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Entanglement entropy in 1+1 dimensional integrable QFTs: some ideas on the entropy of disconnected regions

Olalla Castro Alvaredo

Centre for Mathematical Science,
City University London

0906.2946 (with B. Doyon)

0810.0219 (with B. Doyon)

0803.1999 (by B. Doyon)

0802.4231 (with B. Doyon)

0706.3384 (with J.L. Cardy and B. Doyon)

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Entanglement entropy in extended quantum systems

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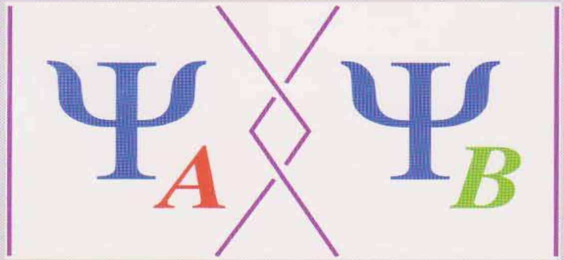

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Entanglement in quantum mechanics

- A quantum system is in an entangled state if performing a localised measurement (in space and time) may instantaneously affect local measurements far away.

  A typical example: a pair of opposite-spin electrons:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \;, \quad \langle\hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle$$

- What is special: Bell's inequality says that this cannot be described by **local variables**.

- A situation that looks similar to $|\psi\rangle$ but without entanglement is a factorizable state:

$$|\hat{\psi}\rangle = \frac{1}{2}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle + |\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle\right) = \frac{1}{2}\left(|\uparrow\rangle + |\downarrow\rangle\right) \otimes \left(|\uparrow\rangle + |\downarrow\rangle\right)$$

- This is particular to **pure states**. Mixed states are described by density matrices

$$\rho = \sum_{\alpha} p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha| \;, \quad \langle\hat{A}\rangle = \mathrm{Tr}(\rho\hat{A})$$

  (for pure states, $\rho = |\psi\rangle\langle\psi|$; for finite temperature, $\rho = e^{-H/kT}$).

How and why to measure (or quantify) quantum entanglement?

- Measuring quantum entanglement is useful: as such measurement may have applications to the design of (still theoretical) quantum computers. It is also a fundamental property of quantum systems.

- In **pure states**, there are various proposals to measure quantum entanglement. Consider the **entanglement entropy** and let us look at a more complicated system:
  - With the Hilbert space a tensor product $\mathcal{H} = s_1 \otimes s_2 \otimes \cdots \otimes s_N = A \otimes \bar{A}$, and a given state $|\mathrm{gs}\rangle \in \mathcal{H}$, calculate the **reduced density matrix**:

$$\rho_A = \mathrm{Tr}_{\bar{A}}(|\mathrm{gs}\rangle\langle\mathrm{gs}|)$$

$$\cdots \mathrm{S}_{\mathrm{i}-1} \otimes \underbrace{\mathrm{S}_{\mathrm{i}} \otimes \mathrm{S}_{\mathrm{i}+1} \otimes \quad \cdots \quad \otimes \mathrm{S}_{\mathrm{i}+\mathrm{L}-1}}_{\mathrm{A}} \otimes \mathrm{S}_{\mathrm{i}+\mathrm{L}} \quad \cdots$$

  - The entanglement entropy is the resulting **von Neumann entropy**:

$$S_A = -\mathrm{Tr}_A(\rho_A \log(\rho_A)) = -\sum_{\substack{\text{eigenvalues of } \rho_A \\ \lambda \neq 0}} \lambda \log(\lambda)$$

Interpretation of the entanglement entropy

- It is the entropy that is measured in a subsystem A, once the rest of the system $\bar{A}$ – the environment – is forgotten.

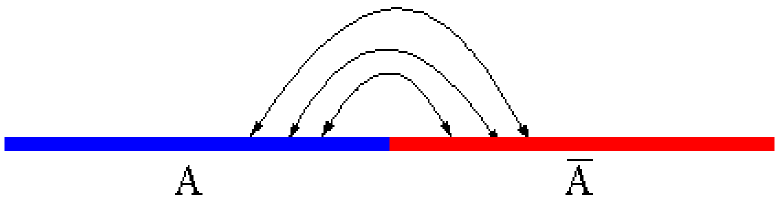
  If we think A is all there is, we will think the system is in a mixed state, with density matrix given by ρ_A. The entropy of ρ_A measures how mixed ρ_A is. This mixing is due to the connections, or entanglement, with the environment.

