SHARED INFORMATION IN STATIONARY STATES

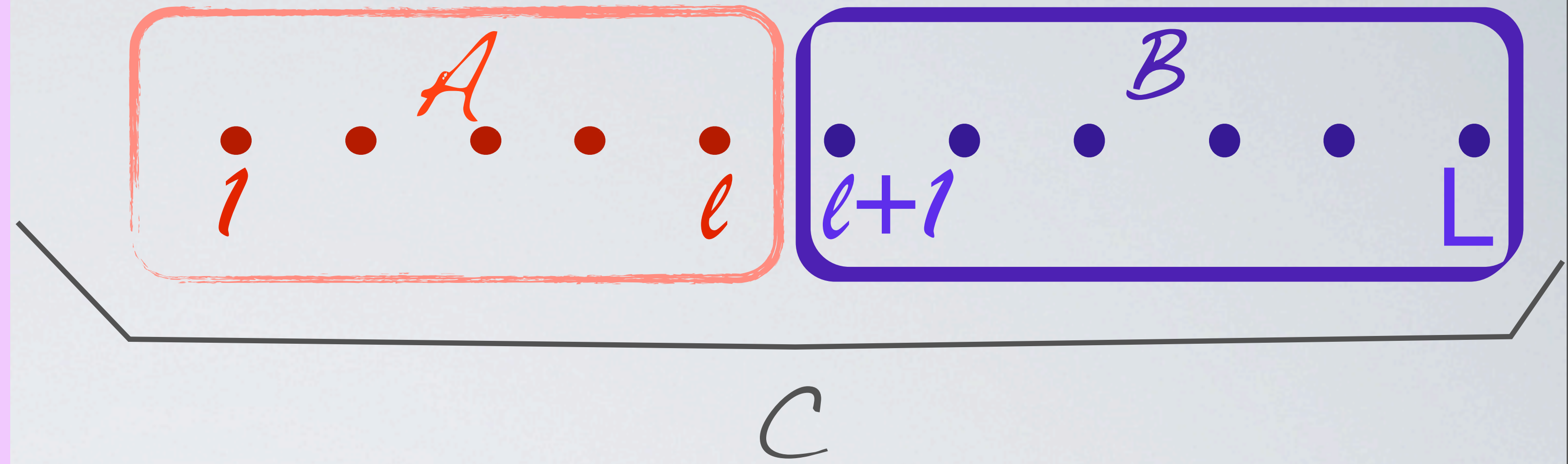
Francisco C. Alcaraz

Universidade de São Paulo - IFSC - São Carlos - BRAZIL

FCA, Sierra G and Rittenberg V - Phys. Rev. E **80**, 030102(R) (2009)
FCA and Rittenberg V - JSTAT P03024 (2010)

- Introduction and motivation
- Estimators for shared information
- Application 1: Model for polymer adsorption
- Aplication 2: The raise and peel model
- Parity effects on the estimators
- Conclusions

Entanglement properties of ground-state wavefunctions of Hermitian Hamiltonians



$$S_{vN}(\mathcal{A}) = S_{vN}(\mathcal{B}) = -\mathrm{Tr}(\rho_{\mathcal{A}} \ln \rho_{\mathcal{A}}) \qquad \rho_{\mathcal{A}} = \mathrm{Tr}_{\mathcal{B}}(\rho)$$

For large system and subsystems $L >> l >> 1$

$S_{vN}(l, L) \sim constant$ ⟵ non critical (gapped)

$S_{vN} \sim \gamma \ln l + C$ ⟵ critical (gapless)

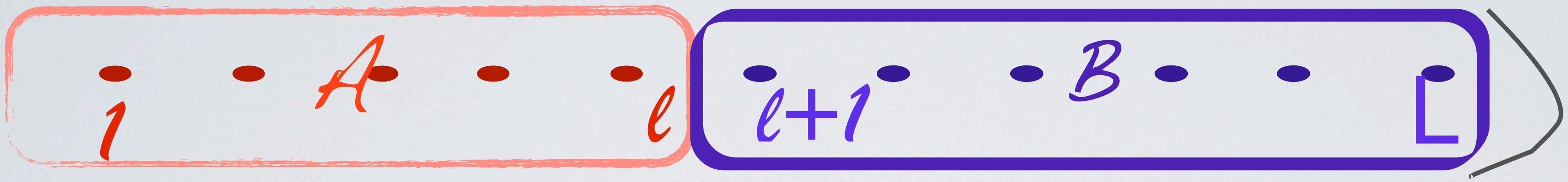
Calabrese and Cardy 2004

critical and conformal invariant - central charge c

$\gamma = \frac{c}{6}$ for open systems $\gamma = \frac{c}{3}$ for periodic systems

$$S_{vN}(l, L) = \gamma \ln \tilde{L}_C + C, \qquad \boxed{\tilde{L}_C = L \sin(\pi l / L)/\pi}$$ ⟵ f.s.s.

Shared information in stationary states of classical systems



C=A+B

$$\frac{d}{dt}|P>= -H|P> \qquad |\Psi> \longrightarrow |P>$$ probability distribution

$$|\Psi(t)>= \sum c_n(t)|\phi_n> \quad \longrightarrow \quad |P(t)>= \sum p_a(t)|\phi_a>$$

c_n complex
$|\phi_n>$ basis- Hilbert space $\longrightarrow$ p_a real (probabilities)
$|\phi_a>$ config. of the system

H Hermitian $\longrightarrow$ H non-Hermitian

Schmidt decomposition
density matrix ρ
entanglement entropy $\longrightarrow$ new measures of shared information

critical violates are law $\longrightarrow$ critical violates are law

Most critical systems conformally invariant $S \sim c/6 \ln l$ $\longrightarrow$ critical system rarelly conformally invariant $c = 0$

Aim

Produce estimators of the shared information among subsystems

★ Similar properties as in the quantum case

a) Vanish if the subsystems are separated

b) If ξ finite $\longrightarrow$ are finite

c) If we have logarithmic behavior for $E(l,L) \sim \gamma_E \ln \tilde{L}_E + C_E$

γ_E is universal and c_E non universal

d) If we have power-law behavior $E(l,L) \sim \gamma_E L^{\delta_E} + D_E$

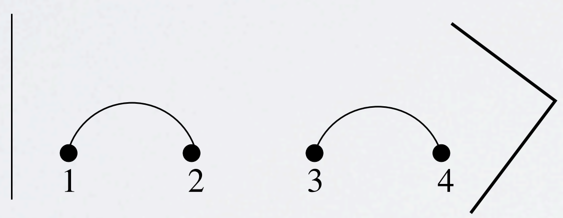
γ_E, δ_E universal and D_E non universal

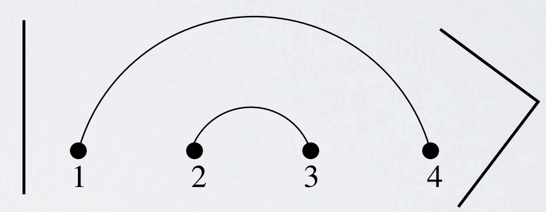
Configuration space ("Hilbert space" for classical model)

Inspiration: Quantum chains spin 1/2 SU(2) symmetric (Hermitian): Ex. Heisenberg chain (open bound. cond)

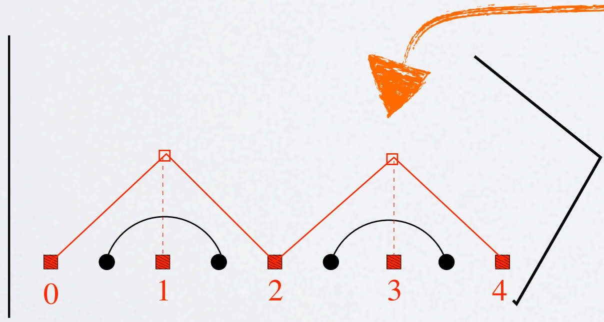
$$H = -J \sum_{i=1}^{L-1} \vec{S}_i \vec{S}_{i+1}$$

