Solvable Critical Dense Polymers on the Cylinder

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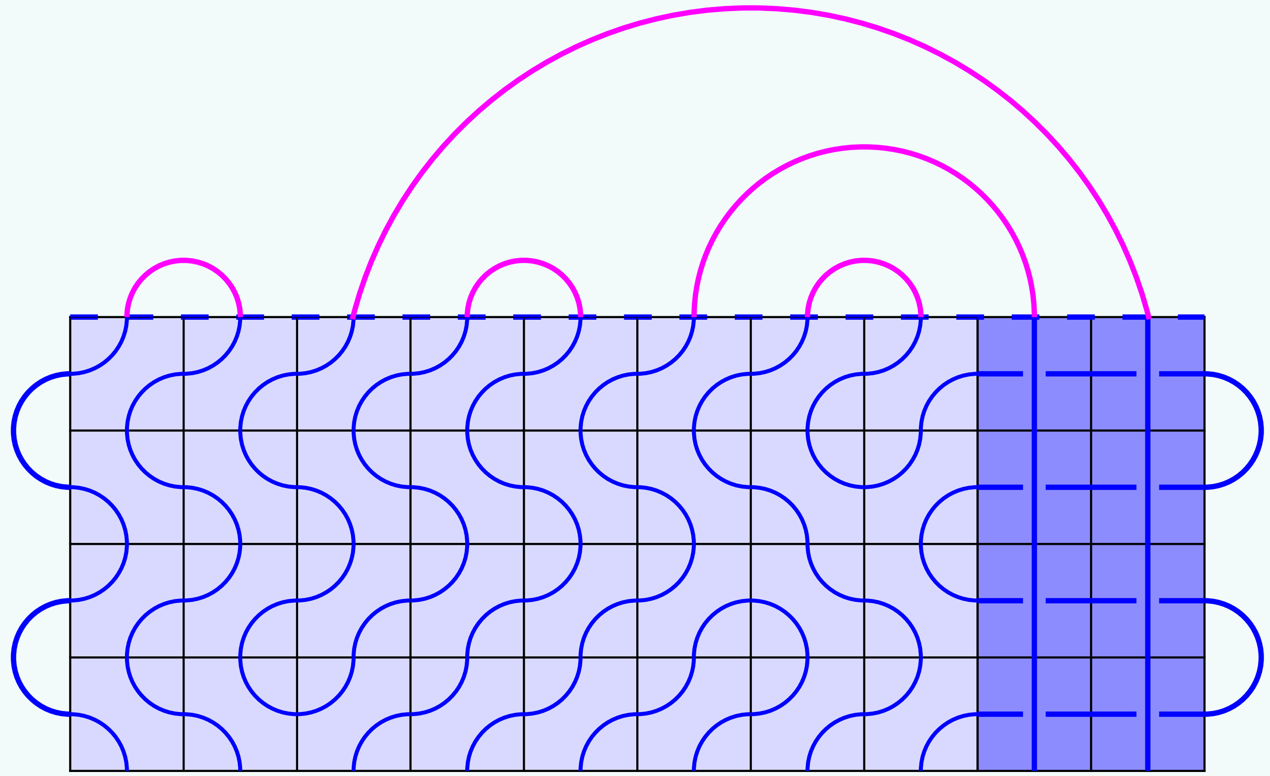


- PAP, J.Rasmussen, J.-B.Zuber, *Logarithmic minimal models*, J.Stat.Mech. P11017 (2006) 1–36.
- PAP, J.Rasmussen, *Solvable critical dense polymers*, J.Stat.Mech. P02015 (2007) 1–32.
- PAP, J.Rasmussen, S.P.Villani, *Solvable critical dense polymers on the cylinder*, P02010 (2010) 1–43; arXiv:0910.4444.

Benasque, 28 June 2010

Background and Overview

- The logarithmic minimal models $\mathcal{LM}(p, p')$ are a family of Yang-Baxter integrable critical lattice models on the square lattice. They are built from nonlocal degrees of freedom. The first few members of this family are critical dense polymers, critical percolation and the critical logarithmic Ising model.

- These models have been studied on the strip, in both the Virasoro and $\mathcal{W}$-extended pictures. In the continuum scaling limit, these theories are described by *logarithmic* CFTs in the sense that they give rise to reducible yet indecomposable representations of the chiral conformal algebras.

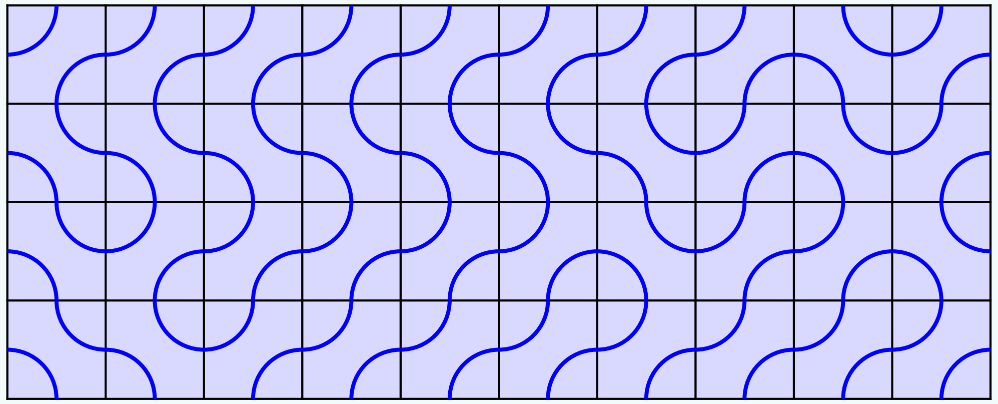
- In the $\mathcal{W}$-extended picture, critical dense polymers $\mathcal{LM}(1, 2)$ is symplectic fermions.

- Consistent CFTs should be well-defined in other topologies, in particular, on the torus. There are known modular invariants which are conjectured to be the torus partition functions but are these really physical? To put it another way, are the physical torus partition functions modular invariant?

- We look into this question in the simplest framework of critical dense polymers on long cylinders where the lattice model can be soved exactly for arbitrary finite-size lattices. The conformal partition functions can therefore be obtained analytically from finite-size corrections using Euler-Maclaurin. The questions of consistency on the torus are then reduced to questions of the *trace operation* closing the cylinder.

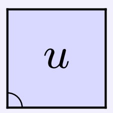
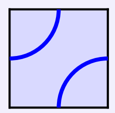
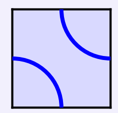
Logarithmic Minimal Models

- To model polymers, percolation etc. on a finite cylinder, we consider a square lattice on a strip, with N columns and M rows of faces, and identify the left and right edges



arcs represent polymer segments
or connectivities

Face operators:

$X(u) =$  $= \sin(\lambda - u)$  $+ \sin u$ 

u is the spectral parameter
related to spatial anisotropy

- The logarithmic minimal model $\mathcal{LM}(p, p')$, $1 \le p < p'$ coprime, is characterized by

$$\text{Crossing parameter : } \quad \lambda = \frac{(p'-p)\pi}{p'}; \qquad \text{Fugacity of contractible loops : } \quad \beta = 2\cos\lambda$$

- Critical dense polymers corresponds to $\mathcal{LM}(1, 2)$

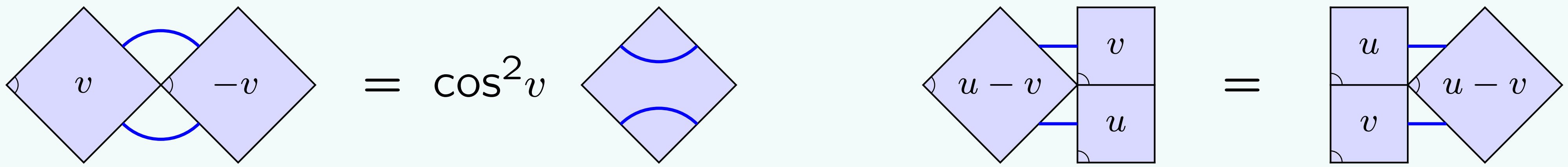
$$\boxed{\lambda = \frac{\pi}{2}, \qquad \sin(\lambda - u) = \cos u, \qquad \beta = 0} \qquad \longrightarrow \qquad \textbf{NO CONTRACTIBLE LOOPS}$$

- The cylinder topology allows for **non-contractible loops** with fugacity α that wind around the cylinder or for ℓ **defects** that propagate along the length of the cylinder.