- It was proposed as a way to understand black hole entropy [Bombelli, Koul, Lee, Sorkin 1986].
- Then it was proposed as a measure of entanglement [Bennet, Bernstein, Popescu, Schumacher 1996].
- Examples:
  - Tensor product state:
    $|\mathrm{gs}\rangle = |A\rangle \otimes |\bar{A}\rangle \Rightarrow \rho_A = |A\rangle\langle A| \Rightarrow S_A = -1\log(1) = 0.$

– The maximally entangled state $|\mathrm{gs}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$:

$$\rho_{1^{\mathrm{st}}\ \mathrm{spin}} = \frac{1}{2}(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|) \Rightarrow S_{1^{\mathrm{st}}\ \mathrm{spin}} = -2\times\left(-\log(\sqrt{2})\right) = \log(2)$$

- In general the entanglement entropy is not "directional", that is $S_A = S_{\overline{A}}$.



- We can think of the entropy as counting the number of "connections" between A and $\bar{A}$.

Scaling limit

- Say $|\mathrm{gs}\rangle$ is a ground state of some local spin-chain Hamiltonian, and that the chain is infinitely long.
- An important property of $|\mathrm{gs}\rangle$ is the **correlation length** ξ:

$$\langle \mathrm{gs}|\hat{\sigma}_i\hat{\sigma}_j|\mathrm{gs}\rangle \sim e^{-|i-j|/\xi} \text{ as } |i-j| \to \infty$$

- If there are parameters in the Hamiltonian (e.g. temperature, magnetic field etc) such that for certain values $\xi \to \infty$. This is a **quantum critical point**.
- We may adjust these parameters in such a way that $L, \xi \to \infty$ in such as way that the length L of A is proportional to ξ: $L/\xi = mr$.
- This is the **scaling limit**, and what we obtain is a **quantum field theory**. m here is a mass scale – we may have many masses m_α associated to many correlation lengths – and r is the dimensionful length of region A in the scaling limit.
- The resulting entanglement entropy has a **universal** part: a part that does not depend very much on the details of the Hamiltonian.

Short- and large-distance entanglement entropy

Consider $\varepsilon = 1/(m_1 \xi)$, a non-universal QFT cutoff with dimenions of length. Then:

- Short distance: $0 \ll L \ll \xi$, logarithmic behavior [Holzhey, Larsen, Wilczek 1994; Calabrese, Cardy 2004]

$$S_A \sim \frac{c}{3} \log \left(\frac{r}{\varepsilon} \right)$$

- Large distance: $0 \ll \xi \ll L$, saturation

$$S_A \sim -\frac{c}{3} \log(m_1 \varepsilon) + U$$

where c is the central charge of the corresponding critical point. One of the main results of our work was to find the next-to-leading order correction to this behavior.

Large distance corrections

[J.L. Cardy, O.C.A., B. Doyon 2007], [O.C.A, B. Doyon 2008], [B. Doyon 2008]

Our result: for any integrable QFT, the entropy with its **first correction to saturation** at large distances is:

$$\boxed{S_A \sim -\frac{c}{3}\log(m\varepsilon) + U - \frac{1}{8}K_0(2rm) + O\left(e^{-3rm}\right)}$$

where m is the mass of the particle.

- The next-to-leading order correction term depends only on the particle spectrum, but not on the interaction between particles (i.e. not the S-matrix).
- In our work we have extended this result to any integrable QFT (even non-integrable [B. Doyon 2008]).
- In a recent work [O.C.A, B. Doyon 2008] we have also computed all the **remaining higher order corrections** for the Ising field theory with and without a boundary.
- In the latter case, we extracted the g**-function** associated to the two different conformal boundary conditions (free and fixed) in the Ising model in the form of a **boundary entropy contribution**.

