta\in\mathcal{B}$$

Periodic bc:

$$\mathcal{R}_{BY}(\theta_j)=1 \qquad \mathcal{S}_{BY}(\theta_k,\theta_j)=S(\theta_k-\theta_j)$$

$$\alpha=1 \qquad \mathcal{B}=\mathbb{R}$$

Integrable boundaries:

$$\mathcal{R}_{BY}(\theta_j)=R_a(\theta_j)R_b(\theta_j) \qquad \mathcal{S}_{BY}(\theta_k,\theta_j)=S(\theta_k-\theta_j)S(\theta_k+\theta_j)$$

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Two assumptions

- The solutions yield real rapidities
- The Hilbert space is spanned by the solutions

The derivation

- Introduce densities $\rho_o(\theta)$ and $\rho_h(\theta)$, $\quad \theta \in \mathcal{B}$
- The constraint:

$$\rho_h(\theta) + \rho_o(\theta) = \frac{1}{2\pi}\sigma(\theta) + \int_{\mathcal{B}} \frac{d\theta'}{2\pi} K_1(\theta,\theta')\rho_o(\theta'),$$

where

$$K_1(\theta,\theta') = -i\frac{\partial}{\partial\theta}\log \mathcal{S}_{BY}(\theta,\theta')$$

$$\sigma(\theta) = \alpha\frac{d}{d\theta}p(\theta) + \Theta(\theta), \qquad (\alpha = 1,2)$$

$$\Theta(\theta) = -i\frac{d}{d\theta}\frac{1}{L}\log\left(R_{BY}(\theta)\mathcal{S}_{BY}(-\theta,-\theta)\right)$$

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Entropy and free energy

Number of micro-canonical configurations:

$$\Omega = \prod_{\theta} \omega(\rho_o(\theta)), \qquad \omega = \binom{L\Delta\theta\rho_o(\theta)}{L\Delta\theta(\rho_o(\theta) + \rho_h(\theta))}$$

Free energy: $F = E - TS = E - T\log\Omega$
Free energy functional:

$$F[\rho(\theta)] = L\sum_{\theta} \Big(e(\theta)\rho_o(\theta) - Ts(\theta)\Big)\Delta\theta$$

$$s(\rho(\theta)) = \rho(\theta)\log(\rho(\theta)) - \rho_o(\theta)\log(\rho_o(\theta)) - \rho_h(\theta)\log(\rho_h(\theta)) + \ldots$$

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The variation procedure

One introduces

$$\frac{\rho_o(\theta)}{\rho_h(\theta)} = e^{-\varepsilon(\theta)}$$

The TBA equation:

$$\varepsilon(\theta) = e(\theta)/T - \int_{\mathcal{B}} \frac{d\theta'}{2\pi}\, K_1(\theta,\theta') \log\left(1 + e^{-\varepsilon(\theta')}\right),$$

The free energy:

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The partition function

Adding the Fredholm determinants

The net result:

$$Z = \frac{\det\left(\hat{1} - \hat{K}_2\right)}{\det\left(\hat{1} - \hat{K}_1\right)} e^{-\beta F_{\min}}$$

where

$$K_1(\theta,\theta') = -i\frac{\partial}{\partial\theta}\log\mathcal{S}_{BY}(\theta,\theta') \qquad K_2(\theta,\theta') = i\frac{\partial}{\partial\theta'}\log\mathcal{S}_{BY}(\theta,\theta')$$

(BP, `arXiv:1003.5542`)

Relativistic massles scattering

- Bethe-equations with boundaries:

$$e^{i2p_jL}R_a(\theta)R_b(\theta)\prod_{k\neq j}S_{LL}(\theta_k-\theta_j)S_{LR}(\theta_k+\theta_j)=1$$

- One extracts:

$$\alpha=2, \qquad R_{BY}(\theta)=R_a(\theta)R_b(\theta)S_{LR}(-2\theta),$$

$$\mathcal{S}_{BY}(\theta,\theta')=S_{LL}(\theta-\theta')S_{LR}(\theta+\theta')$$

- Integration kernels for the Fredholm determinants:

$$K_1(\theta,\theta')=\varphi_{LL}(\theta-\theta')+\varphi_{LR}(\theta+\theta')$$
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Relativistic massles scattering – Examples

Flow from the tricritical Ising to Ising

$$S_{LL} = 1 \qquad S_{LR} = -\tanh(\theta/2 - i\pi/4)$$

$$\varphi(\theta) = -i\frac{d}{d\theta}S_{LR}(\theta) = \frac{1}{\cosh\theta}$$

The boundary independent part:

$$\log g_0 = \sum_{j=1}^{\infty}\frac{1}{2j-1}\left(\prod_{i=1}^{2j-1}\int\frac{d\theta_i}{2\pi}\frac{1}{1+e^{\varepsilon(\theta_i)}}\right)\varphi(\theta_1+\theta_2)\varphi(\theta_2+\theta_3)\ldots\varphi(\theta_{2j-1}+\theta_1)$$

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Relativistic massles scattering – Examples

The flow $M_{3,5} + \Phi_{2,1} \to M_{2,5}$

$$S_{LL} = S_{LY} \qquad S_{LR} = (S_{LY})^{-1}$$

where

$$S_{LY}(\theta) = \frac{\sinh\theta + i\sin(\pi/3)}{\sinh\theta - i\sin(\pi/3)}$$

The boundary independent part:

$$\log g_0 = \frac{1}{2}\log\frac{\det\left(1-\hat{K}_2\right)}{\det\left(1-\hat{K}_1\right)} =$$

$$= -\frac{1}{2}\sum_{n=1}^{\infty}\sum_{a_1\ldots a_n}\frac{1}{n}\left(\prod_{i=1}^{n}\int\frac{d\theta_i}{2\pi}\frac{1}{1+e^{\varepsilon_{a_i}(\theta_i)}}\right)\varphi(\theta_1+\theta_2)\varphi(\theta_2-\theta_3)\ldots\varphi(\theta_n-\theta_1)$$

The summations run over $a_i = 1, 2$ and the pseudo-energies are given by

$$\varepsilon_1(\theta) = \varepsilon(\theta) \qquad \varepsilon_2(\theta) = \varepsilon(-\theta)$$

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This is the end

Thank you for your attention!