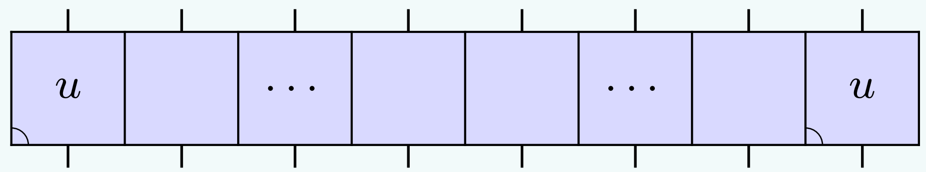
Cylinder Temperley-Lieb Algebra and Transfer Matrix

- The cylinder TL algebra is a diagrammatic algebra built up from elementary faces.

Local inversion identity and Yang-Baxter equation



Transfer matrix

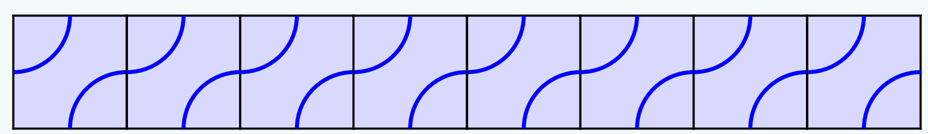
$\boldsymbol{T}(u) =$ 

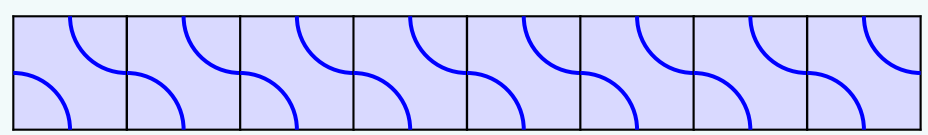
- Commuting family

$$\boxed{\boldsymbol{T}(u)\boldsymbol{T}(v) - \boldsymbol{T}(v)\boldsymbol{T}(u) = 0}$$

multiplication is vertical concatenation of diagrams

- Particular elements of the algebra are the *shift* operator Ω and its inverse Ω^{-1}

$$\lim_{u\to 0} \boldsymbol{T}(u) = \Omega =$$ 

$$\lim_{u\to \lambda} \boldsymbol{T}(u) = \Omega^{-1} =$$ 

- The cylinder TL algebra is infinite-dimensional.
- Act on **vector spaces** of states to obtain **matrix realizations** and spectra.

Periodic Temperley-Lieb Algebra and its Enlargement

- The face operators, in column j, are related to the (loop representation) periodic TL algebra by

$$X_j(u) = \cos u \, I + \sin u \, e_j$$

- The periodic TL algebra is generated by the identity I and the generators $e_0, \ldots, e_{N-1}$ subject to $e_j \equiv e_{j \bmod N}$ and the relations

$$\boxed{\begin{array}{rclrcll} e_j^2 &=& \beta e_j, & e_j e_{j\pm1} e_j &=& e_j, & j = 0, 1, \ldots, N-1 \\ & & e_j e_k &=& e_k e_j, & j-k \neq 0, \pm 1 \ (\mathrm{mod}\ N) & \end{array}}$$

- For N even, non-contractible loops can be removed in pairs

$$EFE \;=\; \alpha^2 E, \qquad FEF \;=\; \alpha^2 F, \qquad E \;=\; e_0 e_2 e_4 \cdots e_{N-2}, \quad F \;=\; e_1 e_3 e_5 \cdots e_{N-1}$$

- For N odd or $\ell > 0$, non-contractible loops cannot appear.

Enlarged periodic TL algebra

- The elements Ω and Ω^{-1} of the cylinder TL algebra are **not** elements of the periodic TL algebra, but may be included.
- Translation on the TL generators is then implemented by conjugation, and the enlarged TL algebra is generated by only three generators

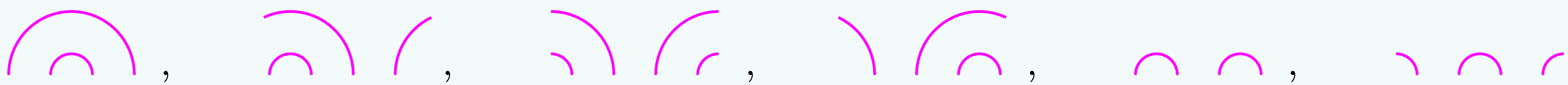
$$e_{j-1} \;=\; \Omega \, e_j \, \Omega^{-1}, \qquad \mathcal{ETL}(N) \;=\; \langle e_0, \Omega, \Omega^{-1} \rangle$$

- Non-contractible loops can now be removed one by one

$$E\Omega^{\pm1}E \;=\; \alpha E, \qquad F\Omega^{\pm1}F \;=\; \alpha F$$

Link States and Defects

- With an even number N of nodes arranged periodically around the upper horizontal edge of the cylinder, a link state specifies how the nodes are linked together.
- Two nodes can be connected by the front of the cylinder or by the back. We can consider the corresponding link states as **distinct** or **identify** them. We work with distinct states.
- For $N = 4$, there are six link states with distinct connectivities

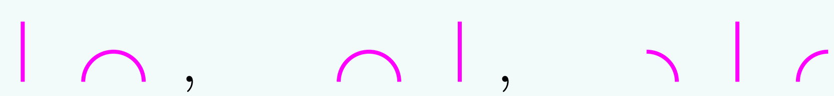


- The enlarged TL algebra becomes finite when acting on link states since then

$$\Omega^N = (\Omega^{-1})^N = I$$

Defects

- A node that is not linked to another node gives rise to a defect which may be viewed as a link to the point (above) at infinity.
- For $N = 3$ and $\ell = 1$, there are three link states



- The dimension of the space of link states with precisely ℓ defects is

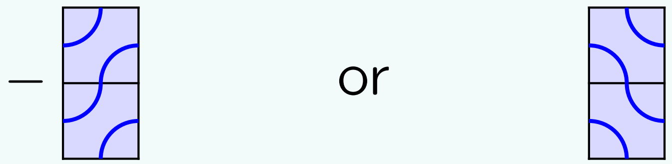
$$\dim(V_N^{(\ell)}) = \binom{N}{\frac{N-\ell}{2}}, \qquad \ell = N \pmod 2$$

Cylinder Inversion Identity

Inversion Identity In the diagrammatic cylinder algebra, the N-tangle $\boldsymbol{T}(u)$ satisfies

$$\boxed{\boldsymbol{T}(u)\boldsymbol{T}(u+\tfrac{\pi}{2}) \;=\; \big(\cos^{2N}u+(-1)^N\sin^{2N}u\big)\boldsymbol{I}+(\cos u\sin u)^N\boldsymbol{J}}$$

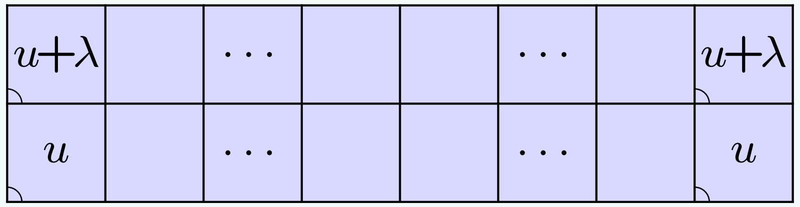
Here $\boldsymbol{J}$ is the N-tangle defined as the sum of the 2^N possible horizontal combinations of N 3-tangles of the type



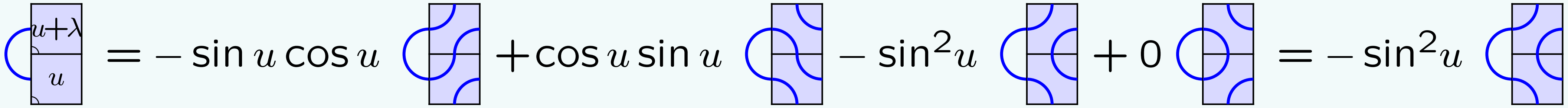
where the left and right vertical edges of $\boldsymbol{J}$ are identified to respect the cylinder topology.

