Fused pair of integrable defects in the sine-Gordon model

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Defects

Consider relativistic massive field theories such as the Toda models

- Purely transmitting defects
- Conservation of a generalised momentum
  - Type I defects (Ed Corrigan, Peter Bowcock, CZ)
    - The a_r Toda models
    - Classical integrability
  - Type II defects (Ed Corrigan, CZ)
    - A bigger set of Toda theories, which incorporates, for instance, the $a_2^{(2)}$ Toda model (or Tzitzéica or Bullough-Dodd or Zhiber-Mikhailov-Shabat model)

The classical type II defect

Consider two relativistic field theories with fields u and v

$$\mathcal{L} = \theta(-x)\mathcal{L}_u + \theta(x)\mathcal{L}_v + \delta(x)\left(2q\lambda_t - D(\lambda, u, v)\right)$$

- Equations of motion

$$\partial^2 u = -U_u \quad x < 0 \qquad \partial^2 v = -V_v \quad x > 0$$

- Defect conditions in $x = 0$

$$2q_x = -D_p \qquad 2p_x - 2\lambda_t = -D_q \qquad 2q_t = -D_\lambda$$

$$q = \frac{u - v}{2} \qquad p = \frac{u + v}{2}$$

- The defect potential D is determined by momentum conservation. It seems that, in the presence of a purely transmitting defect, constraints that follow from momentum conservation are 'equivalent' to the restriction imposed by integrability

Conservation of momentum

Start from the definition of momentum

$$P = \int_{-\infty}^{0} dx\, u_t u_x + \int_{0}^{\infty} dx\, v_t v_x$$

If it exists a functional $\Omega(u, v, \lambda)$ such that $P_t \equiv -\Omega_t$, then $P + \Omega|_{x=0}$ is conserved and it represents the total momentum of the system

Constraints on U, V, Ω:

$$D_p = \Omega_\lambda \qquad D_\lambda = \Omega_p \qquad D_p D_q - \Omega_q D_\lambda = 2(U - V)$$

that is

$$D = f(p + \lambda, q) + g(p - \lambda, q) \qquad \Omega = f(p + \lambda, q) - g(p - \lambda, q)$$

$$f_\lambda g_q - g_\lambda f_q = U - V$$

- Single scalar field theories: Liouville, free massless case, the sinh/sine-Gordon model, free massive case, the Tzitzéica model

$$\text{Tzitzéica potential: } U = e^{2u} + 2\, e^{-u}$$

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Consider λ and its conjugate momentum $\pi_\lambda = 2\, q$

- Then, the Poisson bracket of the defect contribution to energy and momentum is related to the potential difference across the defect, that is

$$f_\lambda g_q - g_\lambda f_q = (U - V) \longrightarrow \{\Omega, D\} = (U - V)$$

The classical sine-Gordon model with a defect

$$U = -(e^{u} + e^{-u}) \qquad V = -(e^{v} + e^{-v})$$

The fields u, v are pure imaginary

- A type I defect

$$\mathcal{L}^{I} = \theta(-x)\mathcal{L}_u + \theta(x)\mathcal{L}_v + \delta(x)\left(\frac{1}{2}\left(uv_t - vu_t\right) - D^{I}(u,v)\right)$$

- A type II defect

$$\mathcal{L}^{II} = \theta(-x)\mathcal{L}_u + \theta(x)\mathcal{L}_v + \delta(x)\left(\left(u - v\right)\lambda_t - D^{II}(u,v,\lambda)\right)$$

Equations of motions for the sine-Gordon fields u $(x < 0)$ and v $(x > 0)$

- Type I: defect conditions in $x = 0$

$$u_x - v_t = -D^I_u \qquad v_x - u_t = D^I_v$$

$$D^I = -\sqrt{2}\,\sigma(e^p + e^{-p}) - \frac{\sqrt{2}}{\sigma}(e^q + e^{-q}) \qquad p = \frac{u+v}{2} \quad q = \frac{u-v}{2}$$