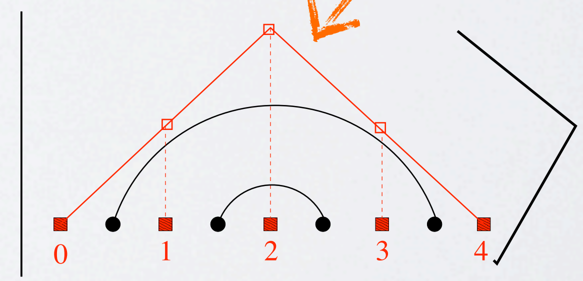
Ground state L=4 sites

$|\Psi >= c_1$ 

$+ \quad c_2$ 

One-to-one correspondence - Dyck paths

$|\Psi >= c_1$ 

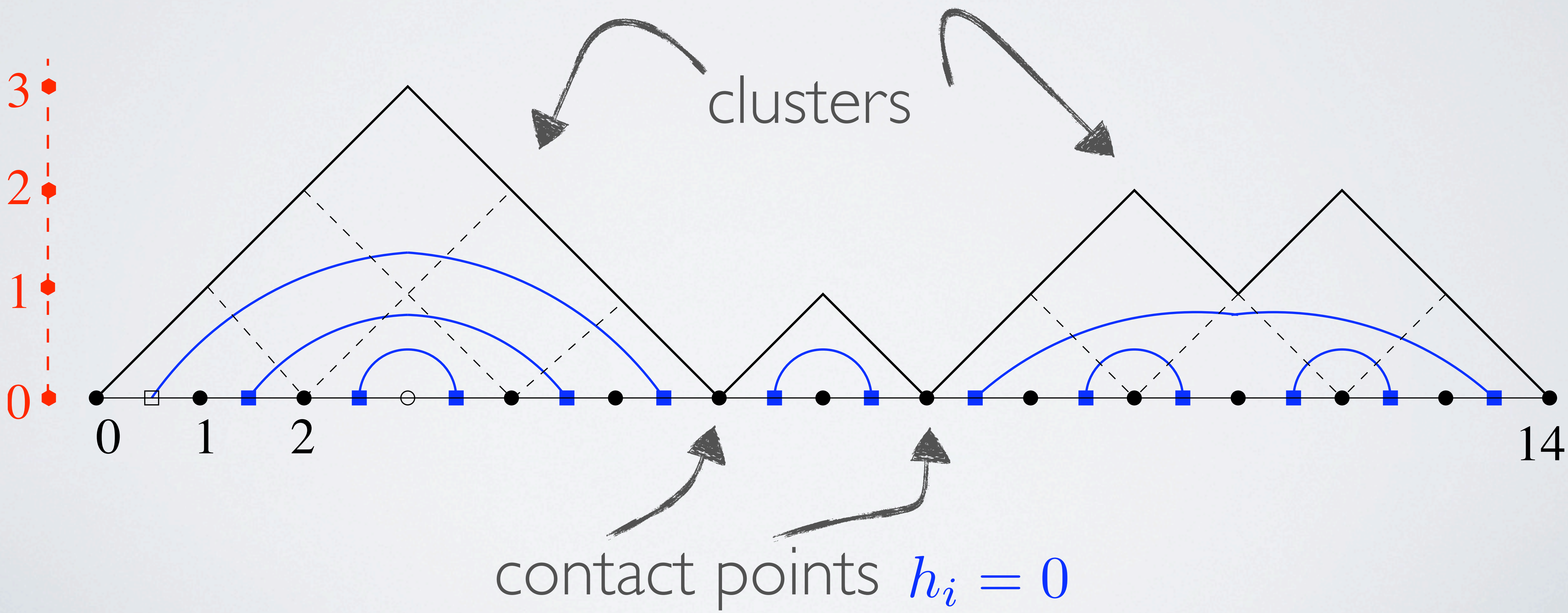
$+ \quad c_2$ 

Dyck Path: restrict solid-on-solid (RSOS) conf. in L+1 sites

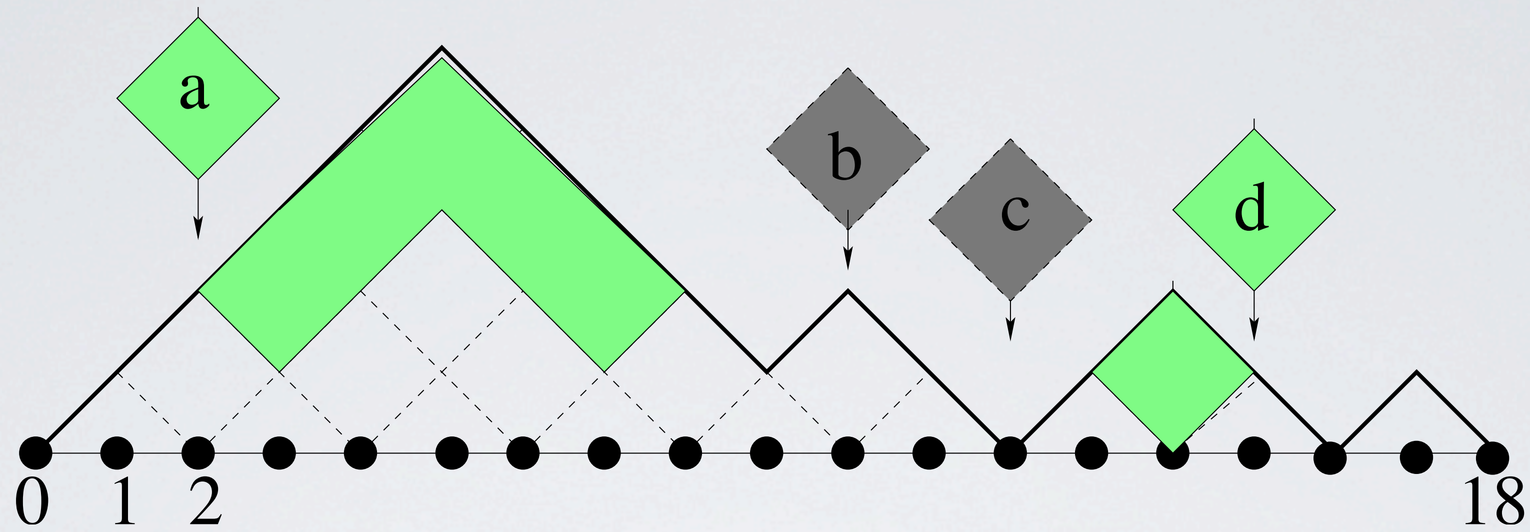
configurations: $(h_1, h_2, \ldots, h_l)$

$$h_{i+1} - h_i = \pm 1, \quad h_0 = h_L = 0 \quad (i = 0, 1, \ldots, L-1)$$

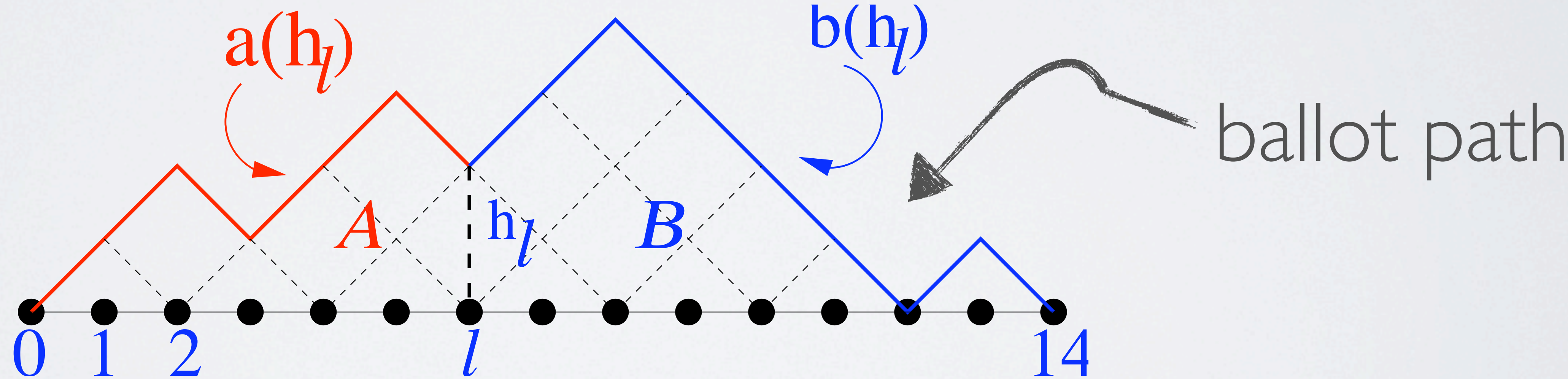
$$Z_1(L) = L!/(L/2)!(L/2+1)!$$



Stochastic model: Dyck paths = config. of an interface



Bipartitions of $\mathcal{C}$ (L+1 sites): Part $\mathcal{A}$ (l+1 sites) and part $\mathcal{B}$ (L-l sites)



if $h_l = 0$ → no shared information

h_l large → large shared information

$P(a(h_l), b(h_l))$ Prob. of Dyck path composed by $a(h_l)$ $b(h_l)$

Marginals $$P(a(h_l)) = \sum_b P(a(h_l), b(h_l))$$

Prob. height h_l at site l

$$P_l(h, L) = \sum_a P(a(h_l)) = \sum_b P(b(h_l))$$

Estimators

- **Mutual information:**

$$I(l,L) = \sum_{h_l, a(h_l), b(h_l)} P(a(h_l), b(h_l)) \ln \frac{P(a(h_l), b(h_l))}{P(a(h_l))P(b(h_l))}$$