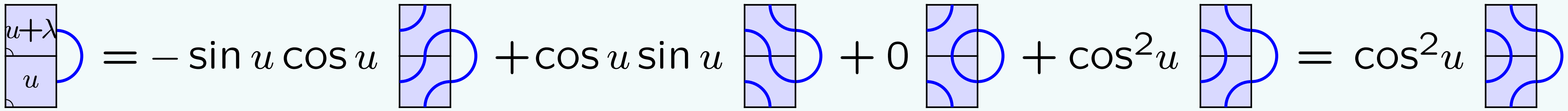
Proof

$$\boldsymbol{T}(u)\boldsymbol{T}(u+\tfrac{\pi}{2}) \;=$$

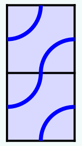
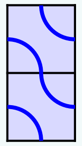


First, we examine the consequences of having a half-arc connecting the two left nodes (or two right nodes) of a 3-tangle



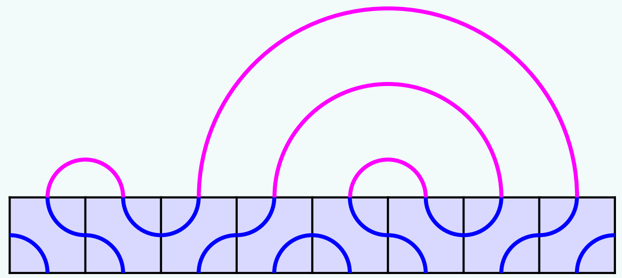


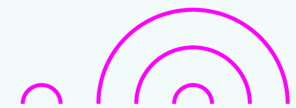
A half-arc is seen to 'propagate' and ultimately lead to the term proportional to $\boldsymbol{I}$. Having accounted for all situations with a half-arc, we are left with the two weighted connectivities

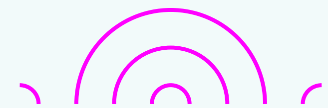
$-\cos u\sin u$  and $\cos u\sin u$  $\square$

Matrix Realizations

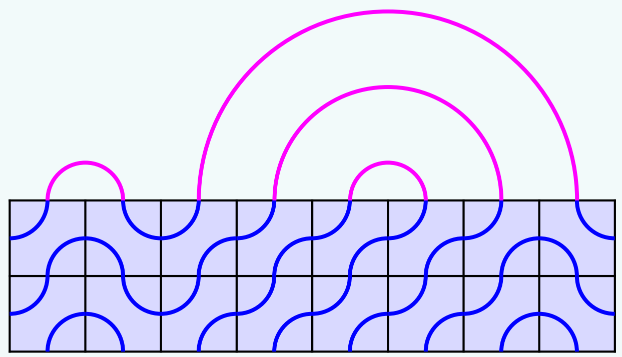
Transfer matrix



initial state: 

resulting state: 

Matrix J



initial state: 

resulting state: α 

- In a given sector defined by a specified number of defects ℓ

$$\boxed{\boldsymbol{J} \;=\; \begin{cases} (-1)^{\frac{N-\ell}{2}}\Big(2+(\alpha^2-4)\delta_{\ell,0}\Big)\boldsymbol{I}, & N,\ell \text{ even} \\ 0, & N,\ell \text{ odd} \end{cases}}$$

Take natural choice

$$\alpha \;=\; 2$$

- Indicative of a **logarithmic** nature of the model, $\boldsymbol{J}$ is **non-diagonalizable** when acting on link states with an arbitrary even number of defects. For $\alpha = 2$, its minimal polynomial implies

$$\boxed{(\boldsymbol{J}^2-4)^2 \;=\; 0 \quad\rightarrow\quad \boldsymbol{R} \;=\; -(\boldsymbol{J}^3-12\boldsymbol{J})/16 \;=\; (-1)^{\boldsymbol{F}}, \qquad \boldsymbol{R}^2 \;=\; \boldsymbol{I}, \qquad R \;=\; (-1)^{\ell/2}}$$

Summary of Finite-Size Corrections Results

Partition function

$$\boxed{Z_{N,M} \;=\; \mathrm{Tr}\,\boldsymbol{T}(u)^M \;=\; \sum_{n\geq 0} T_n(u)^M \;=\; \sum_{n\geq 0} e^{-M\mathcal{E}_n(u)}}$$

Finite-size corrections from conformal invariance

$$\mathcal{E}_0 \;=\; N f_{bulk} - \frac{\pi c}{6N}\sin 2u, \qquad \mathcal{E}_n - \mathcal{E}_0 \;=\; \frac{2\pi i}{N}\left[(\Delta + k)e^{-2iu} - (\bar{\Delta} + \bar{k})e^{2iu}\right]$$

Bulk free energy

$$f_{bulk} \;=\; \tfrac{1}{2}\log 2 - \frac{1}{\pi}\int_0^{\pi/2} \log\left(\frac{1}{\sin t} + \sin 2u\right) dt$$

Conformal data results

$$\boxed{c \;=\; -2, \qquad \Delta, \bar{\Delta} \;=\; \Delta_t \;=\; \frac{t^2 - 1}{8}, \qquad t = \ell/2 \in \tfrac{1}{2}\mathbb{Z}}$$

- In the scaling limit, the conformal partition functions are sesquilinear forms in characters

$$Z(q) \;=\; \sum_{\Delta,\bar{\Delta}} \mathcal{N}_{\Delta,\bar{\Delta}}\, \chi_{\Delta}(q)\chi_{\bar{\Delta}}(\bar{q}) \qquad \left(q \;=\; \exp(2\pi i\tau), \quad \tau \;=\; -\frac{M}{N}e^{-2iu}\right)$$