They are Bäcklund transformations

- Type II: defect conditions in $x = 0$

$$u_x - 2\lambda_t = -D^{II}_u \qquad v_x - 2\lambda_t = D^{II}_v \qquad u_t - v_t = -D^{II}_\lambda$$

$$D^{II} =$$

$$-\sqrt{2}\sigma(e^{p/2+\lambda/2}(e^{q/2-\tau} + e^{-q/2+\tau}) + e^{-p/2-\lambda/2}(e^{q/2+\tau} + e^{-q/2-\tau}))$$

$$-\frac{\sqrt{2}}{\sigma}(e^{p/2-\lambda/2}(e^{q/2+\tau} + e^{-q/2-\tau}) + e^{-p/2+\lambda/2}(e^{q/2-\tau} + e^{-q/2+\tau}))$$

They are not Bäcklund transformations

- Energy, momentum and topological charge are conserved quantities
- For the sine-Gordon model, the type II defect can be thought of two type I defects fused at the same point in space:

$$\begin{aligned} & \delta(x)\left(\frac{u\lambda_t-\lambda u_t}{2}+\frac{\lambda v_t-v\lambda_t}{2}-D^I(u,\lambda,\sigma_1)-D^I(\lambda,v,\sigma_2)\right) \\ = \; & \delta(x)\left((u-v)\lambda_t-D^{II}(u,v,\lambda)\right) \end{aligned}$$

$$\sqrt{\sigma_1\sigma_2}=\sigma\equiv e^{-\eta} \qquad \sqrt{\frac{\sigma_1}{\sigma_2}}=e^{-\tau}$$

- The complete Lagrangian density for the type II defect is not equivalent to the sum of the Lagrangian densities for two type I defects
- The type II defect cannot be split into two separated type I defects
- Once two type I defects are fused a soliton cannot propagate between them, and therefore the transfer of topological charge between them is suppressed

Solitons & Type II defects

A defect with no discontinuity

$$e^{u/2} = \frac{1+E}{1-E} \qquad e^{v/2} = \frac{1+zE}{1-zE}$$

$$E = e^{ax+bt+c} \qquad a = \sqrt{2}\cosh\theta \qquad b = -\sqrt{2}\sinh\theta$$

$$u(0,-\infty) = v(0,-\infty) = 2\pi i \quad \longrightarrow \quad \lambda(-\infty) = 0, 2\pi i$$

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with a $2\pi i$ discontinuity

$$e^{u/2} = \frac{1+E}{1-E} \qquad e^{v/2} = -\frac{1+zE}{1-zE}$$

$$u(0,-\infty) = 2\pi i \qquad v(0,-\infty) = 0 \quad \longrightarrow \quad \lambda(-\infty) = 0, 2\pi i$$

For a general value of t the defect conditions fix the expressions for z and $\lambda(t)$

Examples

- No discontinuity - $\lambda(-\infty) = 2\pi i$:

$$
\begin{aligned}
z &= \coth\left(\frac{\eta+\tau-\theta}{2}\right)\coth\left(\frac{\eta-\tau-\theta}{2}\right) \\
\lambda(t) &= \dots \\
\mathcal{E} &= 8\sqrt{2}\cosh\theta + [-8\sqrt{2}\cosh\eta\cosh\tau]
\end{aligned}
$$

If η and τ are real and positive this is the lowest energy configuration

Examples

- No discontinuity - $\lambda(-\infty) = 2\pi i$:

$$\begin{aligned} z &= \coth\left(\frac{\eta+\tau-\theta}{2}\right) \coth\left(\frac{\eta-\tau-\theta}{2}\right) \\ \lambda(t) &= \ldots \\ \mathcal{E} &= 8\sqrt{2}\cosh\theta + [-8\sqrt{2}\cosh\eta\cosh\tau] \end{aligned}$$

