The vertex operator algebra of conformal loop ensembles

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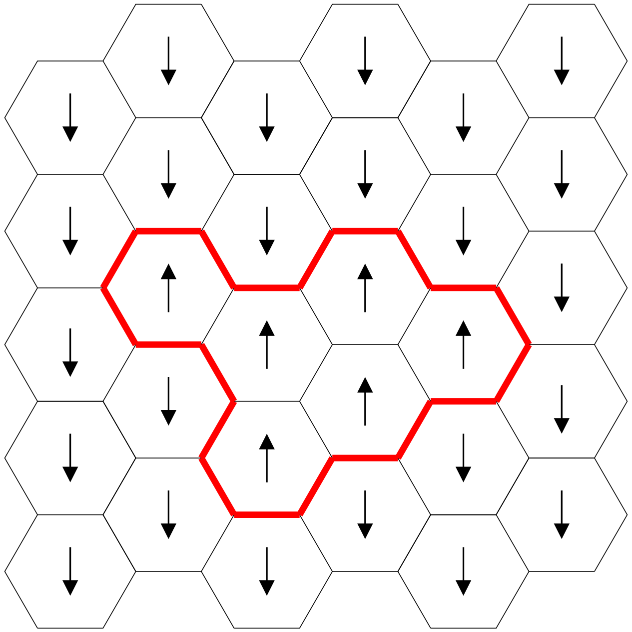
Scaling limits and emergent behaviours

Example: the Ising model

Microscopic model: measure on functions σ from faces of a lattice (ex: hexagonal) to some set (ex: spin $\{\uparrow,\downarrow\} = \{+1,-1\}$), with properties of locality, homogeneity



$$\mu(\sigma) = \exp\left[\beta \sum_{\text{neighbouring faces } j,k} \sigma(j)\sigma(k)\right]$$

Critical point $\beta = \beta_c$: Emergent universal large-distance correlations!

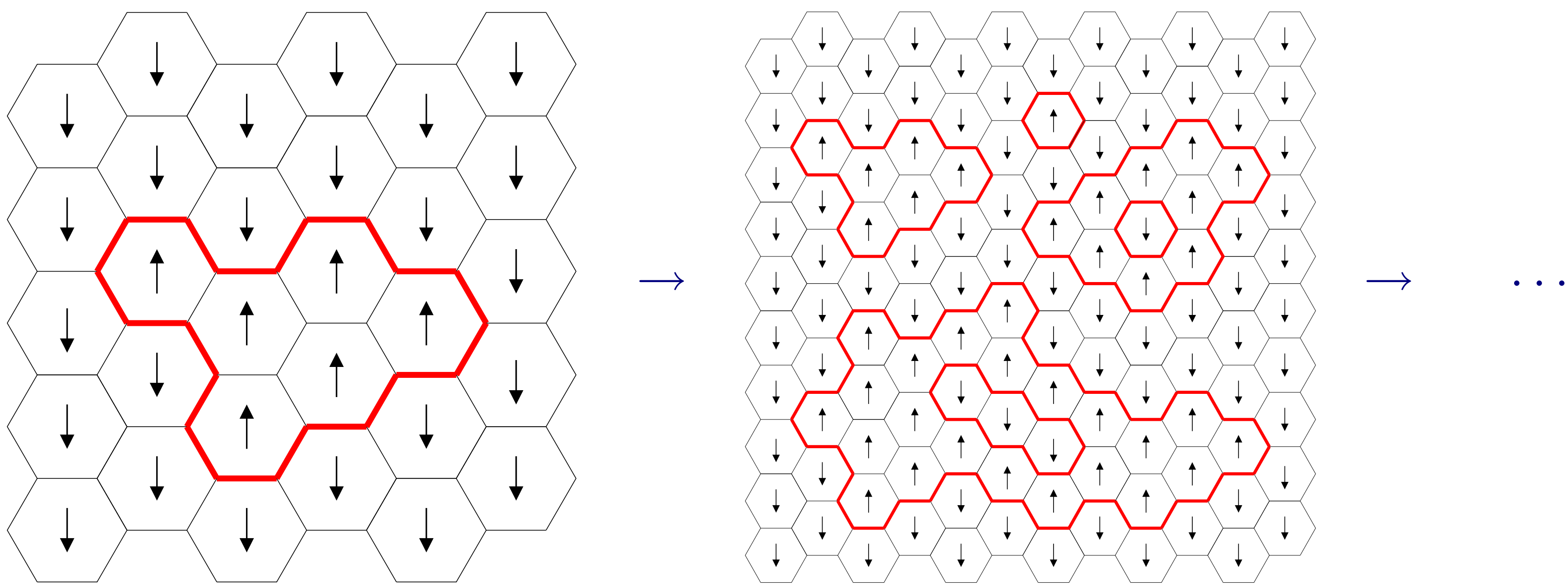
Quantum field theory, a theory for emergent correlations:

The scaling limit of expectations is:

$$\lim_{\varepsilon \to 0} \varepsilon^{-1/4}\, \mathbb{E}^{(\beta=\beta_c-\alpha\varepsilon)}[\sigma(x/\varepsilon)\sigma(y/\varepsilon)] = C^{(\alpha)}(x,y)$$

($x, y \in \mathbb{R}^2$). The coefficient $C^{(\alpha)}(x,y)$ is a correlation function in a QFT

$$C^{(\alpha)}(x,y) = \langle \mathcal{O}(x)\mathcal{O}(y)\rangle^{(\alpha)}$$



The basic ingredients of QFT are

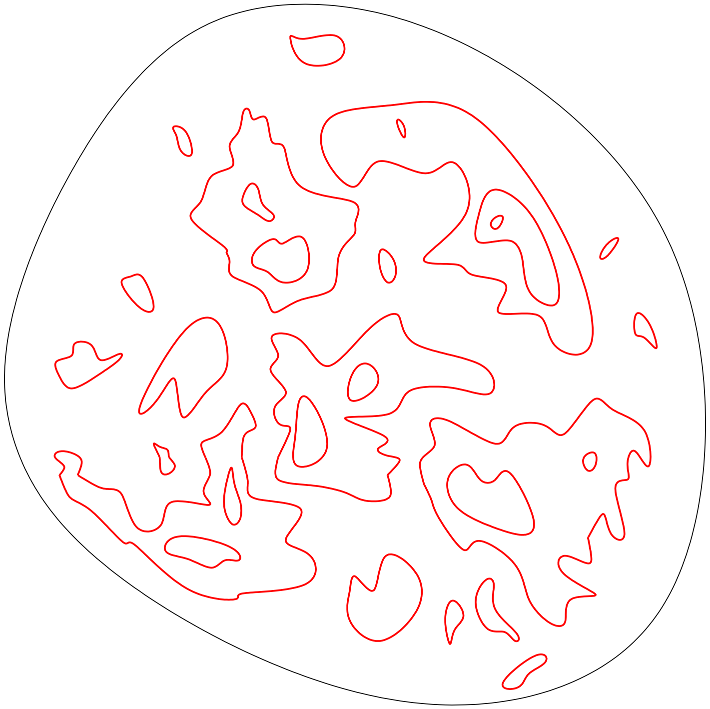
- Local fields $\mathcal{O}(x) \Leftrightarrow$ local variables $1,\ \sigma(k),\ \sigma^2(k),\ \sigma(k)\sigma(\text{neighbour of } k),\ \ldots$
- correlation functions $\langle\cdot\rangle \Leftrightarrow$ expectations of products of local variables $\mathbb{E}[\cdot]$

Some questions:

1. Are there emergent random objects?
2. What is the measure theory for them?
3. Can we reproduce the QFT local correlations from this theory?
4. Can we prove that it emerges from the microscopic theory?

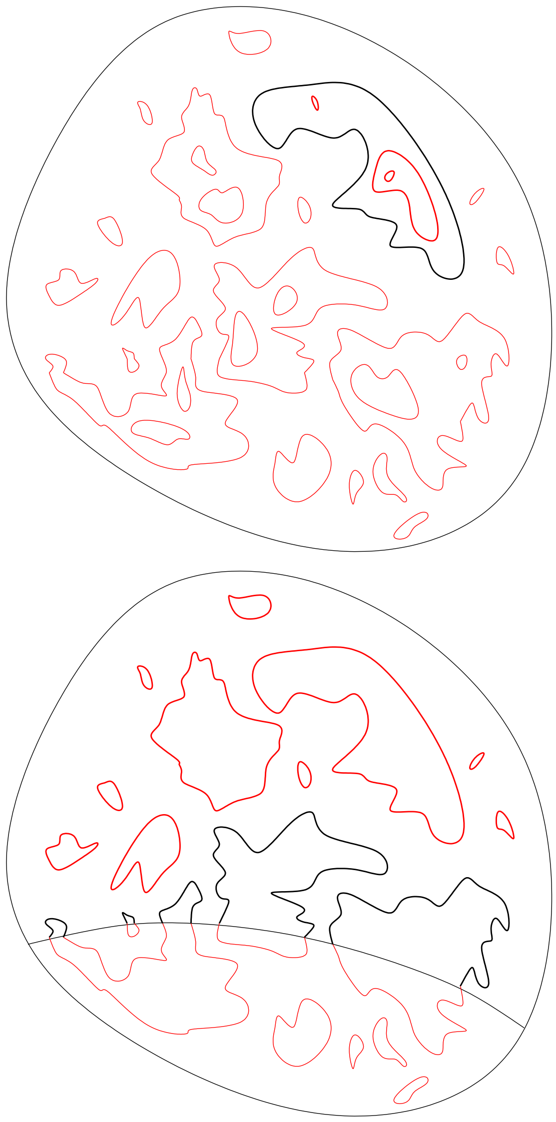
Conformal loop ensembles

Conformal loop ensembles: Consider the set $\mathcal{S}_D$ whose elements are collections of at most a countable infinity of self-avoiding, disjoint loops lying on a simply connected domain D.