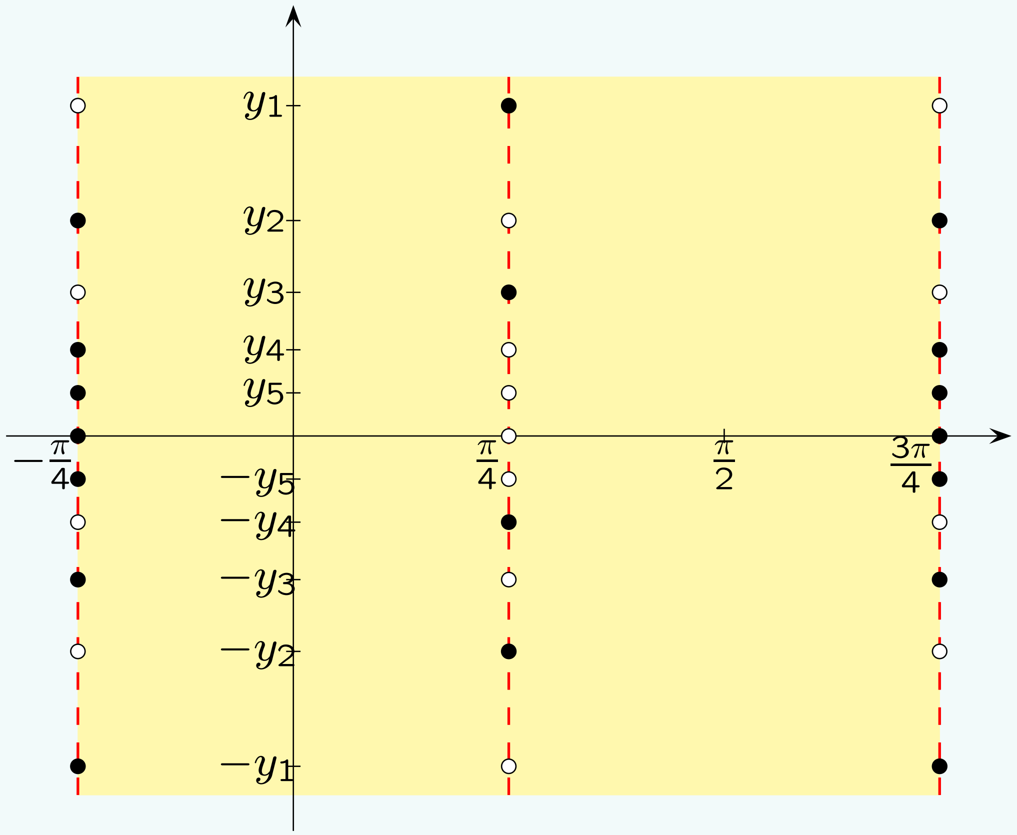
where the characters are of the form

$$\chi_{\Delta}(q) \;=\; q^{-c/24}\sum_{k=0}^{\infty} d_{\Delta}(k)\, q^{\Delta + k} \;=\; \begin{cases} \mathrm{ch}_{r,s}(q) = \dfrac{q^{-c/24 + \Delta_{2r-s}}(1 - q^{rs})}{\prod_{n=1}^{\infty}(1 - q^n)}, & N, \ell \text{ even} \\[2ex] \mathrm{ch}_t(q) = \dfrac{q^{-c/24 + \Delta_t}}{\prod_{n=1}^{\infty}(1 - q^n)}, & N, \ell \text{ odd} \end{cases}$$

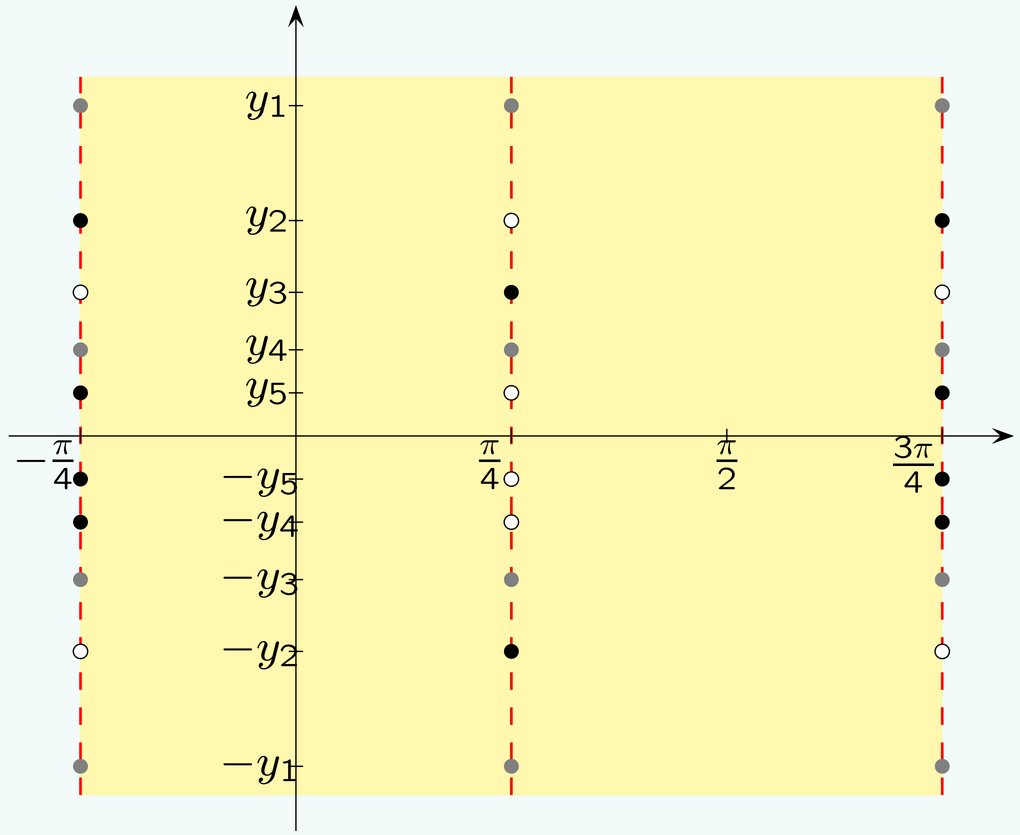
Spectra: Sector-by-Sector

Sector-by-Sector Inversion Identity and Patterns of Zeros in Complex u-Plane:

$$T(u)T(u+\tfrac{\pi}{2}) \;=\; \begin{cases} \cos^{2N}u-\sin^{2N}u, & N,\ell \text{ odd}, \quad \big\{\mathbb{Z}_4 \text{ Sectors} \\ \big(\cos^N u+(-1)^{(N-\ell)/2}\sin^N u\big)^2, & N,\ell \text{ even}, \quad \begin{cases}\text{Ramond } (\ell/2 \text{ even}) \\ \text{Neveu-Schwarz } (\ell/2 \text{ odd})\end{cases} \end{cases}$$



N,ℓ odd



N,ℓ even

- The y-ordinates of 1-strings u_j and 1-string energies E_j are

$$y_j \;=\; -\tfrac{1}{2}\log\tan\frac{E_j\pi}{N}, \qquad E_j=\begin{cases}\tfrac{1}{2}(j-\tfrac{1}{2}), & j=1,2,\ldots,N; \quad \mathbb{Z}_4 \\ j-\tfrac{1}{2}, & j=1,2,\ldots,N/2; \quad \text{Ramond} \\ j, & j=1,2,\ldots,N/2-1; \quad \text{Neveu-Schwarz}\end{cases}$$

$\mathbb{Z}_4$ Sectors (N, ℓ Odd)

- Factorizing and sharing out the zeros in the upper and lower half-planes, while treating the zeros on the real axis as belonging to the upper half-plane, we get

$$\begin{aligned} T(u) &= \epsilon\,\frac{(-i)^{N/2}e^{-Niu}}{2^{N-1/2}}\prod_{j=1}^{N}\Big(e^{2iu}+i\epsilon_j\tan\frac{(2j-1)\pi}{4N}\Big), \qquad \epsilon^2=\epsilon_j^2=\bar{\epsilon}_j^2=1 \\ &= \frac{(-i)^{N/2}}{2^{N-1/2}e^{Niu}}\prod_{j=1}^{\frac{N+1}{2}}\Big(e^{2iu}+i\epsilon_j\tan\frac{(2j-1)\pi}{4N}\Big)\prod_{j=1}^{\frac{N-1}{2}}\Big(\bar{\epsilon}_j e^{2iu}+i\cot\frac{(2j-1)\pi}{4N}\Big) \end{aligned}$$