If η and τ are real and positive this is the lowest energy configuration

- $2\pi i$ discontinuity - $\lambda(-\infty) = 0$:

$$\begin{aligned} z &= \coth\left(\frac{\eta+\tau-\theta}{2}\right) \tanh\left(\frac{\eta-\tau-\theta}{2}\right) \\ \lambda(t) &= \ldots \\ \mathcal{E} &= 8\sqrt{2}\cosh\theta + [8\sqrt{2}\sinh\eta\sinh\tau] \end{aligned}$$

The type II defect as a soliton-like object

Consider the defect with a $2\pi i$ discontinuity and set

$$\eta + \tau = \vartheta \pm \pi i \qquad \eta - \tau = \vartheta$$

- Defect contribution to the energy $\mathcal{E}|_{x=0}$

$$8\sqrt{2}\cosh(\vartheta) \quad \text{- energy of a soliton}$$

- Delay z

$$\tanh^2(\vartheta - \theta)/2 \quad \text{- classical delay for two scattering solitons}$$

- The transmission factor linearized about this configuration, namely
$$u = 2\pi i + e^{-i(\omega t - \kappa x)} \qquad v = T\, e^{-i(\omega t - \kappa x)}$$
$$\omega = \sqrt{2}\cosh\theta \quad \kappa = \sqrt{2}\sinh\theta$$
becomes
$$T = \frac{\sinh(\theta - \vartheta) + i}{\sinh(\theta - \vartheta) - i}$$
which is the classical limit of the transmission factor of a soliton - ϑ - scattering with the lightest breather - θ -

The quantum sine-Gordon model with defects

Triangle equations for purely transmitting defect: it expresses the compatibility between the S-matrix and the T-matrix [*Delfino,Mussardo,Simonetti*]

$$S_{a\,b}^{mn}(\Theta)\,T_{n\alpha}^{t\beta}(\theta_1)\,T_{m\beta}^{s\gamma}(\theta_2) = T_{b\,\alpha}^{n\beta}(\theta_2)\,T_{a\,\beta}^{m\gamma}(\theta_1)\,S_{mn}^{st}(\Theta)$$

- $S(\Theta)$: soliton/soliton scattering matrix [*A. Zamolodchikov, Al. Zamolodchikov*]

$$\begin{aligned} S_{++}^{++} &= S_{--}^{--} = a\,\rho_s \\ S_{+-}^{-+} &= S_{-+}^{+-} = b\,\rho_s \\ S_{+-}^{+-} &= S_{-+}^{-+} = c\,\rho_s \end{aligned}$$

$$q = -e^{-i\pi\gamma} = e^{-4i\pi^2/\beta^2} \qquad \gamma = \frac{4\pi}{\beta^2} - 1$$

- $T(\theta_p)$: soliton/defect transmission matrix

- The T-matrix must be unitary
- Conservation of the topological charge: $a + \alpha = b + \beta$
  α and β are either both odd or both even integers
  Hence the transmission matrix has the following form:

$$T_{a\,\alpha}^{b\,\beta}(\theta) = \begin{pmatrix} A(\theta)\,\delta_\alpha^\beta & B(\theta)\,\delta_\alpha^{\beta-2} \\ D(\theta)\,\delta_\alpha^{\beta+2} & D(\theta)\,\delta_\alpha^\beta \end{pmatrix}$$

- It acts on a $V \otimes \mathcal{V}$ where V is a two dimensional space and $\mathcal{V}$ is an infinite dimensional space

Explicit solutions related to defects...