A conformal loop ensemble can be seen as a family of measures μ_D on the sets $\mathcal{S}_D$ for all simply connected domains D, with three defining properties.

1. Conformal invariance. For any conformal transformation $f : D \to D'$, we have $\mu_D = \mu_{D'} \cdot f$.
2. Nesting. The measure μ_D restricted on a loop $\gamma \subset D$ and on all loops outside γ is equal to the CLE measure μ_{D_γ} on the domain $D_\gamma \subset D$ delimited by γ.
3. Conformal restriction. Given a domain $B \subset D$ such that $D \setminus B$ is simply connected, consider $\tilde{B}$, the closure of the set of points of B and points that lie inside loops that intersect B. Then the measure on each component C_i of $D \setminus \tilde{B}$, obtained by restriction on loops that intersect B, is μ_{C_i}.



[Sheffield, Werner 2005 –]

Some properties:

- One-parameter family of measures: $\kappa \in (8/3, 4]$
- Fractal dimension of loops: $1 + \kappa/8$
- Almost every point is almost surely surrounded by infinitely many loops
- Should describe all central charges between 0 and 1 : $c = \frac{(6-\kappa)(3\kappa-8)}{2\kappa}$

A fundamental field of CFT: the stress-energy tensor

Conformal field theory: with g conformal on a domain D of $\hat{\mathbb{C}}$, there exists a map $\mathcal{O} \mapsto g \cdot \mathcal{O}$ such that

$$\left\langle \prod_i \mathcal{O}_i(z_i) \right\rangle_D = \left\langle \prod_i (g \cdot \mathcal{O}_i)(g(z_i)) \right\rangle_{g(D)}$$

For primary fields, $(g \cdot \mathcal{O})(g(z)) = (\partial g)^h (\bar{\partial}\bar{g})^{\tilde{h}} \mathcal{O}(g(z))$, with $h, \tilde{h} \in \mathbb{R}^+$. Locality and basic QFT concepts: existence of stress-energy tensor $T(w)$, with conformal Ward identities:

$$\left\langle T(w) \prod_i \mathcal{O}(z_i) \right\rangle_D \sim \sum_i \left(\frac{h_i}{(w - z_i)^2} + \frac{1}{w - z_i} \frac{\partial}{\partial z_i} \right) \left\langle \prod_i \mathcal{O}(z_i) \right\rangle_D$$

T is not a primary field, there is a central charge $c \in \mathbb{R}$:

$$(g{\cdot}T)(g(w)) = (\partial g(w))^2 T(g(w)) + \frac{c}{12}\{g, w\}\,, \quad \{g, w\} = \left(\frac{\partial^3 g(w)}{\partial g(w)} - \frac{3}{2}\left(\frac{\partial^2 g(w)}{\partial g(w)}\right)^2\right)$$

Boundary condition $T = \bar{T}$ on $\mathbb{R}$ [Cardy 1984] and analyticity arguments [BPZ 1984] $\Rightarrow$ exact w dependence of $\langle T(w) \prod_i \mathcal{O}(z_i)\rangle_D$.

There’s more: OPE, associativity, commutativity,...

$\Longrightarrow$ Vertex operator algebra [Kac, Lepowsky, ...].

A **vertex operator algebra** $(V, Y, \mathbf{1}, \omega)$ is a $\mathbb{Z}$–graded quasi-finite vector space

$$V = \coprod_{n \in \mathbb{Z}} V_{(n)}; \text{ for } v \in V_{(n)}, \text{ wt } v = n,$$

equipped with a linear map $Y(\cdot, x)$:

$$\begin{aligned} Y(\cdot, x) : V &\longrightarrow (\text{End } V)[[x, x^{-1}]] \\ v &\longmapsto Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1}, \quad v_n \in \text{End } V, \end{aligned}$$

where $Y(v, x)$ is called the **vertex operator** associated with v, and two particular vectors, $\mathbf{1}$, $\omega \in V$, called respectively the **vacuum vector** and the **conformal vector**, with the some properties, mainly:

vacuum property:

$$Y(\mathbf{1}, x) = 1_V \quad (1_V \text{ is the identity on } V);$$

creation property:

$$Y(v, x)\mathbf{1} \in V[[x]] \quad \text{and} \quad \lim_{x \to 0} Y(v, x)\mathbf{1} = v \ ;$$

Virasoro algebra conditions: Let

$$L(n) = \omega_{n+1} \ \text{ for } n \in \mathbb{Z}, \ \text{ i.e., } \ Y(\omega, x) = \sum_{n \in \mathbb{Z}} L(n) x^{-n-2} \ .$$