- An elementary excitation is governed by $\epsilon_j=-1$ or $\bar{\epsilon}_j=-1$ and corresponds to a 1-string at position j. Taking the ratio with precisely one $\epsilon_j=-1$ or $\bar{\epsilon}_j=-1$ to all $\epsilon_j,\bar{\epsilon}_j=+1$ gives

$$\lim_{M,N\to\infty}\left(\frac{e^{2iu}-i\tan\frac{(2j-1)\pi}{4N}}{e^{2iu}+i\tan\frac{(2j-1)\pi}{4N}}\right)^M = \exp[-(j-\tfrac{1}{2})\pi i\,\delta\,e^{-2iu}] = q^{E_j}$$

$$\lim_{M,N\to\infty}\left(\frac{-e^{2iu}+i\cot\frac{(2j-1)\pi}{4N}}{e^{2iu}+i\cot\frac{(2j-1)\pi}{4N}}\right)^M = \exp[(j-\tfrac{1}{2})\pi i\,\delta\,e^{2iu}] = \bar{q}^{E_j}$$

$$\delta=\frac{M}{N}=\text{fixed aspect ratio}$$

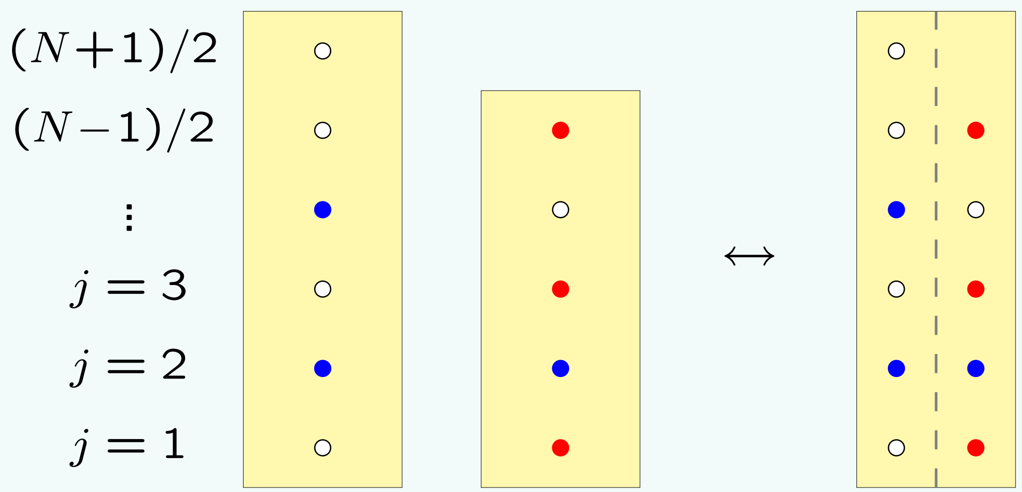
Partition functions:

$$\boxed{Z_\ell(q) = (q\bar{q})^{-c/24+\Delta_{1/2}}\sum_{\epsilon,\bar{\epsilon}} q^{\sum_{j=1}^{(N+1)/2}\delta_{\epsilon_j,-1}E_j}\,\bar{q}^{\sum_{j=1}^{(N-1)/2}\delta_{\bar{\epsilon}_j,-1}E_j}}$$

- The sum is restricted by selection rules determined by *physical combinatorics*.

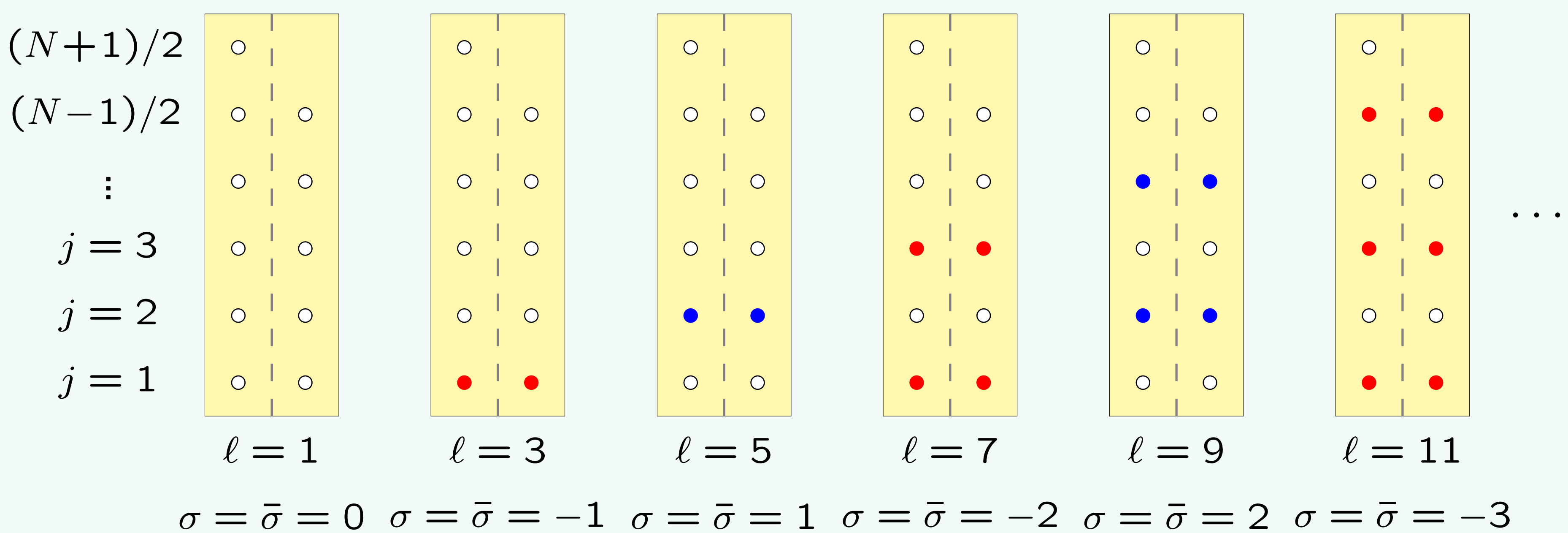
Single and Paired Single-Column Diagrams

- The right column corresponds to the 1-strings in the lower-half u-plane (associated with $\bar{q}$). The left column corresponds to the 1-strings in the upper half-plane (associated with q), including the real axis, but rotated through 180 degrees.



$$\leftrightarrow \quad q^{\frac{3}{4}+\frac{7}{4}}\,\bar{q}^{\frac{1}{4}+\frac{3}{4}+\frac{5}{4}+\frac{9}{4}} = q^{\frac{5}{2}}\,\bar{q}^{\frac{9}{2}}, \qquad E_j, \bar{E}_j = \tfrac{1}{2}(j-\tfrac{1}{2})$$

Groundstate configurations: $\mathbb{Z}_4$ sectors (ℓ odd)