Partition functions on multi-sheeted Riemann surfaces

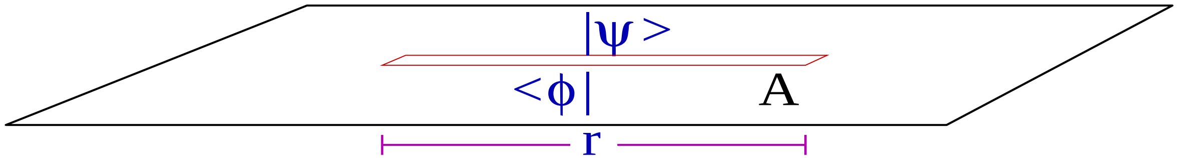
[Callan, Wilczek 1994; Holzhey, Larsen, Wilczek 1994]

- We can use the “replica trick” for evaluating the entanglement entropy:

$$S_A = -\mathrm{Tr}_A(\rho_A \log(\rho_A)) = -\lim_{n\to 1}\frac{d}{dn}\mathrm{Tr}_A(\rho_A^n)$$

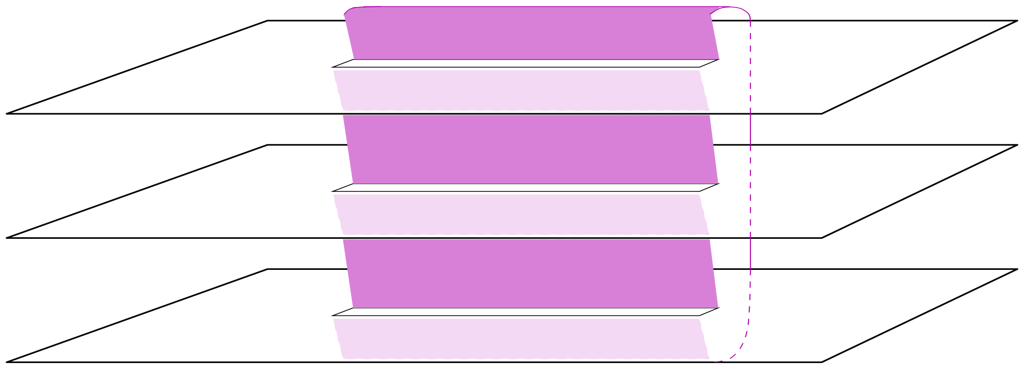
- For integer numbers n of replicas, in the scaling limit, this is a partition function on a Riemann surface:

$${}_A\langle\phi|\rho_A|\psi\rangle_A \quad \sim$$



$$\mathrm{Tr}_A(\rho_A^n) \quad \sim \quad Z_n = \int [d\varphi]_{\mathcal{M}_n} \exp\left[-\int_{\mathcal{M}_n} d^2x\, \mathcal{L}[\varphi](x)\right]$$

$$\mathcal{M}_n \quad :$$



Branch-point twist fields

[J.L. Cardy, O.C.A, B. Doyon 2007]

- Consider many copies of the QFT model on the usual $\mathbb{R}^2$:

$$\mathcal{L}^{(n)}[\varphi_1, \ldots, \varphi_n](x) = \mathcal{L}[\varphi_1](x) + \ldots + \mathcal{L}[\varphi_n](x)$$

- There is an obvious symmetry under cyclic exchange of the copies:

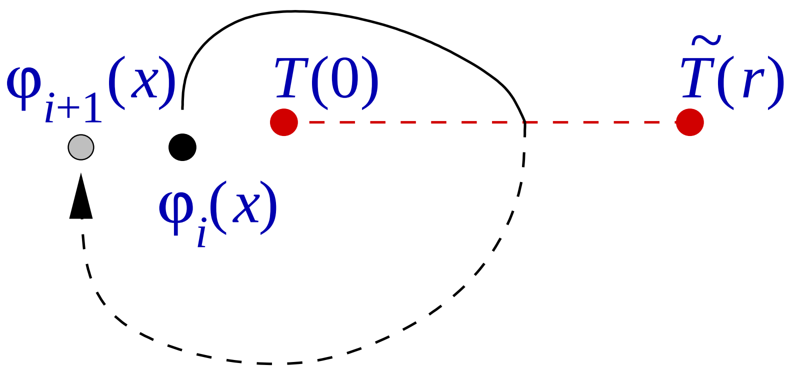
$$\mathcal{L}^{(n)}[\sigma\varphi_1, \ldots, \sigma\varphi_n] = \mathcal{L}^{(n)}[\varphi_1, \ldots, \varphi_n]\,, \quad \text{with} \quad \sigma\varphi_i = \varphi_{i+1 \bmod n}$$