- Transmission matrix for a type I defect [*Konik, LeClair; BCZ*]

$$T_{I}\,{}_{a\alpha}^{b\beta}(\theta,\eta)$$

- Transmission matrix for two type I defects placed somewhere along the x-axis

$$T_{I-I}\,{}_{a\,\alpha\,\gamma}^{b\,\beta\,\delta}(\theta,\eta_1,\eta_2)$$

  Four defect labels reveals the presence of two defects: exchange of topological charge between the defects as the soliton passes between them

- But a type II defect required a completely new solution

A new solution for the $STT = TTS$ equation

$$T_{a\,\alpha}^{b\,\beta}(\theta) = \rho(\theta)\ \times$$
$$\begin{pmatrix} (a_+ Q^\alpha + a_- Q^{-\alpha}\, x^2)\, \delta_\alpha^\beta & x\, (b_+ Q^\alpha + b_- Q^{-\alpha})\, \delta_\alpha^{\beta-2} \\ x\, (c_+ Q^\alpha + c_- Q^{-\alpha})\, \delta_\alpha^{\beta+2} & (d_+ Q^\alpha\, x^2 + d_- Q^{-\alpha})\, \delta_\alpha^\beta \end{pmatrix}$$

$$x = e^{\gamma\theta} \qquad Q = 1/\sqrt{q} = e^{2i\pi^2/\beta^2} \qquad a_\pm\, d_\pm - b_\pm\, c_\pm = 0$$

The free constants and the scalar factor ρ can be constraint by additional requirements:

- Crossing

$$T_{a\,\alpha}^{b\,\beta}(\theta) = \tilde{T}_{\bar{b}\,\alpha}^{\bar{a}\,\beta}(i\pi - \theta)$$

  with

$$T_{a\,\alpha}^{b\,\beta}(\theta)\, \tilde{T}_{b\,\beta}^{c\,\gamma}(-\theta) = \delta_a^c \delta_\alpha^\gamma$$

- Unitarity

$$\sum_{b,\beta} T_{a\,\alpha}^{b\,\beta}(\theta)\, \bar{T}_{c\,\gamma}^{b\,\beta}(\theta) = \delta_{ac}\delta_\alpha^\gamma$$

- Crossing does not provide any further constraints on the free constants, but forces the function ρ to satisfy the following relation:

$$\rho(\theta)\rho(\theta + i\pi)\, Q^2 \Delta(\theta) = 1$$

with

$$\Delta(\theta) = a_+ d_- \, Q^{-2} \left(1 - \frac{b_+ c_-}{a_+ d_-}\, x^2\right) \left(1 - \frac{b_- c_+}{a_+ d_-}\, Q^4 x^2\right)$$

Experience with a similar calculation for the type I defect suggests to set

$$\frac{b_- c_+}{a_+ d_-} = -Q^{-4}\, e^{-2\gamma\eta_1} \qquad \frac{b_+ c_-}{a_+ d_-} = -e^{-2\gamma\eta_2}$$

Then, it is possible to find a solution for the scalar factor ρ

- On the contrary, unitarity provides constraints on the constants ($a_+ = 1$)

$$a_+ = d_- = 1 \qquad a_- = b_- c_- \qquad d_+ = b_+ c_+$$

$$c_- = -\bar{b}_+ \qquad c_+ = -\bar{b}_- \, Q^{-4}$$

and the scalar factor

$$\rho(\theta + i\pi) = \bar{\rho}(\theta)$$

These constraints allow to write

$$b_- \, \bar{b}_- = e^{-2\gamma\eta_1} \qquad b_+ \, \bar{b}_+ = e^{-2\gamma\eta_2}$$

then η_1 and η_2 are real parameters

Transmission matrix for a fused pair of type I defects

$$T_{II}\,{}^{b\,\beta}_{a\,\alpha}(\theta, b_+, b_-) = \rho_{II}(\theta,\eta_1,\eta_2)\ \times$$
$$\begin{pmatrix} (Q^{\alpha} - b_-\,\bar{b}_+\,Q^{-\alpha}\,x^2)\,\delta_\alpha^\beta & x\,(b_+\,Q^{\alpha} + b_-\,Q^{-\alpha})\,\delta_\alpha^{\beta-2} \\ -x\,(\bar{b}_-\,Q^{\alpha-4} + \bar{b}_+\,Q^{-\alpha})\,\delta_\alpha^{\beta+2} & (-b_+\bar{b}_-\,Q^{\alpha-4}\,x^2 + Q^{-\alpha})\,\delta_\alpha^\beta \end{pmatrix}$$