Then

$$[L(m), L(n)] = (m-n)L(m+n) + c_V \frac{m^3 - m}{12} \, \delta_{n+m,0} \, 1_V$$

for $m, n \in \mathbb{Z}$, where $c_V \in \mathbb{C}$ is the central charge, *Jacobi identity:*

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y(u,x_1)Y(v,x_2)-x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y(v,x_2)Y(u,x_1)$$
$$=x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y(Y(u,x_0)v,x_2)$$

where

$$\delta(x)=\sum_{n\in\mathbb{Z}}x^n, \qquad (x_1-x_2)^{-n}=\sum_{k=0}^{\infty}\frac{(n)_k}{k!}x_1^{-n}\left(\frac{x_2}{x_1}\right)^k$$

Virasoro vertex operator algebra:

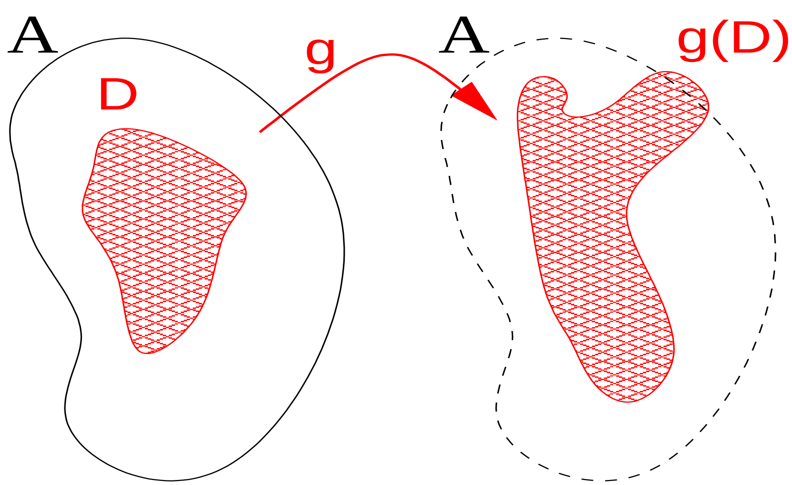
- $V =$ Virasoro highest-weight (or Verma) module
- State-field correspondence: $(\partial^{n_1} T \cdots \partial^{n_k} T) \mapsto L(-2-n_1) \cdots L(-2-n_k)\mathbf{1}$
- Product of vertex operators reproduce correlation functions, e.g.:

$$\left\langle (TT)(w_1)T(w_2)\prod_i \mathcal{O}(z_i) \right\rangle_D = \left(v_{\{z_i\},D}\,,\, Y(L(-2)^2\mathbf{1}, w_1)Y(L(-2)\mathbf{1}, w_2)\mathbf{1}\right)$$

A general analytic set-up with Virasoro vertex operator algebra structure [BD 2010]

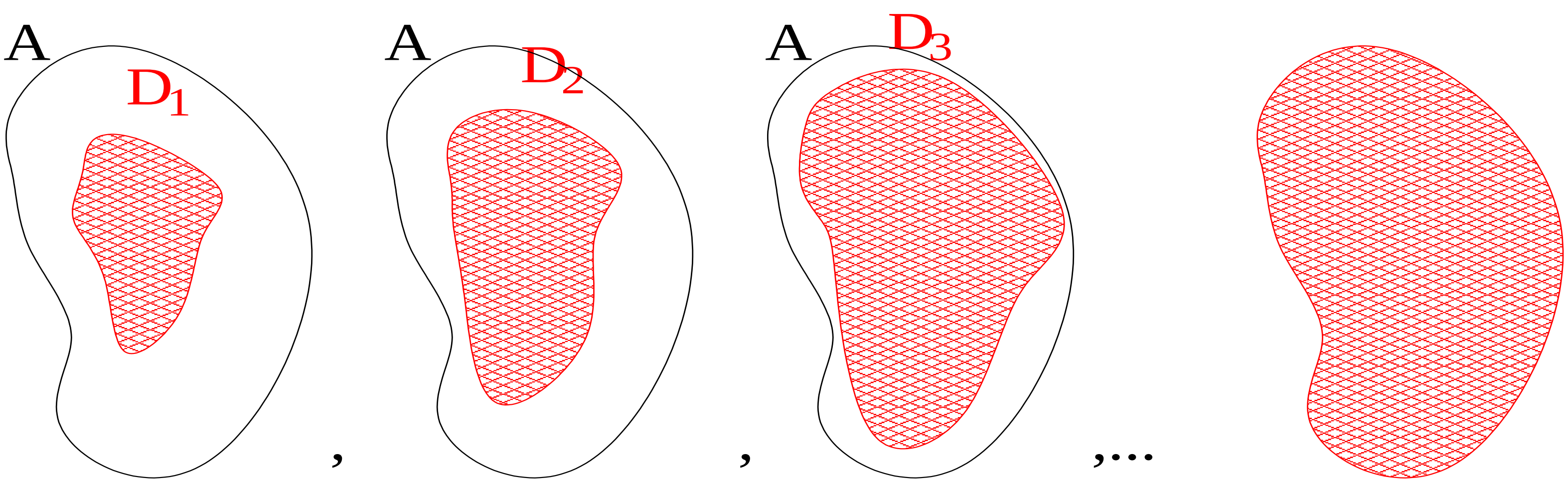
Local manifold of conformal maps around the identity:

Consider a simply connected bounded domain A and the set of maps g that are conformal on some domain (below: the domain D) inside A.