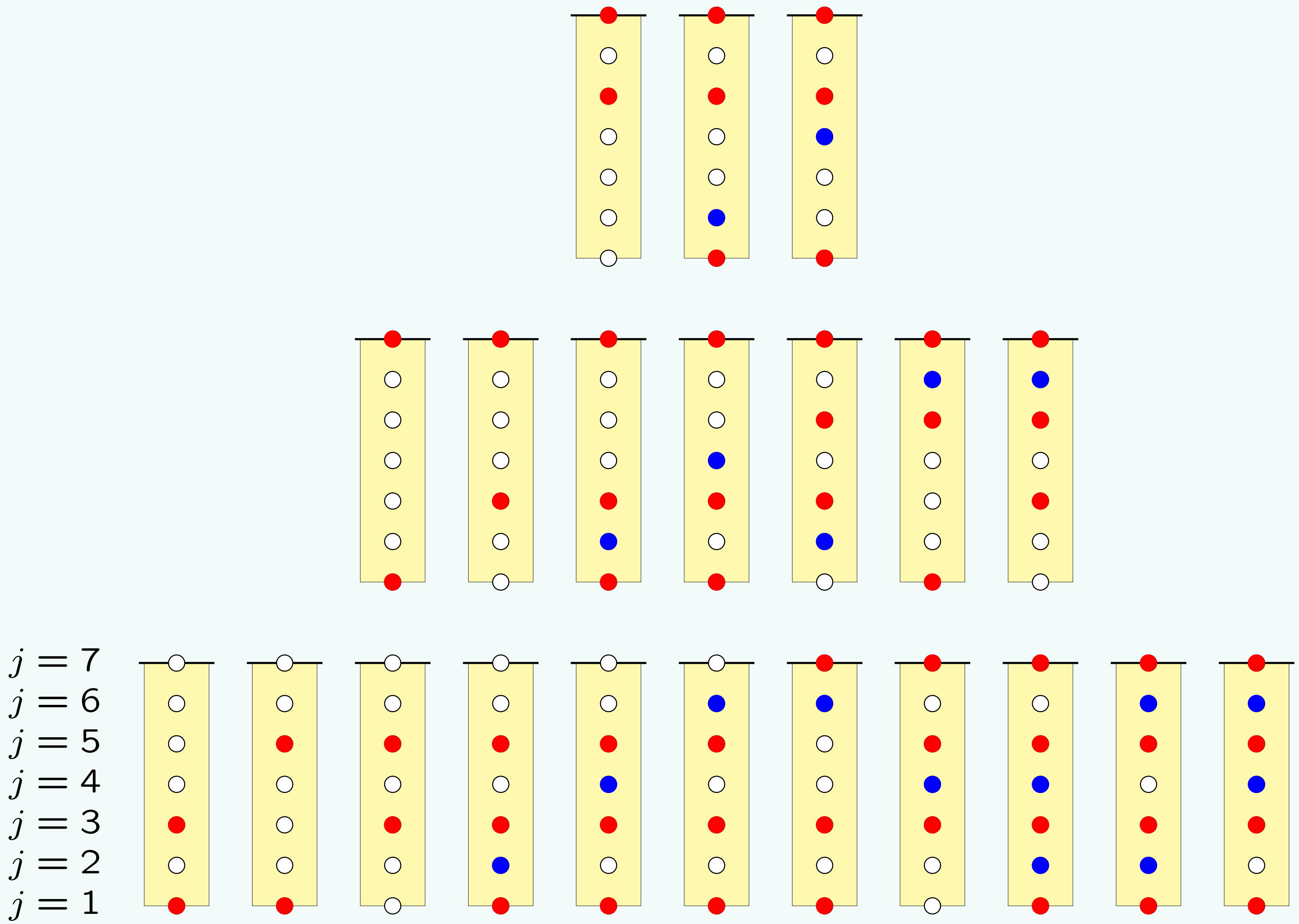


- The quantum numbers $\sigma, \bar{\sigma}$ are given by the excess of blue (even j) over red (odd j).

$\mathbb{Z}_4$ Sectors: Physical Combinatorics I

- The building blocks of the spectra in the $\mathbb{Z}_4$ sectors consist of the q-binomials

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{bmatrix} n \\ \lfloor n/2 \rfloor - \sigma \end{bmatrix}_q = q^{-\frac{1}{2}\sigma(\sigma+\frac{1}{2})} \sum_{\substack{\text{columns} \\ \text{for fixed } \sigma}} q^{\sum_j m_j E_j}, \qquad \begin{cases} \sigma = \lfloor n/2 \rfloor - m \\ E_j = \frac{1}{2}(j - \frac{1}{2}) \end{cases}$$



$$\begin{bmatrix} 7 \\ 5 \end{bmatrix}_q = 1 + q + 2q^2 + 2q^3 + 3q^4 + 3q^5 + 3q^6 + 2q^7 + 2q^8 + q^9 + q^{10} \quad (\sigma = -2)$$

$\mathbb{Z}_4$ Sectors: Physical Combinatorics II

- As q-binomials, $\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{bmatrix} n \\ n-m \end{bmatrix}_q$, but they have **different combinatorial interpretations** because they have different quantum numbers σ.

- At any given position, the number of 1-strings plus the number of 2-strings is exactly 1.

- The single-columns with quantum number σ are generated combinatorially by starting with the minimum energy configuration of 1-strings for given σ.

Empirical Selection Rules:

- Empirically determined selection rules dictate that, in a sector with ℓ defects, the quantum numbers of the groundstate satisfy

$$\boxed{\sigma \;=\; \bar{\sigma} \;=\; \begin{cases} (\ell-1)/4, & \ell = 1 \text{ mod } 4 \\ -(\ell+1)/4, & \ell = 3 \text{ mod } 4 \end{cases} \qquad \ell = |4\sigma+1| = 1,3,5,7,\ldots}$$

- The energy of these groundstates is $E(\sigma)+E(\bar{\sigma}) = \frac{1}{16}(\ell^2-1)$.

- Excitations, incrementing the energy by one unit, are generated either by inserting a pair of 1-strings at positions $j=1$ and $j=2$ or incrementing the position j of a 1-string by 2 units. The excitations satisfy the empirically determined selection rules

$$\boxed{\sigma+\bar{\sigma} \;=\; \begin{cases} \frac{1}{2}(\ell-1), & \ell = 1 \text{ mod } 4 \\ -\frac{1}{2}(\ell+1), & \ell = 3 \text{ mod } 4 \end{cases} \qquad \tfrac{1}{2}(\sigma-\bar{\sigma}) \in \mathbb{Z}}$$

$\mathbb{Z}_4$ Finitized Partition Functions $(N,\ell$ Odd)

Finitized partition functions

$$Z_\ell^{(N)}(q) = \begin{cases} (q\bar{q})^{-c/24} \sum_{k\in\mathbb{Z}} q^{\Delta_{2k+\ell/2}} \begin{bmatrix} \frac{N+1}{2} \\ \frac{N-\ell}{4}-k \end{bmatrix}_q \bar{q}^{\Delta_{2k-\ell/2}} \begin{bmatrix} \frac{N-1}{2} \\ \frac{N-\ell}{4}+k \end{bmatrix}_{\bar{q}} \\ (q\bar{q})^{-c/24} \sum_{k\in\mathbb{Z}} q^{\Delta_{2k+\ell/2}} \begin{bmatrix} \frac{N+1}{2} \\ \frac{N+\ell+2}{4}+k \end{bmatrix}_q \bar{q}^{\Delta_{2k-\ell/2}} \begin{bmatrix} \frac{N-1}{2} \\ \frac{N+\ell-2}{4}-k \end{bmatrix}_{\bar{q}} \end{cases} \qquad N-\ell = 0 \bmod 4$$

$$Z_\ell^{(N)}(q) = \begin{cases} (q\bar{q})^{-c/24} \sum_{k\in\mathbb{Z}} q^{\Delta_{2k+\ell/2}} \begin{bmatrix} \frac{N+1}{2} \\ \frac{N-\ell+2}{4}-k \end{bmatrix}_q \bar{q}^{\Delta_{2k-\ell/2}} \begin{bmatrix} \frac{N-1}{2} \\ \frac{N-\ell-2}{4}+k \end{bmatrix}_{\bar{q}} \\ (q\bar{q})^{-c/24} \sum_{k\in\mathbb{Z}} q^{\Delta_{2k+\ell/2}} \begin{bmatrix} \frac{N+1}{2} \\ \frac{N+\ell}{4}+k \end{bmatrix}_q \bar{q}^{\Delta_{2k-\ell/2}} \begin{bmatrix} \frac{N-1}{2} \\ \frac{N+\ell}{4}-k \end{bmatrix}_{\bar{q}} \end{cases} \qquad N-\ell = 2 \bmod 4$$