$$\rho_{II}(\theta,\eta_1,\eta_2) = \frac{f_{II}(z_1,z_2)}{2\pi}\,e^{-\gamma(\theta-\eta_1)/2}\,e^{-\gamma(\theta-\eta_2)/2}\,e^{i\pi\gamma/2}$$

$$z_p = \frac{i\gamma(\theta-\eta_p)}{\pi} \quad p = 1,2 \qquad \gamma = \frac{4\pi}{\beta^2} - 1$$

$$f_{II}(z_1,z_2) = \Gamma(1/2 - z_1)\Gamma(1/2 - z_2)\ \times$$
$$\prod_{k=1}^{\infty} \frac{\Gamma(1/2 + z_1 + (2k-1)\gamma)\Gamma(1/2 - z_1 + 2k\gamma)}{\Gamma(1/2 + z_1 + 2k\gamma)\Gamma(1/2 - z_1 + (2k-1)\gamma)}\ \times$$
$$\frac{\Gamma(1/2 + z_2 + (2k-1)\gamma)\Gamma(1/2 - z_2 + 2k\gamma)}{\Gamma(1/2 + z_2 + 2k\gamma)\Gamma(1/2 - z_2 + (2k-1)\gamma)}$$

- This solution is supposed to describe the scattering of a soliton and a type II defect in its stable configuration with even topological charge labels
- The degrees of freedom are represented by the two real constants η_1 and η_2 and the relative phase between b_- and b_+ with

$$b_- \, \bar{b}_- = e^{-2\gamma\eta_1} \qquad b_+ \, \bar{b}_+ = e^{-2\gamma\eta_2}$$

- There are two defect 'resonance' states, representing the absorption and emission of a soliton, at:

$$\theta = \eta_1 + \frac{i\pi}{2\gamma} \qquad \theta = \eta_2 + \frac{i\pi}{2\gamma}$$

  - In the classical limit $\beta \to 0$ $(1/\gamma \to 0)$ their energies coincide with the classical energies of a soliton with rapidity η_1 or η_2
  - In the classical limit these poles coincide with the rapidity at which the classical soliton delay diverges, provided $\eta_1 = \eta + \tau$ and $\eta_2 = \eta - \tau$ - though a normalization could be required

- This solution is similar to $T_{I-I}\,{}^{b\,\beta\,\delta}_{a\,\alpha\,\gamma}(\theta,\eta_1,\eta_2)$. The essential difference is represented by the presence of two extra defect labels for T_{I-I}

The sine-Gordon model has breather 'poles' at:

$$\Theta = i\pi\left(1-\frac{n}{\gamma}\right) \qquad n = 1, 2, \cdots < \gamma$$

By using the bootstrap relation

$$c^{n}_{a\,\bar{a}}\,{}^{n}T(\theta)\delta^{\beta}_{\alpha} = \sum_{b} T^{\bar{b}\,\gamma}_{\bar{a}\,\alpha}(\theta_{\bar{a}})\,T^{b\,\beta}_{a\,\gamma}(\theta_a)\,c^{n}_{b\,\bar{b}} \qquad c^{n}_{+\,-} = (-)^n c^{n}_{-\,+}$$

the transmission factor for the lightest breather is

$${}^{1}T(\theta) = -\frac{\sinh\left(\frac{\theta-\eta_1}{2}-\frac{\pi i}{4}\right)}{\sinh\left(\frac{\theta-\eta_1}{2}+\frac{\pi i}{4}\right)}\frac{\sinh\left(\frac{\theta-\eta_2}{2}-\frac{\pi i}{4}\right)}{\sinh\left(\frac{\theta-\eta_2}{2}+\frac{\pi i}{4}\right)}$$

- Is coincide with the transmission factor for the classical problem linearized around the lowest energy configuration with $\eta_1 = \eta + \tau$ and $\eta_2 = \eta - \tau$