- Whenever we have an internal symmetry in a QFT we can associate a twist field to it. We will call the fields associated to the $\mathbb{Z}_n$ symmetry introduced here **branch point twist fields**.

- Another twist field $\tilde{\mathcal{T}}$ is associated to the inverse symmetry σ^{-1}, and we have

$$\begin{aligned}\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_{\mathcal{L}^{(n)}} &\propto \int_{C_{0,r}} [d\varphi_1 \cdots d\varphi_n]_{\mathbb{R}^2} \exp\left[-\int_{\mathbb{R}^2} d^2x\, \mathcal{L}^{(n)}[\varphi_1,\ldots,\varphi_n](x)\right] \\ &= Z_n = \mathrm{Tr}_A(\rho_A^n)\end{aligned}$$

$C_{0,r}$:

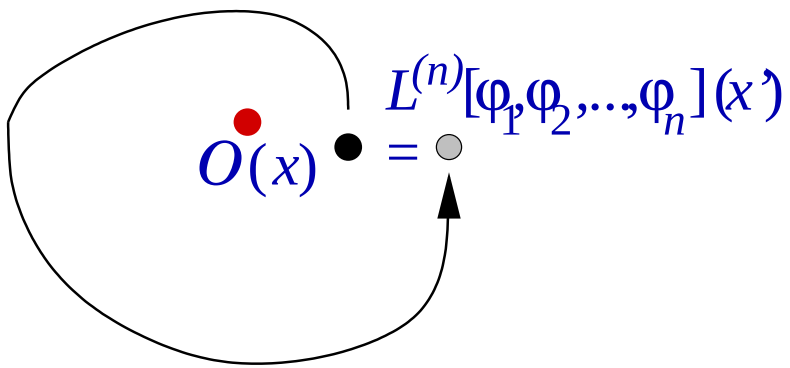


Locality in QFT

- A field $\mathcal{O}(x)$ is **local** in QFT if measurements associated to this field are quantum mechanically independent from measurements of the **energy density** (or **Lagrangian density**) at space-like distances. That is, equal-time commutation relations vanish:

$$[\mathcal{O}(\mathrm{x}, t=0), \mathcal{L}^{(n)}(\mathrm{x}', t=0)] = 0 \quad (\mathrm{x} \neq \mathrm{x}') \ .$$

- This means that:



- **Branch-point twist fields are local fields in the n-copy theory.**

Short- and large-distance entanglement entropy revisited

Hence we have

$$Z_n = D_n \varepsilon^{2d_n} \langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_{\mathcal{L}^{(n)}} \;, \quad S_A = -\lim_{n\to 1} \frac{d}{dn} Z_n$$

where D_n is a normalisation constant, and d_n is the scaling dimension of $\mathcal{T}$ [Calabrese, Cardy 2004]:

$$d_n = \frac{c}{12}\left(n - \frac{1}{n}\right)$$

- Short distance: $0 \ll L \ll \xi$, logarithmic behavior

$$\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_{\mathcal{L}^{(n)}} \sim r^{-2d_n} \;\Rightarrow\; S_A \sim \frac{c}{3}\log\left(\frac{r}{\varepsilon}\right)$$

- Large distance: $0 \ll \xi \ll L$, saturation

$$\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_{\mathcal{L}^{(n)}} \sim \langle \mathcal{T}\rangle^2_{\mathcal{L}^{(n)}} \;\Rightarrow\; S_A \sim -\frac{c}{3}\log(m_1\varepsilon) + U$$

Form factors and two-point functions in integrable models

- The two-point function of branch-point twist fields can be decomposed as follows, giving a **large-distance expansion**:

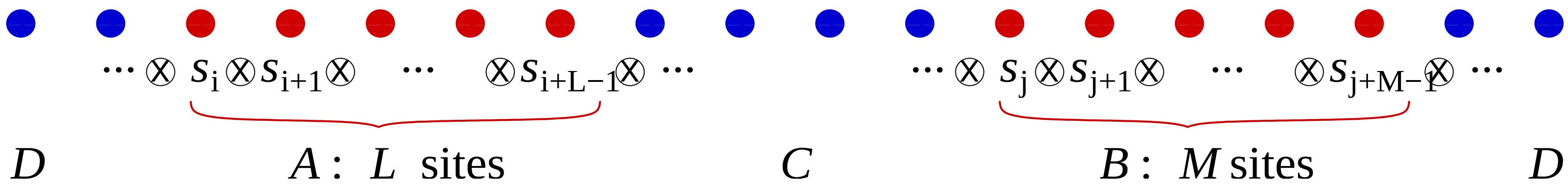
$$\begin{aligned}\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle &= \langle \mathrm{gs}|\mathcal{T}(0)\tilde{\mathcal{T}}(r)|\mathrm{gs}\rangle \\ &= \sum_{\text{state } k} \langle \mathrm{gs}|\mathcal{T}(0)|k\rangle\langle k|\tilde{\mathcal{T}}(r)|\mathrm{gs}\rangle\end{aligned}$$

where $\sum_k |k\rangle\langle k|$ is a sum over a complete set of states in the Hilbert space of the theory.

- The matrix elements $\langle \mathrm{gs}|\mathcal{T}(0)|k\rangle$ are called **form factors**.
- For integrable models, an specific program exists (**form factor program**) that allows their exact (non-perturbative) computation.
- However the program needs to be modified to include twist fields correctly.
- Main challenge: analytic continuation from $n = 1, 2, \ldots$ to $n \in [1, \infty)$.

The entropy of disconnected regions

- Consider again a quantum spin chain which we now divide into four different regions (periodic boundary conditions)



- A problem of current interest is finding the entanglement entropy of the region $A \cup B$ with respect to the rest of the system (where A and B are now "disconnected")

$$S_{A \cup B} = -\mathrm{Tr}_{A \cup B}(\rho_{A \cup B} \ln(\rho_{A \cup B}))$$

with

$$\rho_{A \cup B} = \mathrm{Tr}_{C \cup D}(|gs\rangle\langle gs|)$$

and $|gs\rangle$ is again the ground state of the chain (a pure state).

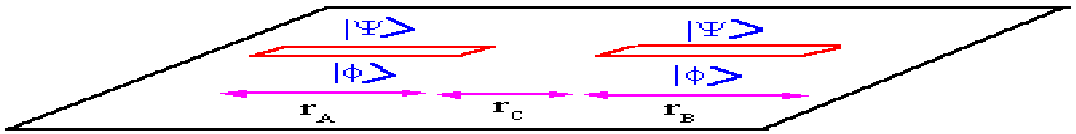
The entropy of disconnected regions and the replica trick

- We can use the “replica trick” for evaluating the entanglement entropy as before:

$$S_{A\cup B} = -\mathrm{Tr}_{A\cup B}(\rho_{A\cup B}\log(\rho_{A\cup B})) = -\lim_{n\to 1}\frac{d}{dn}\mathrm{Tr}_{A\cup B}(\rho^n_{A\cup B})$$

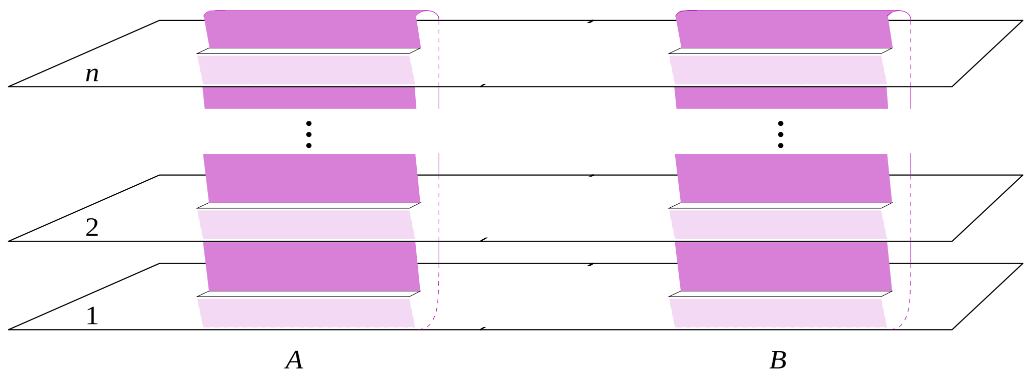
- For integer numbers n of replicas, in the scaling limit, this is again a partition function on a (more complicated) Riemann surface:

$${}_{A\cup B}\langle\phi|\rho_{A\cup B}|\psi\rangle_{A\cup B} \quad \sim$$



$$\mathrm{Tr}_{A\cup B}(\rho^n_{A\cup B}) \quad \sim \quad Z_n = \int [d\varphi]_{\mathcal{M}_n}\exp\left[-\int_{\mathcal{M}_n} d^2x\,\mathcal{L}[\varphi](x)\right]$$

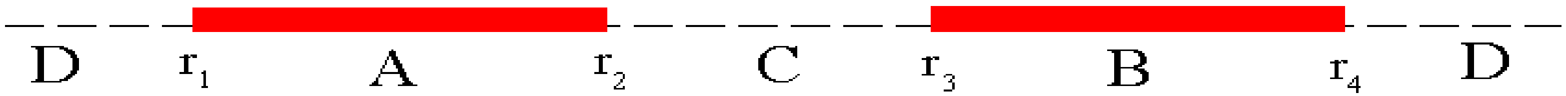
$$\mathcal{M}_n \quad :$$



The entropy of disconnected regions from twist fields

- Similarly to the previous case, the partition function of the n-sheeted Riemann surface can be expressed in terms of a correlation function of twist fields. In this case it is a four point function:

$$
\begin{aligned}
&\langle \mathcal{T}(r_1)\tilde{\mathcal{T}}(r_2)\mathcal{T}(r_3)\tilde{\mathcal{T}}(r_4)\rangle_{\mathcal{L}^{(n)}} \\
&\propto \int_{C_{0,r}} [d\varphi_1 \cdots d\varphi_n]_{\mathbb{R}^2} \exp\left[-\int_{\mathbb{R}^2} d^2x\, \mathcal{L}^{(n)}[\varphi_1,\ldots,\varphi_n](x)\right] \\
&= Z_n = \mathrm{Tr}_{A\cup B}(\rho^n_{A\cup B})
\end{aligned}
$$



- From a computational point of view, it is much more challenging to compute a four-point function than a two-point function, even in the two-particle approximation.
- Some limiting cases, such as $r_{1,2} \ll r_{3,4}$ can be analyzed more easily.

Extensivity

- A property of the entropy of disconnected regions that has been recently investigated is its extensivity.
- Extensivity implies that the entropy of disconnected regions can be expressed in terms of the individual entropies of connected domains

$$S_{A\cup B} = S_A + S_B + S_C + S_D - \frac{1}{2}(S_{A\cup C} + S_{B\cup C} + S_{A\cup D} + S_{B\cup D})$$

- In our twist field picture it would imply a relationship of the type

$$\lim_{n\to 1}\frac{d}{dn}\langle\mathcal{T}(r_1)\tilde{\mathcal{T}}(r_2)\mathcal{T}(r_3)\tilde{\mathcal{T}}(r_4)\rangle \sim \lim_{n\to 1}\frac{d}{dn}\Big[\langle\mathcal{T}(r_1)\tilde{\mathcal{T}}(r_2)\rangle + \langle\mathcal{T}(r_3)\tilde{\mathcal{T}}(r_4)\rangle$$
$$+\langle\mathcal{T}(r_1)\tilde{\mathcal{T}}(r_4)\rangle + \langle\mathcal{T}(r_2)\tilde{\mathcal{T}}(r_3)\rangle - \langle\mathcal{T}(r_1)\tilde{\mathcal{T}}(r_3)\rangle - \langle\mathcal{T}(r_2)\tilde{\mathcal{T}}(r_4)\rangle\Big]$$