Standard estimator for shared information

- **Interdependency:**

$$H_h(l,L) = -\sum_h P_l(h,L) \ln P_l(h,L)$$

Shannon entropy for heights h_l

New estimator: imitates the entanglement entropy

Renyi Interdependencies

$$R_n(l, L) = 1/(1-n) \ln \sum_h P_l(h, L)^n, \quad n = 2, 3, \ldots$$

- **Valence bond entanglement entropy:**

$$h(l, L) = \sum_h h P_l(h, L)$$

Average height at separation site.

Used in the context of SU(2) spin -1/2 quantum chains (Chhajlany, Tomczak, Wojcic 2007, Jacobsen, Saleur 2008)

- Density of contact points:

$$D(l,L) = -\ln P_l(0,L)$$

If $\rho(l,L) = P_l(0,L)$ small $\longrightarrow$ large clusters

- Separation Shanon entropy:

$$S(l,L) = H(L) - H(l) - H(L-l)$$

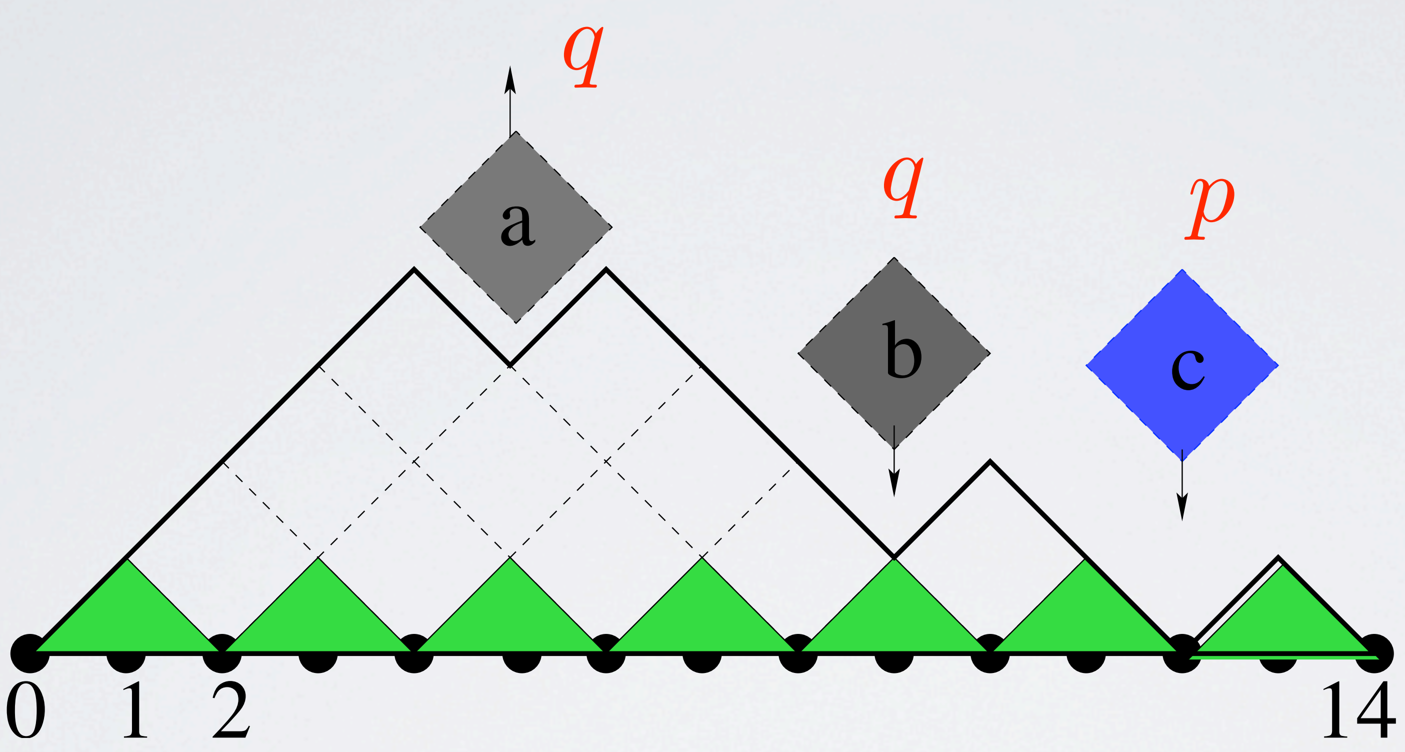
$$H(M) = -\sum_k P_k \ln P_k$$

Measures the increase of disorder in $\mathcal{C}$ due to $\mathcal{A}$ and $\mathcal{B}$

All the estimators vanishes when $\mathcal{A}$ and $\mathcal{B}$ separated

Stochastic models

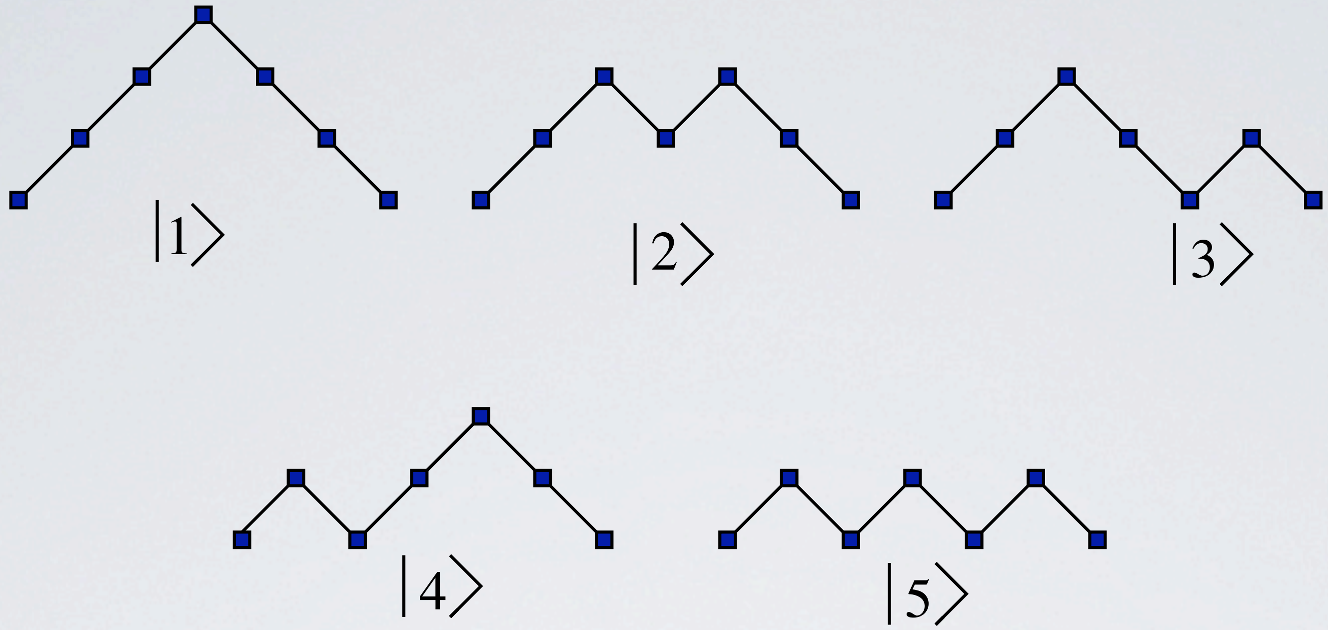
Model for polymer adsorption



$$q/p = K = u^{-1}$$

$u = 1$ is the Rouse Model (Rouse, 1953)

L=6



$$|P(t)>=\sum_{a=1}^{5} P_a(t)|a>, \quad P_a = \lim_{t\to\infty} P_a(t), \quad |0>=\sum_{a=1}^{5} P_a|a>$$