A-topology:

- growing domains D_n tend to A
- compact convergence: uniform convergence on any compact subset



$$\lim_{n\to\infty} \sup(g_n(z) - z : z \in D_n) = 0$$

- Topology **preserved under conformal maps** $G : A \longrightarrow B$ between simply connected domains $A,\ B$.
- Leads to **manifold structure:** A certain restriction of the A-topology gives a homeomorphism to the vector space $\mathtt{H}(A)$ of holomorphic functions on A with compact convergence topology (A^*-manifold).
- Family $(g_\eta : \eta > 0) \in \mathtt{F}(A)$:

$$\lim_{\eta\to 0} g_\eta = \mathrm{id} \quad (A\text{-topology}), \quad \lim_{\eta\to 0} \frac{g_\eta(z) - z}{\eta} = h(z) \quad \exists \quad (\text{compactly for } z \in A).$$

- Tangent space in general is $\mathtt{H}^{>}(A)$: holomorphic functions $h(z)$ on A except for $O(z^2)$ as $z \longrightarrow \infty$ if $\infty \in A$.

Derivatives:

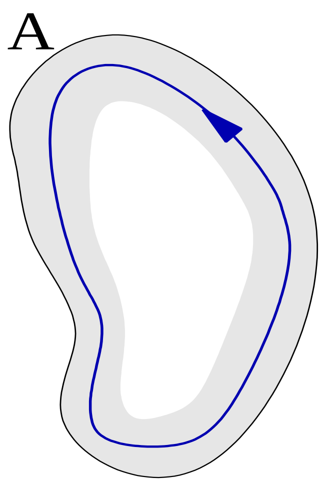
Derivative of a function on A^*-manifold at id = element of the **cotangent space** at id.

Need **continuous dual** $\mathtt{H}^{>*}(A)$ **(space of continuous linear functionals) of** $\mathtt{H}^{>}(A)$.

Any continuous linear functional $\Upsilon : \mathtt{H}^{>}(A) \longrightarrow \mathbb{R}$ is of the form

$$\Upsilon(h) = \oint_{\partial A^-} \mathrm{d}z\, \alpha(z) h(z) + \oint_{\partial A^-} \bar{\mathrm{d}}\bar{z}\, \bar{\alpha}(\bar{z}) \bar{h}(\bar{z})$$

for some α holomorphic on an annular neighbourhood of ∂A inside A.



Arbitrariness of α: functional Υ is characterised by a **class of functions:**

$$\mathcal{C} = \left\{\alpha + u : u \in \mathtt{H}^{<}(A)\right\}$$

where $\mathtt{H}^{<}(A)$: holomorphic functions $h(z)$ on A with $O(z^{-4})$ as $z \longrightarrow \infty$ if $\infty \in A$.

- Function $f : \Omega \longrightarrow \mathbb{R}$
- Point $\Sigma \in \Omega$
- Action $g(\Sigma) \in \Omega$ for any g in A-neighbourhood of id.

A-differentibility: for any $(g_\eta : \eta > 0) \in \mathtt{F}(A)$,

$$\lim_{\eta \to 0} \frac{f(g_\eta(\Sigma)) - f(\Sigma)}{\eta} = \nabla^A f(\Sigma) h, \qquad \nabla^A f(\Sigma) \in \mathtt{H}^{>*}(A)$$

Some definitions and notations:

- $\nabla^A f(\Sigma)$: the **conformal A-derivative of f at Σ**
- $\nabla_h f(\Sigma) = \nabla^A f(\Sigma) h$: the **directional derivative of f at Σ in the direction h**
- $\Delta^A f(\Sigma)$: the **holomorphic A-class of f at Σ**

Transformation under conformal maps:

- A-differentiability of f at Σ $\quad\Leftrightarrow\quad$ $g(A)$-differentiability of $f \circ g^{-1}$ at $g(\Sigma)$
- "Holomorphic dimension-2" transformation property for the holomorphic A-class:

$$\Delta^A f(\Sigma) = (\partial g)^2 \left(\Delta^{g(A)}(f \circ g^{-1})(g(\Sigma))\right) \circ g.$$