Some simple limit cases

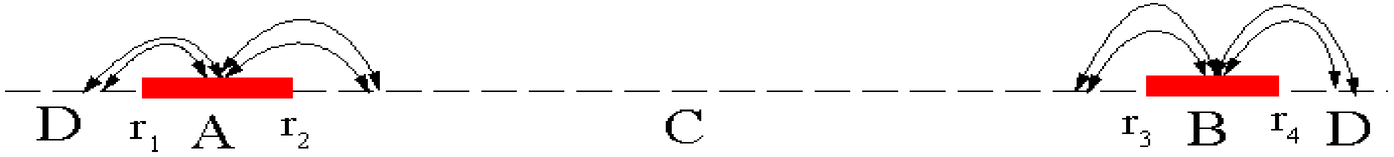
- When $r_1, r_2 \ll r_3, r_4$ the 4-point function factorizes as

$$\lim_{n\to 1} \frac{d}{dn} \langle \mathcal{T}(r_1)\tilde{\mathcal{T}}(r_2)\mathcal{T}(r_3)\tilde{\mathcal{T}}(r_4)\rangle \to \lim_{n\to 1} \frac{d}{dn} \langle \mathcal{T}(r_1)\tilde{\mathcal{T}}(r_2)\rangle\langle \mathcal{T}(r_3)\tilde{\mathcal{T}}(r_4)\rangle$$

$$= \underbrace{\lim_{n\to 1} \langle \mathcal{T}(r_3)\tilde{\mathcal{T}}(r_4)\rangle}_{=1} \underbrace{\lim_{n\to 1} \frac{d}{dn} \langle \mathcal{T}(r_1)\tilde{\mathcal{T}}(r_2)\rangle}_{=-S_A}$$

$$+ \underbrace{\lim_{n\to 1} \langle \mathcal{T}(r_1)\tilde{\mathcal{T}}(r_2)\rangle}_{=1} \underbrace{\lim_{n\to 1} \frac{d}{dn} \langle \mathcal{T}(r_3)\tilde{\mathcal{T}}(r_4)\rangle}_{=-S_B} = -S_A - S_B,$$

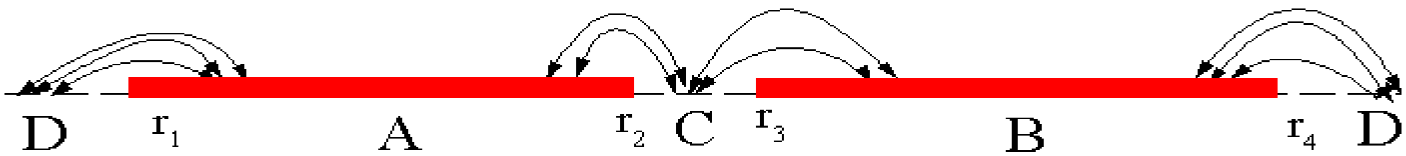
- Regions A and B are so far from each other (the size of region C tends to ∞ whilst the sizes of regions A and B are fixed and finite) that the bipartite entropy becomes simply the sum of the entanglement entropy between each of the regions and the rest of the system.



- When $r_1 \ll r_2$ and $r_3 \ll r_4$ the 4-point function factorizes as

$$\lim_{n\to 1} \frac{d}{dn} \langle \mathcal{T}(r_1)\tilde{\mathcal{T}}(r_2)\mathcal{T}(r_3)\tilde{\mathcal{T}}(r_4)\rangle \to \lim_{n\to 1} \frac{d}{dn} \langle \mathcal{T}(r_1)\rangle\langle\tilde{\mathcal{T}}(r_2)\mathcal{T}(r_3)\rangle\langle\tilde{\mathcal{T}}(r_4)\rangle$$
$$= \lim_{n\to 1}\langle\mathcal{T}\rangle^2 \frac{d}{dn}\langle\tilde{\mathcal{T}}(r_2)\mathcal{T}(r_3)\rangle + \lim_{n\to 1}\langle\tilde{\mathcal{T}}(r_2)\mathcal{T}(r_3)\rangle \frac{d}{dn}\langle\mathcal{T}\rangle^2$$
$$= -S_C + 2\lim_{n\to 1}\frac{d\langle\mathcal{T}\rangle}{dn}$$

- In this case the size of regions A and B tends to ∞ while the size of region C is fixed and finite. Region C only sees the infinite regions A and B and its entanglement with them is S_C. To this we add the two “boundary” contributions from the entanglement of A and B with D.