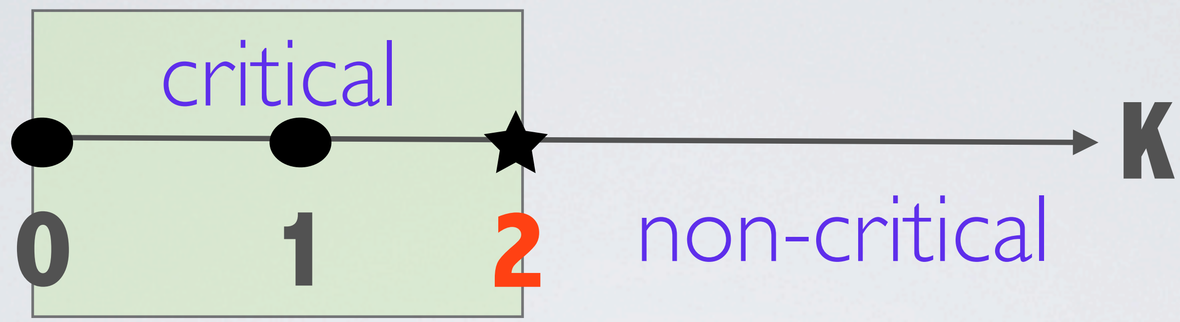
$$H = \begin{array}{c|ccccc} & |1\rangle & |2\rangle & |3\rangle & |4\rangle & |5\rangle \\ \hline \langle 1| & & & & & \\ \langle 2| & & & & & \\ \langle 3| & & & & & \\ \langle 4| & & & & & \\ \langle 5| & & & & & \end{array} \left(\begin{array}{rrrrr} 1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -\mathrm{u} & -\mathrm{u} & 0 \\ 0 & -1 & 1+\mathrm{u} & 0 & -\mathrm{u} \\ 0 & -1 & 0 & 1+\mathrm{u} & -\mathrm{u} \\ 0 & 0 & -1 & -1 & 2\mathrm{u} \end{array}\right). \qquad u = \frac{1}{K}$$

$$|0\rangle = |1\rangle + |2\rangle + K(|3\rangle + |4\rangle) + K^2|5\rangle$$

$$|0\rangle = \sum_{\psi} K^{m(\psi)}|\psi\rangle$$

contact points

The phase diagram (Owczarek 2009)



K=1 (Random Walker)

$$P_l(h,L) \sim \frac{4}{\sqrt{\pi}} \frac{z^2 e^{-z^2}}{\sqrt{\tilde{L}_{RW}}}, \quad z = h/\sqrt{\tilde{L}_{RW}}, \quad \tilde{L}_{RW}/2 = l(1-\frac{l}{L})$$

$$I(l,L) = H_h(l,L) \sim \frac{1}{2} \ln \tilde{L}_{RW} + C, \quad C \approx 0.303007$$

$$R_n(l,L) \sim 1/2 \ln \tilde{L}_{RW} + C_n$$

$$h(l,L) \sim \frac{4}{\sqrt{2\pi}} \tilde{L}_{RW}^{1/2}$$

$$D(l,L) = S(l,L) \sim \frac{3}{2} \ln \tilde{L}_{RW} + \ln \frac{\pi}{2\sqrt{2}}$$

In the finite-scaling regime: dependence on $\tilde{L}_{RW} = l(1 - l/L)$

The information diverges

$$E(l,L) \sim \gamma_E \ln \tilde{L}_{RW} + C_E$$

$$E(l,L) \sim \gamma_E \tilde{L}_{RW}^{\delta_E} + D_E$$

For $1 << l << L$

$$E(l,L) \sim \gamma_E \ln l + C_E$$

$$E(l,L) \sim \gamma_E l^{\delta_E}$$

$\gamma_I = \gamma_H = 1/2$ and $\gamma_D = \gamma_S = 3/2$ $\quad \gamma_h = 4/\sqrt{2\pi}, \quad \delta_h = 1/2$

Question: Are γ_E universals ?

Yes, $0 < K < 2$ $\longrightarrow$ γ_E same value (analytical and MC)

Finite- size scaling function $\tilde{L}_{RW} = l(1 - l/L)$

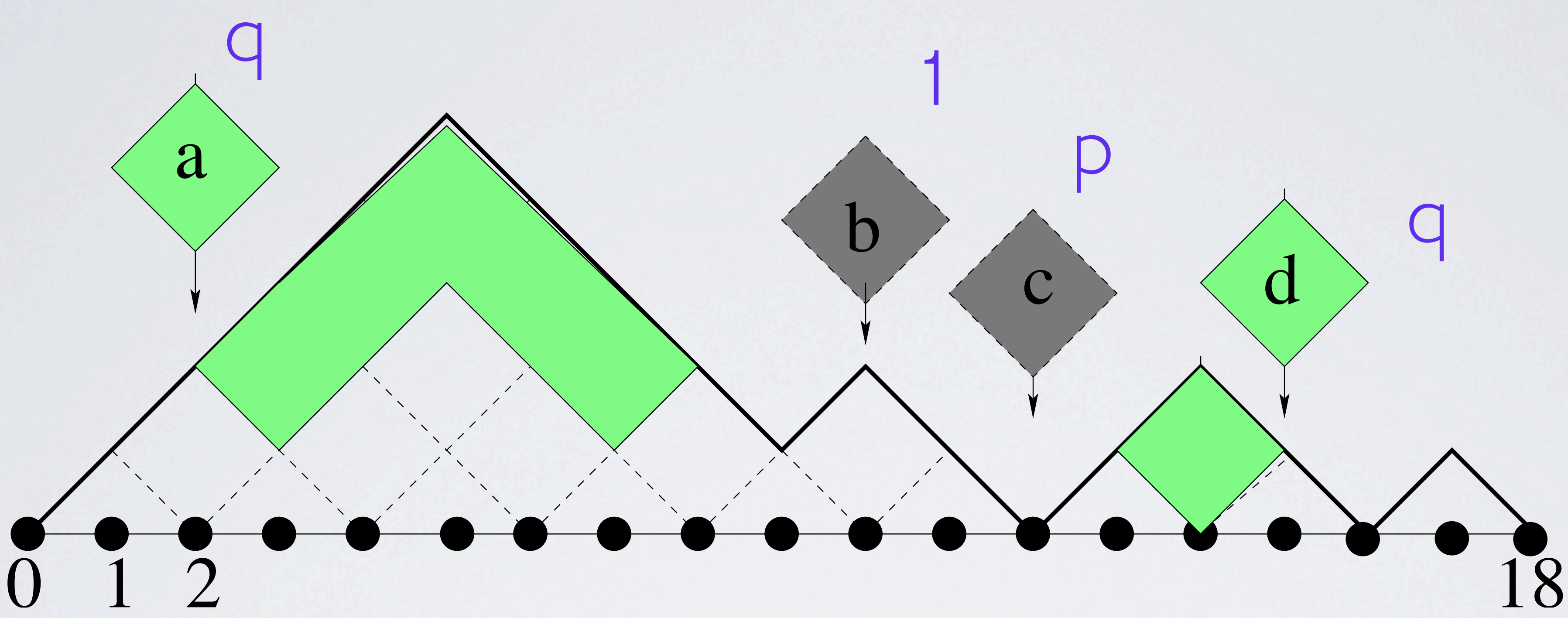
K =2 (critical)

The shared information is distinct and smaller:

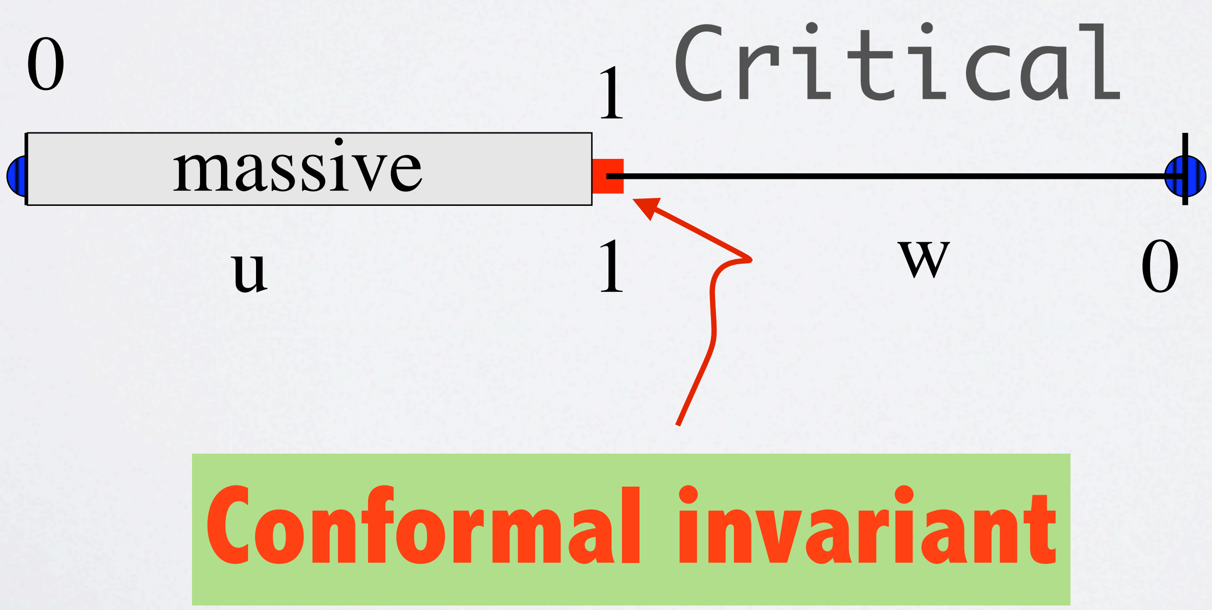
Separation entropy: $S(l,L) \sim 1/2 \ln(l(1 - l/L)) + C_S$

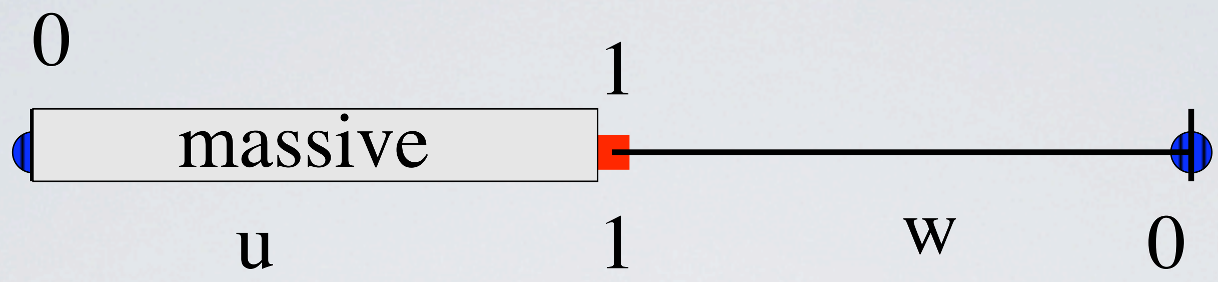
K >2 (non critical) The shared information is finite

Raise and peel model (de Gier, Nienhuis,Pearce,Rittenberg, 2003; FCA, Rittenberg, 2007 [review])

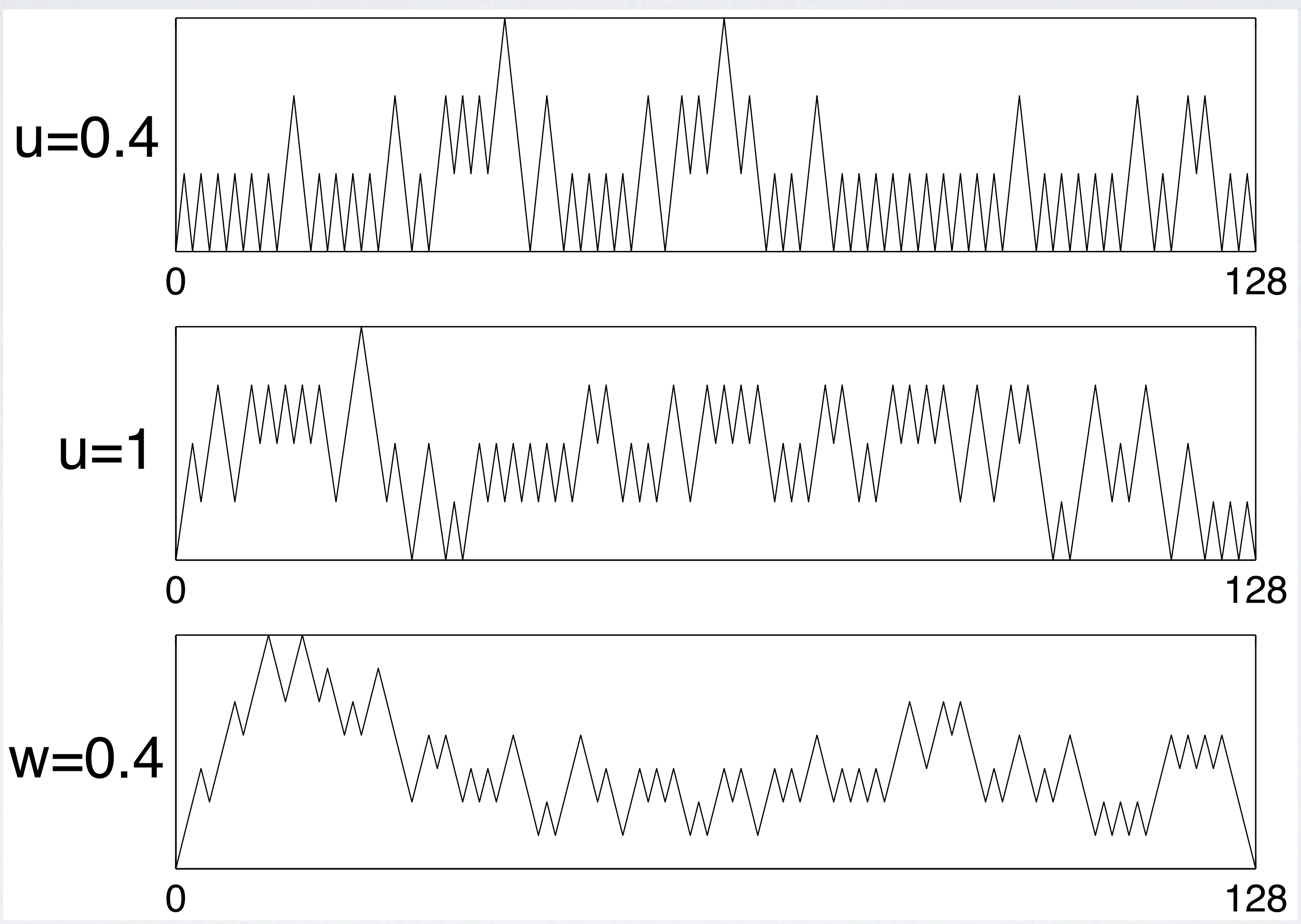


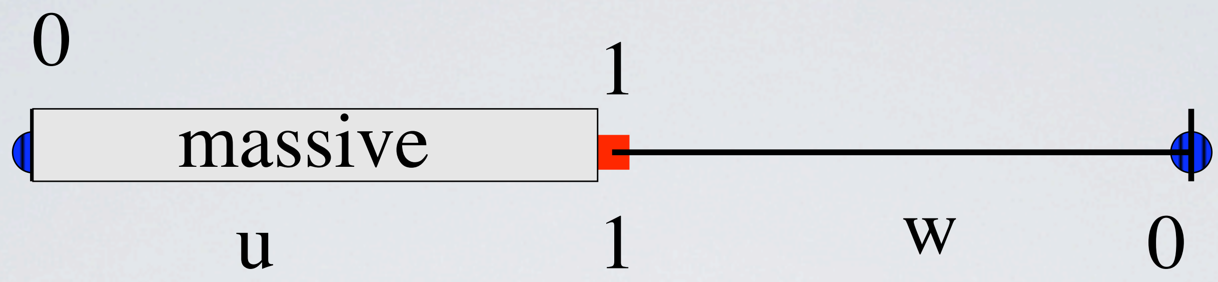
$u = p/q$





Typical configurations (L=128)





★ At $u = 1$ we have a conformally invariant stochastic model - central charge $c = 0$

H spin 1/2 XXZ quantum chain with $U_q(sl(2))$ symmetry with

$$\Delta = q + 1/q = 1/2, \quad q = \exp(i\pi/3)$$

The link patterns correspond to $U_q(sl(2))$ singlets

★ $u > 1$ Almost a single cluster.