The global holomorphic A-derivative

If f is **globally invariant**, i.e. invariant under möbius maps, then:

$$\Delta_z^{[A]} f(\Sigma) := \text{unique function in } \Delta^A f(\Sigma) \text{ holomorphic on } \hat{\mathbb{C}} \setminus A$$

- Exists and only depends on the sector $[A]$
- Holomorphic for $z \in \hat{\mathbb{C}} \setminus \cap[A]$
- $O(z^{-4})$ as $z \to \infty$
- "Holomorphic dimension-2" transformation property for G a möbius map:

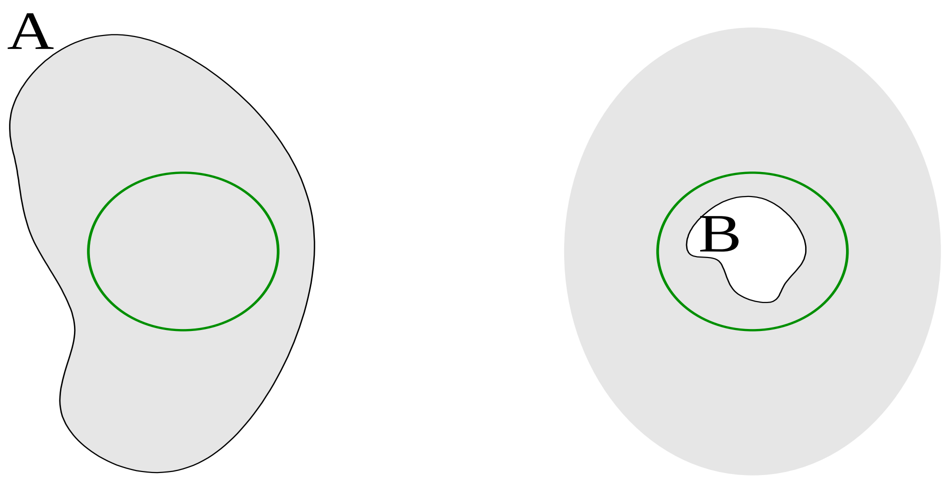
$$\Delta_z^{[A]} f(\Sigma) = (\partial G(z))^2 \Delta_{G(z)}^{[G(A)]} f(G(\Sigma))$$

Sectors:

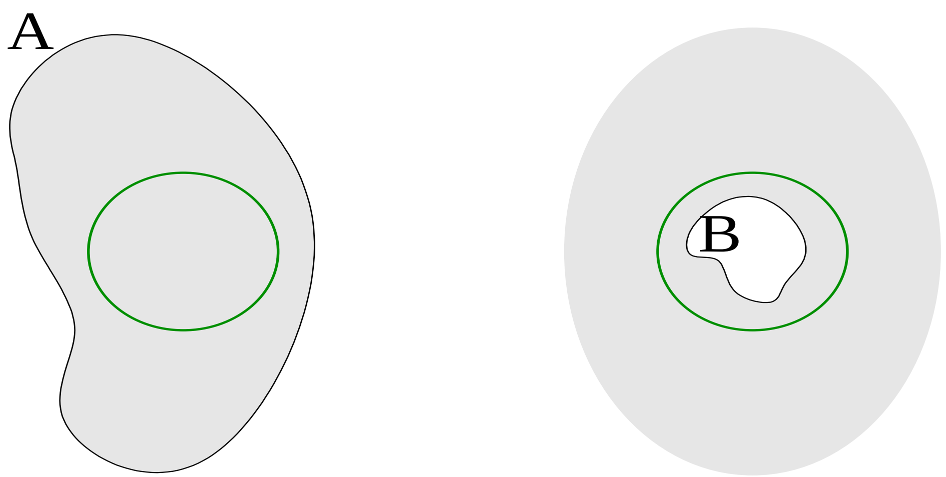
- Consider set Ξ of all domains A such that f is A-differentiable.
- Equivalence relation: domains with intersecting complements are equivalent, complete by transitivity.
- Denote by $[A]$ the equivalence class, or **sector** containing A

$\Rightarrow$ Ξ **is divided into sectors where global holomorphic derivatives are the same**

Example: $\Sigma =$ a circle, $\Omega =$ a space of smooth loops. Two natural sectors: $[A] =$ bounded sector, $[B] =$ another sector:



Consider two domains A and B such that $\hat{\mathbb{C}} \setminus A \subset B$.