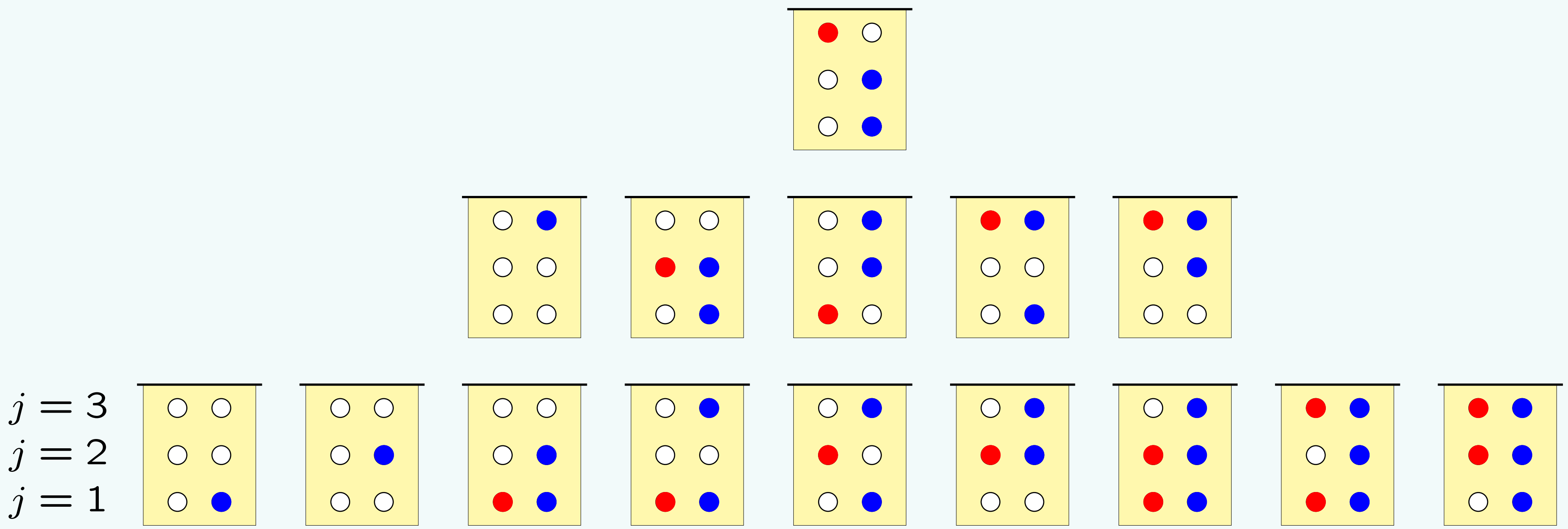
- For given mod 4 parities of $N-\ell$, these expressions are equivalent as partition functions but, in each case, the first form is used for the combinatorial interpretation when $\ell = 1$ mod 4 and the second form when $\ell = 3$ mod 4.
- We observe that

$$\sum_{\ell\in 2\mathbb{N}-1}^{\ell\le N} Z_\ell^{(N)}(q) = \tfrac{1}{2}(q\bar{q})^{-\frac{c}{24}-\frac{3}{32}}\left[\prod_{n=1}^{\frac{N+1}{2}}(1+q^{\frac{2n-1}{4}})\prod_{n=1}^{\frac{N-1}{2}}(1+\bar{q}^{\frac{2n-1}{4}}) + \prod_{n=1}^{\frac{N+1}{2}}(1-q^{\frac{2n-1}{4}})\prod_{n=1}^{\frac{N-1}{2}}(1-\bar{q}^{\frac{2n-1}{4}})\right]$$

Physical Combinatorics: Ramond Sectors

- The building blocks of the spectra in the Ramond sectors consist of the q-binomials

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{bmatrix} n \\ \lfloor n/2 \rfloor - \sigma \end{bmatrix}_q = q^{-\frac{1}{2}\sigma^2} \sum_{\substack{\text{double-columns} \\ \text{for fixed } \sigma}} q^{\sum_j m_j E_j}, \qquad \begin{cases} \sigma = \lfloor n/2 \rfloor - m = \#\text{right} - \#\text{left} \\ E_j = j - \frac{1}{2} \end{cases}$$



$$\begin{bmatrix} 6 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + 2q^3 + 3q^4 + 2q^5 + 2q^6 + q^7 + q^8 \quad (\sigma = 1)$$

- As q-binomials, $\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{bmatrix} n \\ n-m \end{bmatrix}_q$, but they have **different combinatorial interpretations**.
- In a sector with ℓ defects, the quantum numbers of the groundstate satisfy

$$\boxed{\sigma = \bar{\sigma} = \ell/4, \qquad \ell = 0, 4, 8, \ldots} \qquad \left(E(\sigma) + E(\bar{\sigma}) = \frac{\ell^2}{16} \right)$$

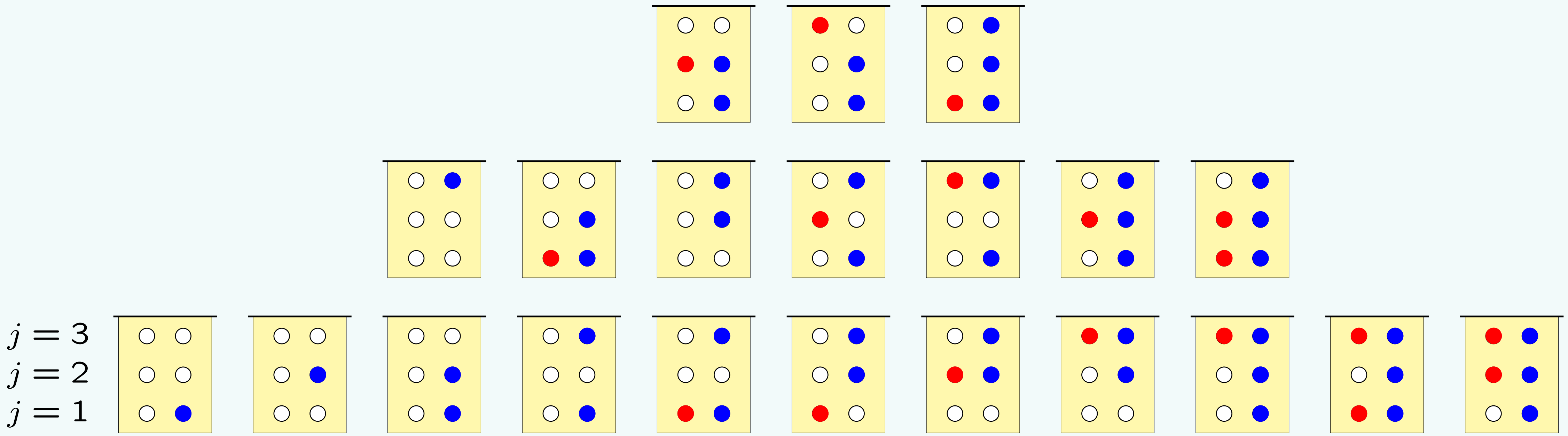
- Excitations are generated either by inserting a left-right pair of 1-strings at position $j = 1$ or incrementing the position j of a 1-string by 1 unit. The selection rules are

$$\boxed{\sigma + \bar{\sigma} = \ell/2, \qquad \tfrac{1}{2}(\sigma - \bar{\sigma}) \in \mathbb{Z}}$$

Physical Combinatorics: Neveu-Schwarz Sectors

- The building blocks of the spectra in the Neveu-Schwarz sectors consist of the q-binomials

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{bmatrix} n \\ \lfloor n/2 \rfloor - \sigma \end{bmatrix}_q = q^{-\frac{1}{2}\sigma(\sigma+1)} \sum_{\substack{\text{double-columns} \\ \text{for fixed } \sigma}} q^{\sum_j m_j E_j}, \qquad \begin{cases} \sigma = \lfloor n/2 \rfloor - m, \ \#\text{right} - \#\text{left} = \sigma, \sigma+1 \\ E_j = j \end{cases}$$



$$\begin{bmatrix} 7 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + 2q^3 + 3q^4 + 3q^5 + 3q^6 + 2q^7 + 2q^8 + q^9 + q^{10} \quad (\sigma = 1)$$