Results

$0 < u < 1$

(non critical) density of clusters finite $\longrightarrow$ shared information finite

$u = 1$

$$E(l, L) \sim \gamma_E \ln \tilde{L}_C + C_E, \quad \tilde{L}_C = L \sin(\pi l/L)/\pi$$

$$E(l, L) \sim \gamma_E \ln l + C_E, \quad 1 << l << L$$

Similar as $S_{vN}(l, L)$ in the quantum case

Results $E(l,L) \sim \gamma_E \ln \tilde{L}_C + C_E, \quad \tilde{L}_C = L\sin(\pi l/L)/\pi$

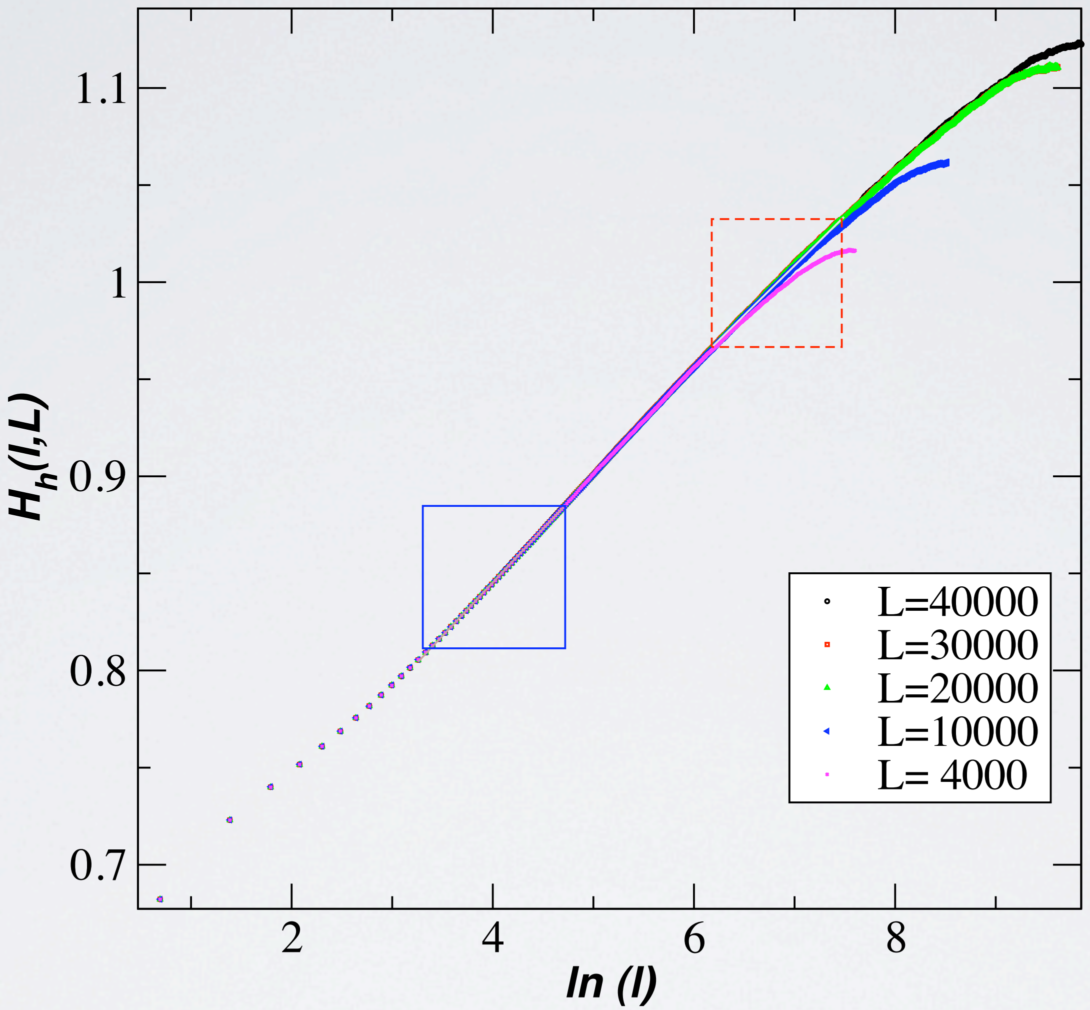
- Mutual information and separation Shannon entropy (not precise L=26)

  $\gamma_I = 0.07 \quad C_I = 0.65 \quad \gamma_S = 0.4 \quad C_S = 0.7$

- The other measures (precise)

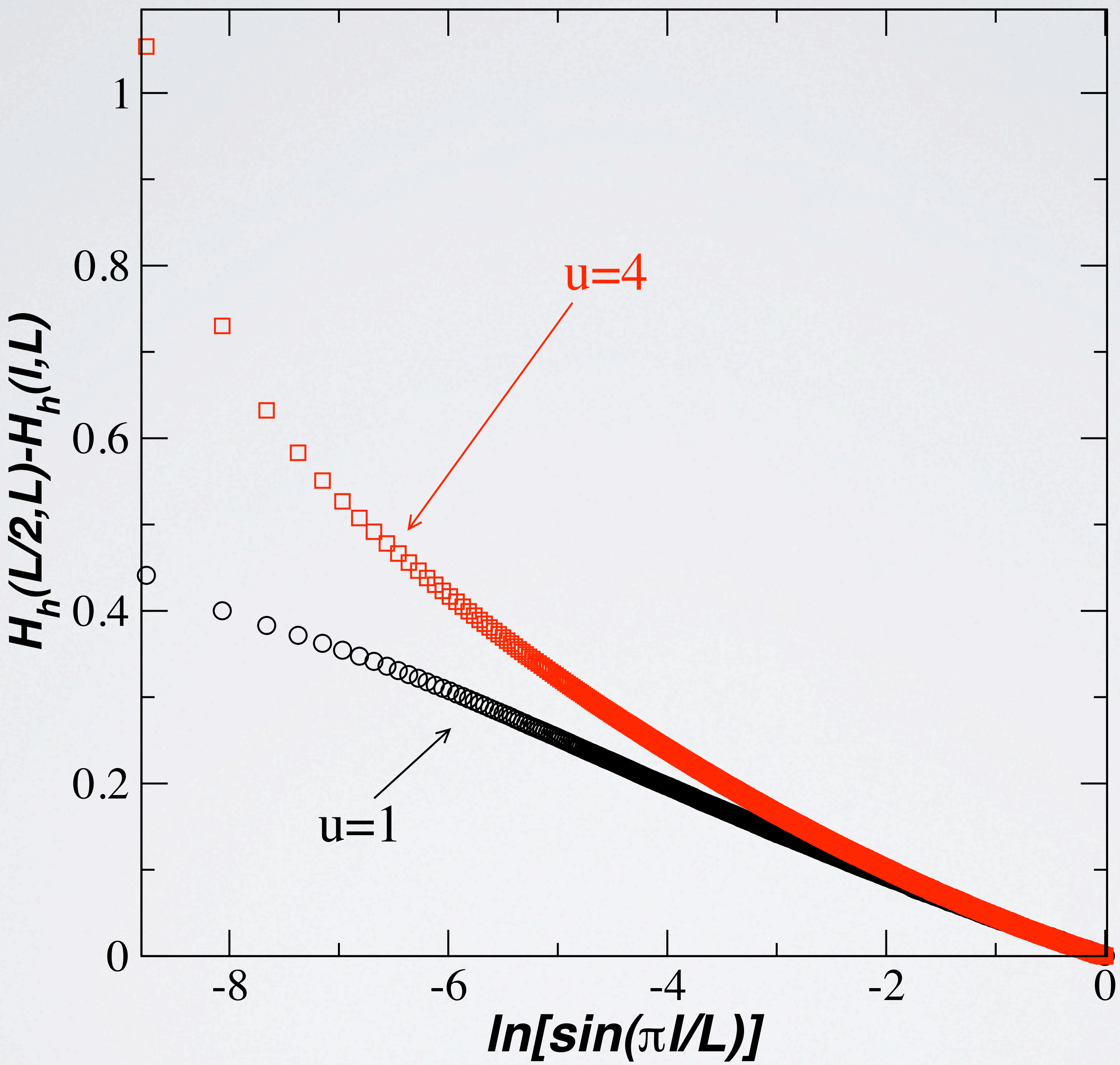
$P_l(h,l)$ Evaluated by MC up to L =40000