If f is A**-invariant**, i.e. invariant under maps conformal on A, then: for $g : A \to A'$,

$$\Delta_z^{[B]} f(\Sigma) = (\partial g(z))^2 \, \Delta_{g(z)}^{[\hat{\mathbb{C}} \setminus g(\hat{\mathbb{C}} \setminus B)]} f(g(\Sigma))$$

Virasoro vertex operator algebra structure of conformal derivatives

Consider

$$h_{n,w}(z) = (w-z)^{n+1}, \quad \Delta[h_{n,w}] = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, e^{-i\theta}\nabla_{e^{i\theta}h_{n,w}}$$

We have $\Delta[h_{-2,w}] = \Delta_w^{[\hat{\mathbb{C}}_w]}$, and the Witt algebra

$$\Delta[h_{n,0}]\Delta[h_{m,0}]f(\Sigma) - \Delta[h_{m,0}]\Delta[h_{n,0}]f(\Sigma) = (n-m)\Delta[h_{n+m,0}]f(\Sigma)$$

Consider a function $Z(\Sigma)$ with the conditions

$$\Delta[h_{n,0}]\Delta[h_{m,0}]\log Z(\Sigma) = \begin{cases} 0 & \left(\begin{array}{l} n \geq 1, -1-n \leq m \leq -2, n+m \neq 0 \\ \text{or} \\ n \leq -2, -1 \leq m \leq -2-n \end{array} \right) \\ \text{const(n)} & (m+n=0, n \geq -1) \end{cases}$$

Virasoro algebra:

$$L(n) = \begin{cases} Z^{-1}\Delta[h_{n,0}]Z & (n \leq -2) \\ \Delta[h_{n,0}] & (n \geq -1) \end{cases}$$

Function $f(\Sigma)$ invariant under maps conformal on $D \ni 0 \;\Rightarrow\;$ highest-weight vector $\mathbf{1}$.

If $\cdot$ represents **Lie action** on $\Delta[h_{n,w}]$ given its transformation property under conformal maps, then

$$Z^{-1} \prod_j \Delta[h_{n_j,w}] \cdot \prod_j \Delta[h_{n'_j,w'}] \cdots Z f = Y\Big(\prod_j L(n_j)\mathbf{1}, w\Big) Y\Big(\prod_j L(n'_j)\mathbf{1}, w'\Big) \cdots \mathbf{1}$$

Relation to CFT:

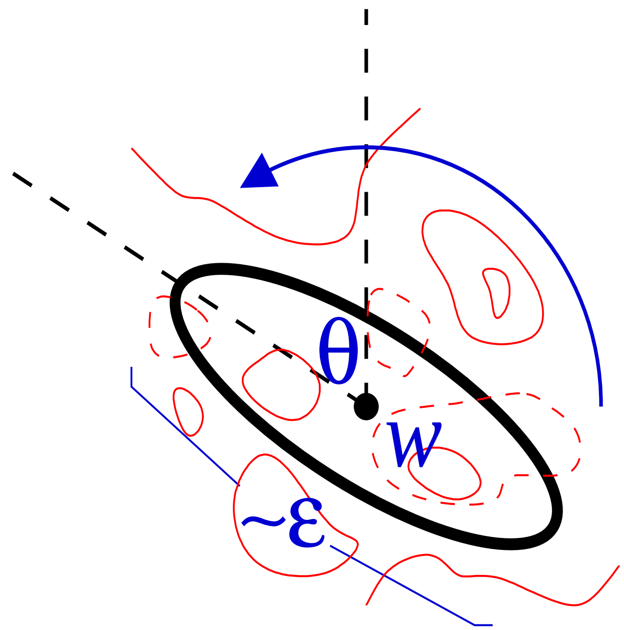
- $f(\Sigma) =$ correlation function $\langle \prod_i \mathcal{O}(z_i) \rangle_D$ with Lie action on fields and domain D
- Insertion of $(\prod_j \partial^{n_j} T)(w)$ given by action of $Z^{-1} \prod_j \Delta[h_{-2-n_j,w}] Z$
- $Z =$ relative partition function

$$\frac{Z_D Z_{\hat{\mathbb{C}} \setminus \overline{C}}}{Z_{D \setminus \overline{C}}}, \quad \overline{C} \subset D$$

Application to CLE [BD 2009,2010]

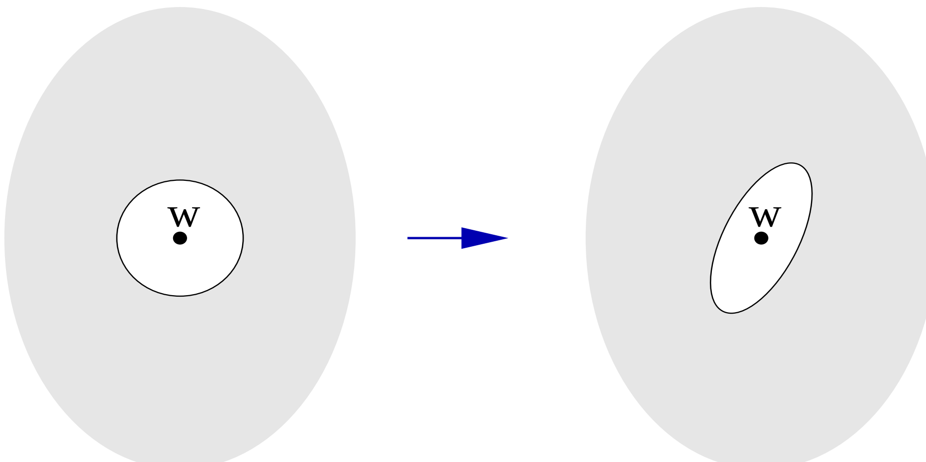
- $f(\Sigma) =$ probability function, or expectations, or limits thereof
- Relate Z-conjugated conformal derivatives to **local random variables**