- As q-binomials, $\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{bmatrix} n \\ n-m \end{bmatrix}_q$, but they have **different combinatorial interpretations**.
- In a sector with ℓ defects, the quantum numbers of the groundstate satisfy

$$\boxed{\sigma = \bar{\sigma} = (\ell-2)/4, \qquad \ell = 2, 6, 10, \ldots} \qquad \left(E(\sigma) + E(\bar{\sigma}) = \frac{\ell^2 - 4}{16} \right)$$

- Excitations are generated either by inserting a left-right pair of 1-strings at position $j = 1$ or incrementing the position j of a 1-string by 1 unit. The selection rules are

$$\boxed{\sigma + \bar{\sigma} = (\ell-2)/2, \qquad \tfrac{1}{2}(\sigma - \bar{\sigma}) \in \mathbb{Z}}$$

Finitized Partition Functions (N, ℓ Even)

Ramond sectors ($\ell/2$ even)

$$\boxed{Z_\ell^{(N)}(q) \;=\; (q\bar{q})^{-c/24}\sum_{k\in\mathbb{Z}} q^{\Delta_{2k+\ell/2}}\begin{bmatrix} 2\lfloor\frac{N+2}{4}\rfloor \\ \lfloor\frac{N+2-\ell}{4}\rfloor-k\end{bmatrix}_q \bar{q}^{\Delta_{2k-\ell/2}}\begin{bmatrix} 2\lfloor\frac{N}{4}\rfloor \\ \lfloor\frac{N-\ell}{4}\rfloor+k\end{bmatrix}_{\bar{q}}}$$

- We observe that

$$Z_0^{(N)} + 2\sum_{\ell\in 4\mathbb{N}}^{\ell\le N} Z_\ell^{(N)}(q) \;=\; \tfrac{1}{2}(q\bar{q})^{-\frac{c}{24}-\frac{1}{8}}\Bigg[\prod_{n=1}^{\lfloor\frac{N+2}{4}\rfloor}(1+q^{n-\frac{1}{2}})^2\prod_{n=1}^{\lfloor\frac{N}{4}\rfloor}(1+\bar{q}^{n-\frac{1}{2}})^2$$

$$+\prod_{n=1}^{\lfloor\frac{N+2}{4}\rfloor}(1-q^{n-\frac{1}{2}})^2\prod_{n=1}^{\lfloor\frac{N}{4}\rfloor}(1-\bar{q}^{n-\frac{1}{2}})^2\Bigg]$$

Neveu-Schwarz sectors ($\ell/2$ odd)

$$\boxed{Z_\ell^{(N)}(q) = (q\bar{q})^{-c/24}\sum_{k\in\mathbb{Z}} q^{\Delta_{2k+\ell/2}}\begin{bmatrix} 2\lfloor\frac{N}{4}\rfloor+1 \\ \lfloor\frac{N+2-\ell}{4}\rfloor-k\end{bmatrix}_q \bar{q}^{\Delta_{2k-\ell/2}}\begin{bmatrix} 2\lfloor\frac{N+2}{4}\rfloor-1 \\ \lfloor\frac{N-\ell}{4}\rfloor+k\end{bmatrix}_{\bar{q}}}$$

- We observe that

$$\sum_{\ell\in 4\mathbb{N}-2}^{\ell\le N} Z_\ell^{(N)}(q) \;=\; (q\bar{q})^{-\frac{c}{24}}\prod_{n=1}^{\lfloor\frac{N}{4}\rfloor}(1+q^n)^2\prod_{n=1}^{\lfloor\frac{N-2}{4}\rfloor}(1+\bar{q}^n)^2$$

Conformal Partition Functions

- In the limit $N \to \infty$, the sesquilinear conformal partition functions are

$$Z_\ell(q) = \begin{cases} \displaystyle\sum_{k=-\infty}^{\infty} \mathrm{ch}_{2k+\ell/2}(q)\,\mathrm{ch}_{2k-\ell/2}(\bar q), & \mathbb{Z}_4\ (\ell \text{ odd}) \\ \displaystyle\sum_{r,\bar r=1}^{\infty} \tfrac{1}{4}(1+(-1)^{r+\bar r})\Big[\max\Big(\tfrac{\ell}{2}, r+\bar r\Big) - \max\Big(\tfrac{\ell}{2}, |r-\bar r|\Big)\Big]\mathrm{ch}_{r,2}(q)\,\mathrm{ch}_{\bar r,2}(\bar q), & \text{Ramond } (\tfrac{\ell}{2} \text{ even}) \\ \displaystyle\sum_{r,\bar r=1}^{\infty} \Big[\max\Big(\tfrac{\ell}{2}, r+\bar r\Big) - \max\Big(\tfrac{\ell}{2}, |r-\bar r|\Big)\Big]\,\mathrm{ch}_{r,1}(q)\,\mathrm{ch}_{\bar r,1}(\bar q), & \text{N-S } (\tfrac{\ell}{2} \text{ odd}) \end{cases}$$

- Summing over the ℓ odd or ℓ even sectors with suitable multiplicities yields

$$\sum_{\ell\in 2\mathbb{N}-1} Z_\ell(q) = \frac{|\vartheta_{\frac{1}{2},2}(q)|^2 + |\vartheta_{\frac{3}{2},2}(q)|^2}{|\eta(q)|^2} \qquad \Big(\vartheta_{s,p}(q) = \sum_{\lambda\in\mathbb{Z}+\frac{s}{2p}} q^{p\lambda^2}, \quad \eta(q) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)\Big)$$

$$Z_0(q) + 2\sum_{\ell\in 4\mathbb{N}} Z_\ell(q) = \frac{|\vartheta_{0,2}(q)|^2 + |\vartheta_{2,2}(q)|^2}{|\eta(q)|^2} = |\hat\chi_{-1/8}(q)|^2 + |\hat\chi_{3/8}(q)|^2$$

$$2\sum_{\ell\in 4\mathbb{N}-2} Z_\ell(q) = \frac{|\vartheta_{1,2}(q)|^2 + |\vartheta_{3,2}(q)|^2}{|\eta(q)|^2} = \frac{2|\vartheta_{1,2}(q)|^2}{|\eta(q)|^2} = 2|\hat\chi_0(q) + \hat\chi_1(q)|^2$$

where $\hat\chi_\Delta(q)$ are $\mathcal{W}$-irreducible characters.

- Due to the special role of the $\ell = 0$ sector, a naive trace does *not* produce a *modular invariant*. However, a *modified trace* does produce the expected modular invariant

$$\boxed{|\hat\chi_{-1/8}(q)|^2 + |\hat\chi_{3/8}(q)|^2 + 2|\hat\chi_0(q) + \hat\chi_1(q)|^2}$$

Summary

- Exact solution of critical dense polymers on the cylinder as a Yang-Baxter integrable lattice model. General solution in $\mathbb{Z}_4$, Ramond and Neveu-Schwarz sectors.

- Transfer matrix defined in the diagrammatic cylinder Temperley-Lieb algebra. Non-contractible loops with fugacity $\alpha = 2$. Contractible loops not allowed since $\beta = 0$.

- Inversion identity in cylinder Temperley-Lieb algebra involving the matrix $\boldsymbol{J}$. Non-trivial Jordan decomposition of $\boldsymbol{J}$ implies existence of involution $\boldsymbol{R}$ with eigenvalues $R = (-1)^{\ell/2}$. Pairs of defects interpreted as fermions.

- Physical combinatorics and selection rules: patterns of zeros, single- and double-column diagrams, several interpretations of q-binomials, finitized partition functions and characters.

- Sesquilinear conformal partition functions are obtained in the continuum scaling limit. The naive trace does not give a modular invariant partition function. The role of modular invariance is not yet fully understood on the torus.