T & S matrices

- Classically the type I defect 'behaves' like 'half' soliton and the type II defect like a soliton with respect to energy, momentum, topological charge and delay
- This 'identification' may be extended to the quantum context since the S matrix is embedded within a T matrix for suitable choices of defect parameters

Consider the new solution shown previously

$$T_{a\,\alpha}^{b\,\beta}(\theta)=\rho(\theta)\ \times$$
$$\begin{pmatrix} (a_+ Q^{\alpha}+a_- Q^{-\alpha}\,x^2)\,\delta_{\alpha}^{\beta} & x\,(b_+ Q^{\alpha}+b_- Q^{-\alpha})\,\delta_{\alpha}^{\beta-2} \\ x\,(c_+ Q^{\alpha}+c_- Q^{-\alpha})\,\delta_{\alpha}^{\beta+2} & (d_+ Q^{\alpha}\,x^2+d_- Q^{-\alpha})\,\delta_{\alpha}^{\beta} \end{pmatrix}$$

$$x=e^{\gamma\theta}\qquad Q=1/\sqrt{q}=e^{2i\pi^2/\beta^2}\qquad a_\pm\,d_\pm-b_\pm\,c_\pm=0$$

Remember that because of the crossing constraints it is possible to find a candidate for the overall scalar function ρ

- Since we want to recover the S matrix we consider the solution $T_{a\,\alpha}^{b\,\beta}$ with α, β odd integers
- For the same reason, we require there is no amplitude for transitions between topological charges ± 1 and ± 3, in other words

$$T_{+\ 1}^{-\ 3} = T_{-\ -1}^{+\ -3} = 0 \quad \longrightarrow \quad b_- = -b_+ \, Q^2 \qquad c_- = -c_+ \, Q^{-2}$$

Hence

$$T_{a\,\alpha}^{b\,\beta} = {}^{-}T_{a\,\alpha\ \text{infinite}}^{b\,\beta} \oplus T_{a\,\alpha\ \text{finite}}^{b\,\beta} \oplus {}^{+}T_{a\,\alpha\ \text{infinite}}^{b\,\beta}$$

In addition we set

$$a_+ = d_- = 1 \qquad a_- = d_+ = -e^{-2\gamma\vartheta} Q^{-2}$$

$$b_+ = c_- = -e^{-\gamma\vartheta} Q^{-2} \qquad c_+ = b_- = e^{-\gamma\vartheta}$$

hence, a single real free parameter survived ϑ.

The non zero elements of the finite T matrix are:

$$T_{+\,+1}^{+\,+1} = T_{-\,-1}^{-\,-1} \;=\; a\,\hat{\rho} \qquad \hat{\rho} = \left(-Q^{-1}\,e^{\gamma(\theta-\vartheta)}\right)\rho$$
$$T_{+\,-1}^{+\,-1} = T_{-\,+1}^{-\,+1} \;=\; b\,\hat{\rho} \qquad T_{+\,-1}^{-\,+1} = T_{-\,+1}^{+\,-1} = c\,\hat{\rho}$$

- Apart from the overall factor $\hat{\rho}$, these elements are precisely the non zero elements of the S-matrix
- The scalar factor $\hat{\rho}$ can be calculated. By using the previously definitions for the free constants and by making the choice $\eta_2 = \bar{\eta}_1$
$$\hat{\rho}(\theta) = \rho_S(\theta - \vartheta) \equiv \rho_S(\Theta)$$
- Hence the finite part of T coincide with the soliton/soliton S matrix and it is unitary, even though the full T matrix is not

- The transmission factor for the lightest breather coincide with the scattering amplitude between a soliton and the lightest breather whose rapidity difference is $\Theta = (\theta - \vartheta)$ [*A.Zamolodchikov, Al. Zamoloschikov*]

$$ {}^{s}T(\Theta) = \frac{\sinh(\Theta) + i \cos\frac{\pi}{2\gamma}}{\sinh(\Theta) - i \cos\frac{\pi}{2\gamma}} $$