$$Z^{-1}\Delta[h_{-2,w}]Z \mapsto$$

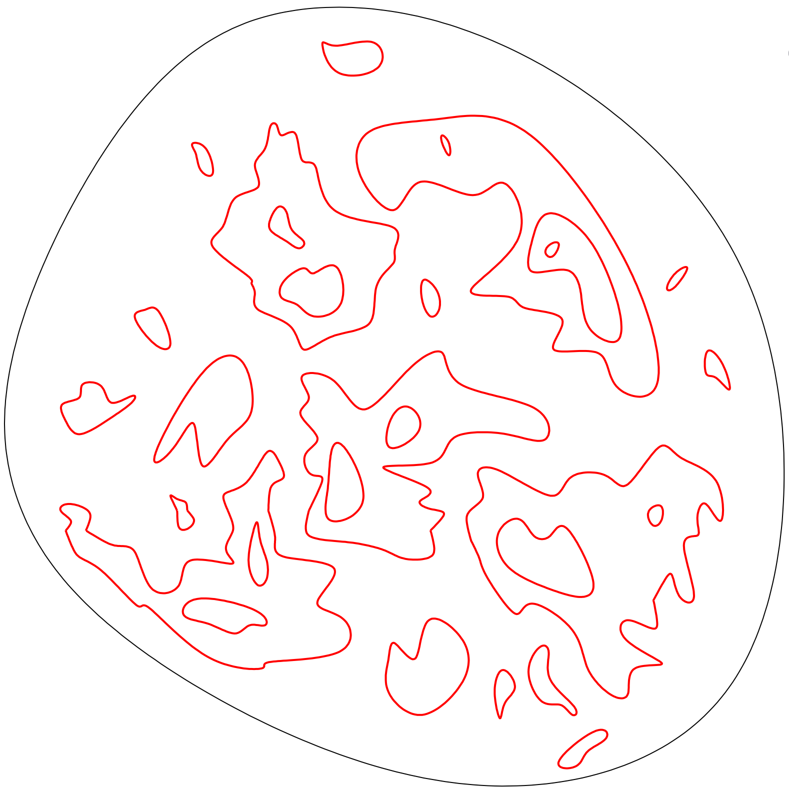


- Universality: anything that is **local** and that **transforms like the stress-energy tensor** in fact **is** the stress-energy tensor (likewise for descendants). Hence "free-field" representation for the whole vertex operator algebra constructed out of "bosons" $\phi(z_1)\phi(z_2) \mapsto n(z_1, z_2) =$ number of loops surrounding both points z_1, z_2.

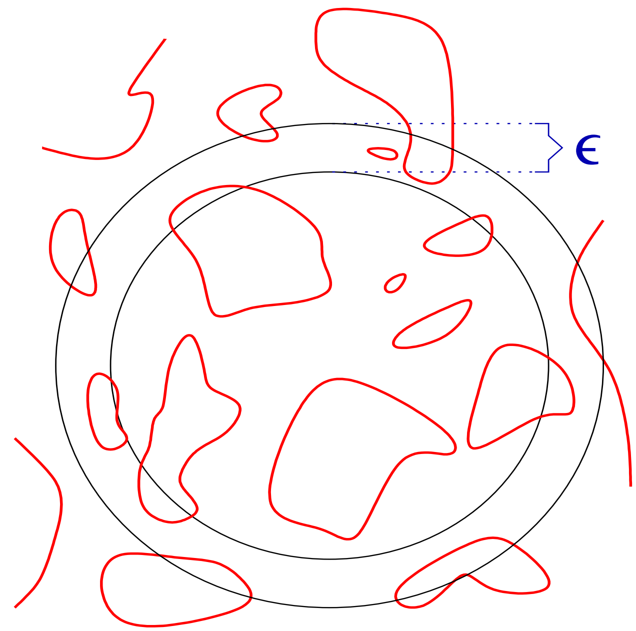
Interpret $\Delta[h_{-2,w}]f(\Sigma)$ **geometrically**: $\mathrm{id} + \varepsilon^2 e^{2i\theta} h_{-2,w}$ gives



CLE:



Renormalised probabilities that no loop crosses boundary ∂A of thickness $\epsilon \to 0$:



Perspectives

- Applications to other probability models of CFT, or to other situations altogheter, where there is a concept of scale invariance (e.g. fractals?)
- Other symmetry currents when internal symmetries are present...