- These two examples are compatible with extensivity of the entanglement entropy!

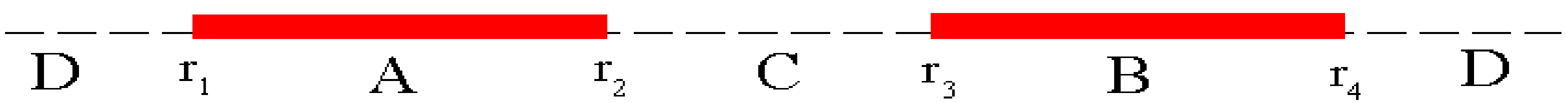
Non-extensivity

- In recent years numerical and analytical evidence of the non-extensivity of the entanglement entropy has been found. In fact, it appears that the entropy is only extensive for the massless free Fermion in 1+1 dimensions [Casini, Huerta 2004; Calabrese, Cardy 2004/2005, Casini, Fosco, Huerta 2005]
- [Caraglio, Gliozzi 2008] found analitical and numerical evidence (critical Ising model) of non-extensivity. [Casini, Huerta 2009] have also shown analytically the non-extensivity of the entropy for a massive free Fermion model and small mass. The entanglement entropy of disconnected regions has been shown to be non-extensive for generic CFTs in [Calabrese, Cardy, Toni 2009]. Numerical evidence for the free compactified Boson was also provided there.
- Numerical computations of the entanglement entropy of disconnected regions, showing its non-extensivity have been carried out for the critical Ising model [Alba, Tagliacozzo, Calabrese 2009] and for the XY integrable spin chain [Fagotti, Calabrese 2010].
- Preliminary results of our work appear to indicate that **extensivity is also violated when looking at higher order corrections** to the entanglement entropy, when the sizes of all regions are large and of the same order of magnitude.

More precisely...

$$\boxed{\begin{aligned} S_{A\cup B} \sim -\frac{2c}{3}\log(m\varepsilon) + 2U &- \frac{e^{-2mr_A}}{8}\sqrt{\frac{\pi}{4mr_A}} - \frac{e^{-2mr_B}}{8}\sqrt{\frac{\pi}{4mr_B}} - \frac{e^{-2mr_C}}{8}\sqrt{\frac{\pi}{4mr_C}} \\ &- \frac{1}{4\pi}\frac{e^{-2m(r_A+r_C)}}{2m(r_A+r_C)} - \frac{1}{4\pi}\frac{e^{-2m(r_B+r_C)}}{2m(r_B+r_C)} + \mathcal{O}(e^{-2m(r_A+r_C)}, e^{-2m(r_B+r_C)}) \end{aligned}}$$

- At order e^{-2mr} extensivity is preserved (we recover the Bessel function $K_0(2mr)$, whose first order expansion for large r gives the blue terms above).
- Extensivity appears to fail at order e^{-4mr}, although there are still corrections of the same order to be obtained!



Conclusions

- The main result of our work has been the derivation of the first **correction to saturation** of the entanglement entropy in any IQFT and the computation of all corrections for the Ising model (with and without boundaries).
- The key ingredients for this are the introduction of branch point twist fields in terms of whose two-point function the entropy can be evaluated. The form factor program has been generalised to accommodate **branch point twist fields**.
- In this talk I have tried to indicate how our approach could be pursued further to investigate the **entanglement entropy of disconnected regions** (two for a start).
- The next-to-leading order correction to the entanglement entropy of two disconnected regions involves multiple integrals over combinations of two- and four-particle form factors of the twist fields whose **analytic continuation** in n is extremely involved.
- Preliminary results suggest that some higher order correction to the entanglement entropy of two disconnected regions are **not compatible with extensivity**.

Thank you for listening!