★ Interdependency $H_h(l,L) = -\sum_n P_l(h,L)\ln P_l(h,L)$



$$H_h(l,L) \sim 0.050 \ln l$$

$$H_h(l,L) - H(L/2,L) \sim 0.050\,\ln[\sin(\pi l/L)]$$



L=40000

★ Valence bond entanglement entropy

$$h(l,L)=\sum_h hP_l(h,L)$$

$$\gamma_h = 0.277 \quad C_h = 0.75$$

Related to a periodic model (Jacobsen ans Saleur, 2008)

$$\gamma_h = \sqrt{3}/2\pi \approx 0.275$$

Second moment of $P_l(h,l)$

$$\kappa_2(l,L) \sim \beta_2 \ln \tilde{L}_C + b_2 \qquad \beta_2 = 0.19, \quad b_2 = 0.25$$

$$\beta_2 = (2\pi\sqrt{3}-9)/\pi^2 \approx 0.190767$$

half of the value of a related periodic model

$\kappa_2(l,L)$ also a possible estimator

★ Density of contact points $D(l,L) = -\ln P_l(o,L)$

$$\rho(l,L) \sim \frac{\alpha}{\tilde{L}_C^{1/3}}, \quad \alpha = -\frac{\sqrt{3}\Gamma(-\frac{1}{6})}{6\pi^{5/6}}$$

(FCA, Pyatov and Rittenberg, 2007

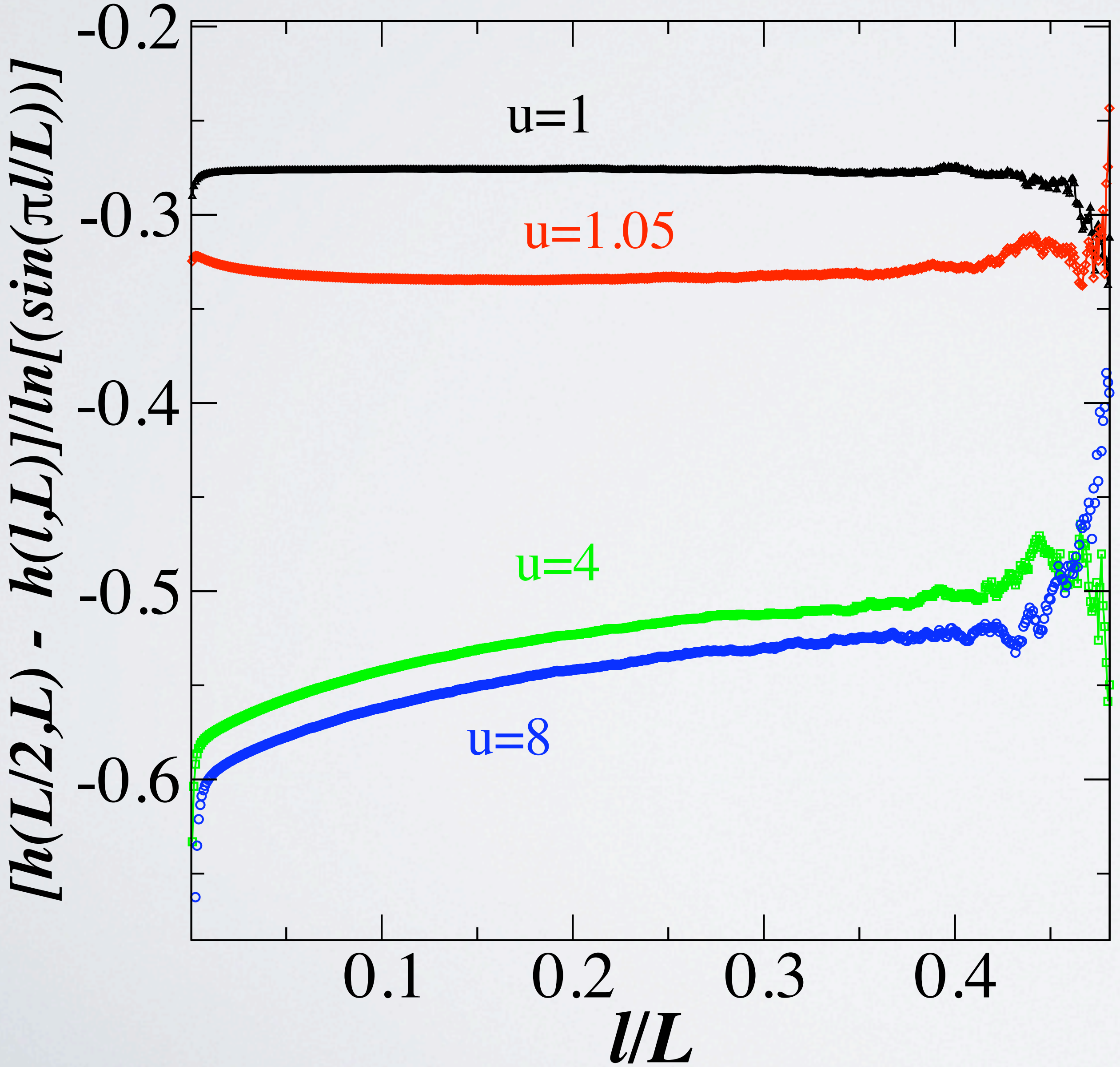
$$D(l,L) = -\ln\rho(l,L) \sim \frac{1}{3}\ln\tilde{L}_C + 0.28349$$