- In the classical limit this expression coincides with the transmission factor for the classical problem linearized around the 'soliton configuration'

The same T matrix labeled by even integers can also describe the lightest breather. Demand

$$T_{+\ 0}^{-\ 2} = T_{-\ \ 0}^{+\ -2} = 0 \quad \longrightarrow \quad b_- = -b_+ \qquad c_- = -c_+$$

and set

$$a_+ = d_- = 1 \qquad a_- = d_+ = -e^{-2\gamma\vartheta} Q^{-2}$$

In addition, for fixing all free constants but one, choose

$$\eta_1 = \vartheta - \frac{i\pi}{2} + \frac{i\pi}{\gamma} \qquad \eta_2 = \vartheta + \frac{i\pi}{2} - \frac{i\pi}{\gamma}$$

which is a choice compatible with the above mentioned constraints, then

- The finite part of the T matrix becomes the scattering amplitude for the lightest breather and a soliton

$$T_{+\ 0}^{+\ 0} = T_{-\ 0}^{-\ 0} = \frac{\sinh(\Theta) - i\cos\frac{\pi}{2\gamma}}{\sinh(\Theta) + i\cos\frac{\pi}{2\gamma}}$$

- The lightest breather transmission factor becomes the scattering amplitude for two lightest breathers

$$^{1}T(\theta) = \frac{\sinh(\Theta) - i\sin\frac{\pi}{\gamma}}{\sinh(\Theta) + i\sin\frac{\pi}{\gamma}}$$

Finally, the choice

$$\eta_1 = \vartheta + \frac{i\pi}{2} \qquad \eta_2 = \vartheta - \frac{i\pi}{2}$$

leads to

$$^{1}T(\theta) = T_{+\ 0}^{+\ 0} = T_{-\ 0}^{-\ 0} = 1$$

Defects & Representation theory

At the quantum level an algebraic setting capable to describe the transmission matrices for the type I and the type II defects for the sine-Gordon model can be constructed [*Weston*]

- $S = \rho_s\, P\, R$: $\quad R(\theta_1/\theta_2) : V^{1/2}_{\theta_1} \otimes V^{1/2}_{\theta_2} \rightarrow V^{1/2}_{\theta_1} \otimes V^{1/2}_{\theta_2}$
  $V^{1/2}_{\theta_i}$ is a representation of $U_q(\hat{sl}_2)$
- $T \simeq \rho\, \mathcal{L}$: $\quad \mathcal{L}(\theta_1/\theta_2) : \mathcal{V}^{\Delta}_{\theta_1} \otimes V^{1/2}_{\theta_2} \rightarrow \mathcal{V}^{\Delta}_{\theta_1} \otimes V^{1/2}_{\theta_2}$
  $\mathcal{V}^{\Delta}_{\theta}$ is a representation of the Borel subalgebra $U_q(b_+)$
- The Borel subalgebra is described in terms of a generalisation of the q-oscillator algebra with a set of parameter Δ
- The intertwiner $\mathcal{L}$ is solution of the following linear equation

$$\mathcal{L}\,\Delta(x) = \Delta'(x)\,\mathcal{L} \longrightarrow STT = TTS \quad \Delta'(x) = P\Delta(x)P \quad x \in U_q(b_+)$$

- Both T_I and T_{II} can be linked with $\mathcal{L}$

Summary

- Both type I and type II defects are purely transmitting defects, which allow momentum conservation
- The type II defect allows to overcome some restrictions imposed by the type I defect
- It is likely that new ideas, hence new types of defects are required to describe all Toda models (with a defect)
- In the sine-Gordon case a type II defect is equivalent to a fused pair of type I defects
- The quantum transmission matrices for these type of defects are infinite dimensional
- For the sine-Gordon case a explicit description in terms of representation theory is available
- For the sine-Gordon case the S matrix is found embedded inside the a type II T matrix