$u > 1$

Dynamical critical exponent $z < 1$

$$E(l, L) \sim \gamma_E \ln \tilde{L}_u + C_E$$

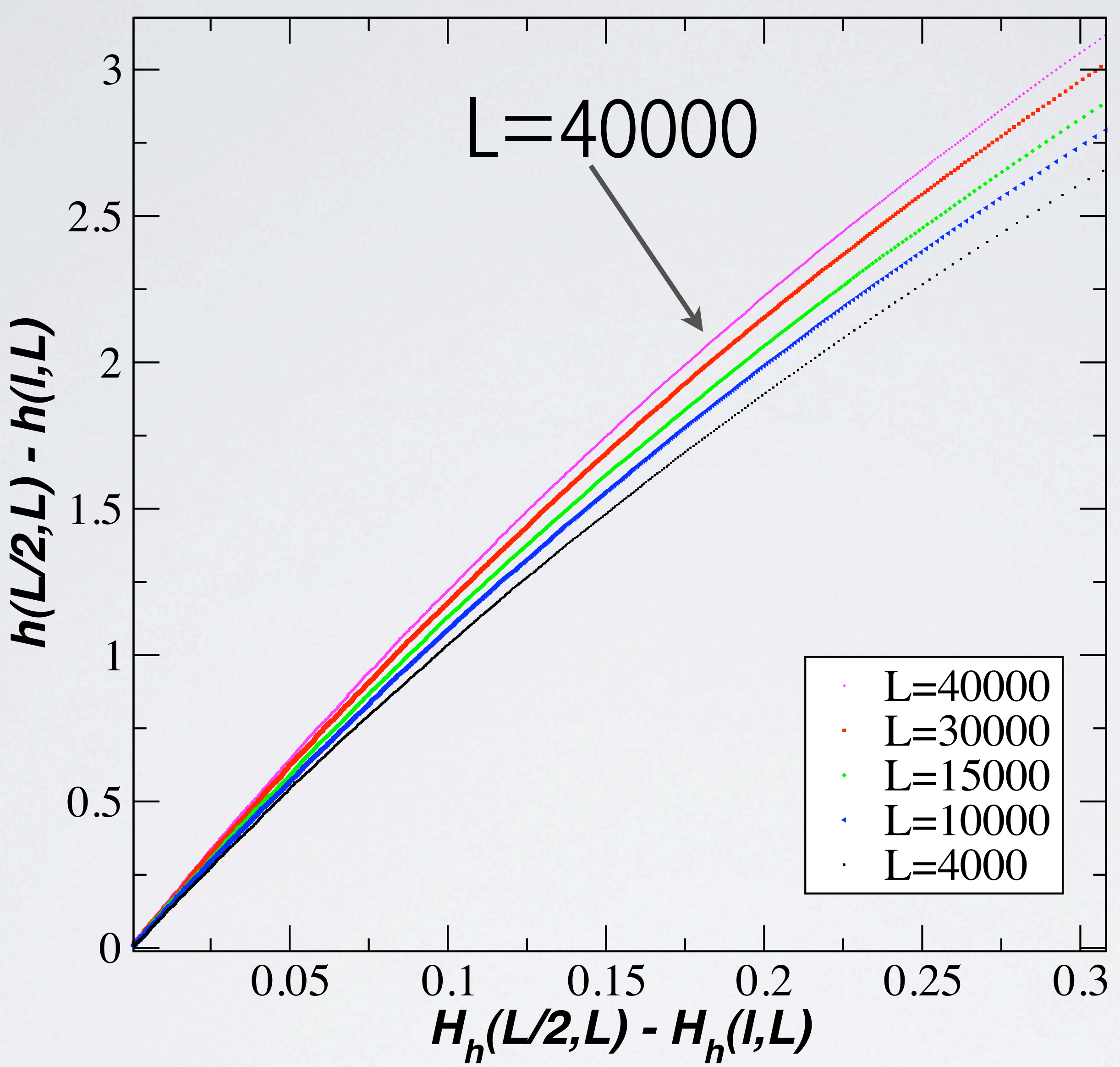
γ_E changes continuously with u



finite-size scaling $\tilde{L}_u(l, L)$ depends on u

All estimators share same $\tilde{L}_u$

Valence bond entropy



Interdependency

Summary of results RPM

	u=1 γ_E	u=1 C_E	u=4 γ_E	u=4 C_E
Mutual information	0.07	0.65	-	-
Interdependency	0.050	0.67	0.09	0.91
Rényi ($n = 2$)	0.05	0.39	0.06	0.09
Valence bond ent.	0.277	0.75	0.63	1.37
Dens. Contact points	0.333	0.284	0.73	0.71

As u increases the shared information increases

Parity effects on subleading contributions in RPM

Motivation: The quantum XXZ chain $-1 \leq \Delta \leq 1$

Affleck, Laflorencie, Sorensen, 2009; Calabrese, Campostrini, Essler, Nienhuis, 2010, Song Rachel, Le Hur, 2010

$$R_n(l, L) = \frac{1}{1-n} \ln \mathrm{Tr}(\rho_A^n)$$

$$\delta R_n(l, L) = R_n(l, L) - R_n(l+1, L) = f_n(l/L)/\tilde{L}_c^{K/n}$$

$$K = \pi/(2 \arccos(\Delta))$$

corrections induced by relevant conical singularites (Cardy, Calabrese 2010)

$$\delta R_n(l, L) = f_n(l/L)/\tilde{L}_c^{K/n}$$

Stochastic model (RPM)

$$\delta E(l,L) = E(l,L) - E(l+1,L)$$

$u = 1$

Interdependency

$$H_h(l,L) = -\sum_n P_l(h,L)\ln P_l(h,L)$$

$$\delta H = \delta R_1 \approx c_1/\tilde{L}_c^{x_1} \quad c_1 \approx 1.3 \quad x_1 = 0.58$$

$x_1 \neq K = 3/2$ $\qquad$ σ_z basis $\neq$ singlet basis

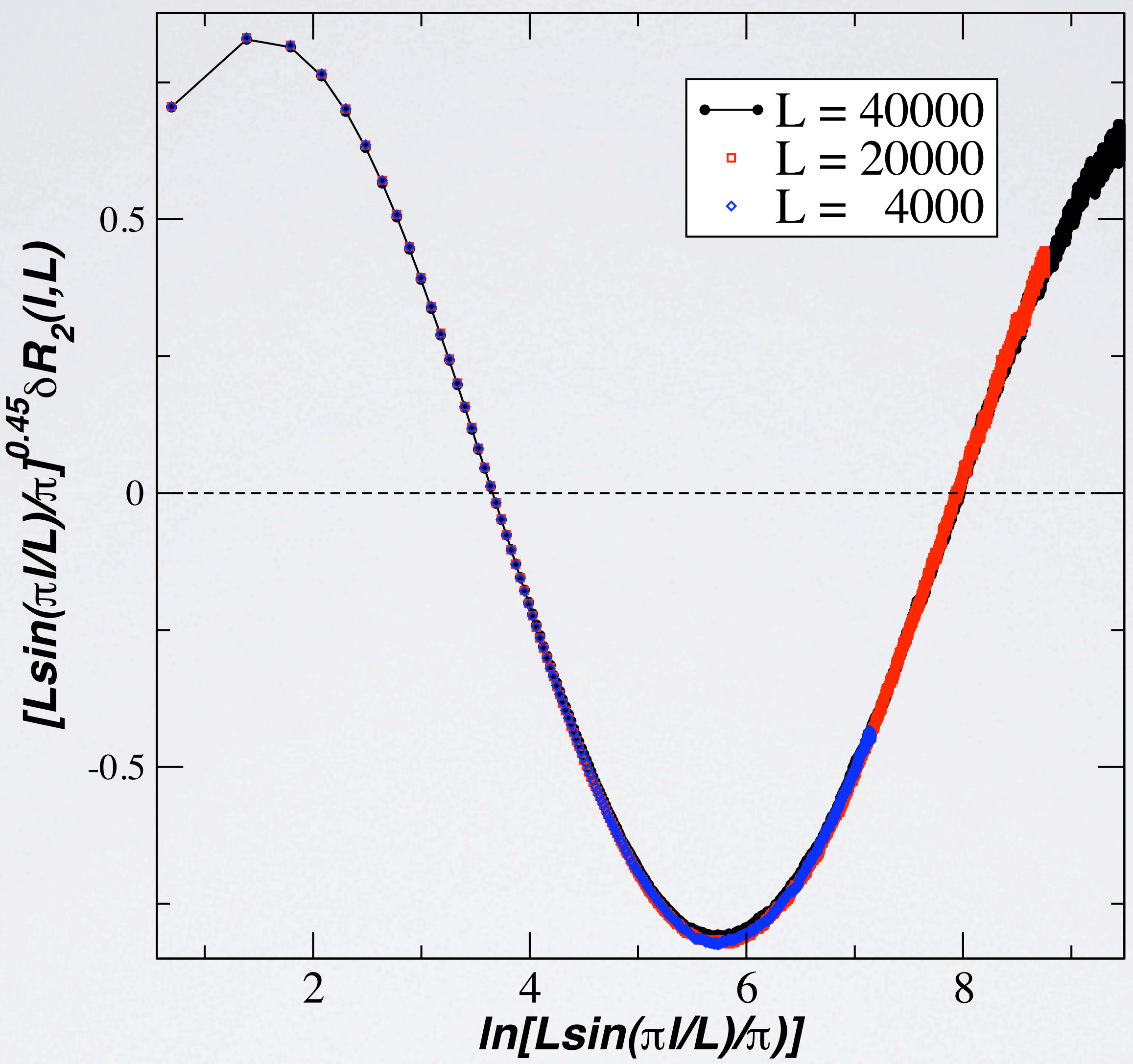
$$\delta R_n \sim 1/\tilde{L}_c^{x_n} F_n(\ln(\tilde{L}_c))$$

oscillating function !

$$x_2 \approx 0.45, \quad x_3 \approx 0.32$$

oscillating function (n=2) !

$$F_n = F_n(\ln(\tilde{L}_c))$$



Valence-bond entanglement

$$h(l,L) = \sum_h h P_l(h,L)$$

$$\delta h(l,L) = c_h/\tilde{L}_c^{x_h} \quad c_h \approx 0.50 \quad x_h \approx 0.99$$

No oscillations

Conclusions

- Estimators introduced for the shared information of subsystems in stationary states of one-dimensional Markov processes
- Some of the estimators can be evaluated by Monte Carlo simulation (distinct from the quantum case)
- Estimators with properties similar to the counterparts in the quantum case

a) Vanish if the subsytems are separated

b) If ξ finite $\longrightarrow$ are finite

c) If we have logarithmic behavior for $E(l,L) \sim \gamma_E \ln \tilde{L}_E + C_E$

γ_E is universal and C_E non universal

d)If we have power-law behavior $E(l,L) \sim \gamma_E L^{\delta_E} + D_E$

γ_E, δ_E universal and D_E non universal

- Analitical results for the exponents in the conformal case?

  $\tilde{L} = \tilde{L}_c(l, L)$ (the same as in the quantum cases)

  γ_h can be inferred from Jacobson, Saleur 2008

  γ_D related to the density of contact points

  γ_I γ_H γ_S no analytic results

- Parity effects show oscillatory effects, explanation?

- Raise and peell in $u > 1$ region (non conformal invariant)

  $\tilde{L}_u$ and $\gamma_E(u)$(analytical resullts?)

- It is possible to apply in general stochastic models

- The finite-size scaling of the estimator (simple to calculate) may detect conformal invariance