Homogenization and corrector for the wave equation

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Problem: We want to study the asymptotic behavior of the solutions of

$$\begin{cases} \partial_t(\rho_n(t,x)\partial_t u_n) - \mathrm{div}_x(A_n(t,x)\nabla_x u_n) = f_n \text{ in } (0,T)\times\Omega \\ u_n(t,x) = 0 \text{ on } (0,T)\times\partial\Omega \\ u_n(0,x) = u_n^0(x), \ \ \rho_n(t,x)\partial_t u_n(0,x) = \vartheta_n(x) \quad \text{in } \Omega, \end{cases}$$

where Ω is a bounded open set of $\mathbb{R}^N$, and the coefficients ρ_n, A_n are uniformly elliptic and bounded.

This problem has been studied by other authors. In order to recall their results (and to state our ones) we need to recall the classical result for elliptic problems.

Theorem (S. Spagnolo 1968, symmetric case,

F. Murat, L. Tartar 1977, general case)

$\Omega \subset \mathbb{R}^N$, open, bounded, $A_n \in L^\infty(\Omega; \mathbb{R}^{N\times N})$, with

$$A_n(x)\xi \cdot \xi \geq \alpha|\xi|^2,\ \ A_n(x)^{-1}\xi \cdot \xi \geq \gamma|\xi|^2,\ \text{a.e. } x \in \Omega, \forall \xi \in \mathbb{R}^N,$$

$\alpha, \gamma > 0$. Then (for a subsequence), $\exists A \in L^\infty(\Omega; \mathbb{R}^{N\times N})$, with

$$A(x)\xi \cdot \xi \geq \alpha|\xi|^2,\ \ A(x)^{-1}\xi \cdot \xi \geq \gamma|\xi|^2,\ \text{a.e. } x \in \Omega, \forall \xi \in \mathbb{R}^N,$$

such that if $f_n \to f$ in $H^{-1}(\Omega)$, u_n solution of

$$\begin{cases} -\mathrm{div}(A_n(x)\nabla u_n) = f_n & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

then
$$u_n \rightharpoonup u \quad \text{in } H_0^1(\Omega)$$
$$A_n \nabla u_n \rightharpoonup A\nabla u \quad \text{in } L^2(\Omega)^N$$

with u solution of
$$\begin{cases} -\mathrm{div}(A(x)\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We say that A_n H-converges to A, and write $A_n \overset{H}{\rightharpoonup} A$.

Remark: We have

$$\left.\begin{array}{r} u_n \rightharpoonup u \quad \text{in } H^1(\Omega) \\ \operatorname{div}(A_n \nabla u_n) \ \text{ compact in } H^{-1}(\Omega) \end{array}\right. \Longrightarrow A_n \nabla u_n \rightharpoonup A \nabla u \quad \text{in } L^2(\Omega)^N$$

$$A_n \overset{H}{\rightharpoonup} A \text{ in } \Omega \ \Longrightarrow A_n \overset{H}{\rightharpoonup} A \text{ in } \omega, \qquad \forall \omega \subset \Omega \text{ open.}$$

Corrector: (F. Murat, L. Tartar 1977)

If $A_n \overset{H}{\rightharpoonup} A$ in Ω, then $\exists P_n \in L^2(\Omega)^{N\times N}$ bounded such that

$$\left.\begin{array}{r} u_n \rightharpoonup u \ \text{ in } H^1(\Omega), \ \ u \text{ smooth} \\ \operatorname{div}(A_n \nabla u_n) \ \text{ compact in } H^{-1}(\Omega) \end{array}\right. \Longrightarrow \nabla u_n - P_n(x) \nabla u \to 0 \ \text{ in } L^2(\Omega)^N$$

The corrector is local. To construct the strong approximation $P_n(x)\nabla u(x)$ in a point x, we only need to know $\nabla u(x)$. Moreover P_n depends locally of Ω.

Return to the wave problem: Take $A_n(t,x)$ symmetric

$$A_n(t,x)\xi\cdot\xi \geq \alpha|\xi|^2,\ \ |A_n(x)| \leq \beta,\ \ \alpha \leq \rho_n(t,x) \leq \beta$$

$$\text{a.e. } (t,x) \in (0,T)\times\Omega, \qquad \forall\xi \in \mathbb{R}^N.$$

Assume $A_n(t,.) \overset{H}{\rightharpoonup} A(t,.)$ a.e. $t \in (0,T)$, $\rho_n \overset{*}{\rightharpoonup} \rho$ in $L^\infty\big((0,T)\times\Omega\big)$

$f_n \to f$ in $L^1(0,T;L^2(\Omega))$, $u_n^0 \rightharpoonup u^0$ in $H_0^1(\Omega)$, $\vartheta_n \rightharpoonup \vartheta$ in $L^2(\Omega)$.

Then, u_n solution of

$$(\mathcal{P}_n)\quad \begin{cases} \partial_t(\rho_n(t,x)\partial_t u_n) - \operatorname{div}_x(A_n(t,x)\nabla_x u_n) = f_n \ \text{ in } (0,T)\times\Omega \\ u_n(t,x) = 0 \ \text{ on } \ (0,T)\times\partial\Omega \\ u_n(0,x) = u_n^0(x),\ \ \rho_n(0,x)\partial_t u_n(0,x) = \vartheta_n(x) \quad \text{in } \Omega, \end{cases}$$

satisfies $u_n \overset{*}{\rightharpoonup} u$ in $L^\infty\left(0,T;H_0^1(\Omega)\right) \cap W^{1,\infty}\big(0,T;L^2(\Omega)\big)$ with

$$(\mathcal{P})\quad \begin{cases} \partial_t(\rho(t,x)\partial_t u) - \operatorname{div}_x(A(t,x)\nabla_x u) = f \ \text{ in } (0,T)\times\Omega \\ u(t,x) = 0 \ \text{ on } \ (0,T)\times\partial\Omega \\ u(0,x) = u^0(x),\ \ \rho(0,x)\partial_t u(0,x) = \vartheta(x) \quad \text{in } \Omega. \end{cases}$$

This resul was proved by:

F. Colombini, S. Spagnolo 1978:

$$\rho_n = 1, A_n \text{ bounded in } W^{1,\infty}(0,T;L^\infty(\Omega)^{N\times N})$$

S. Brahim-Otsmane, G. Francfort, F. Murat 1992:

$$\rho_n, A_n \text{ independent of the time variable } t.$$

Our aim: To consider ρ_n, A_n less smooth (possibly non continuous) in t.

Remark: Some smoothness is necessary in order to have existence (and uniqueness) of solution for $(\mathcal{P}_n)$.

A.E. Hurd, D.H. Sattinger 1968: Problem $(\mathcal{P}_n)$ has not solution in general for $\rho_n = 1, A_n$ constant in the two sides of an hyperplan not parallel to $t = 0$.

F. Colombini, S. Spagnolo 1989: Problem $(\mathcal{P}_n)$ has not solution in general for $\rho_n = 1, A_n \in C^{0,\alpha}([0,T]\times\overline{\Omega};\mathbb{R}^{N\times N}), \alpha < 1$.

We generalize the above homogenization result to the case

$$\rho_n \text{ bounded in } BV(0,T;L^\infty(\Omega))$$

$$A_n \text{ bounded in } BV\big(0,T;L^\infty(\Omega;\mathbb{R}^{N\times N})\big).$$

For the right-hand side f_n we just assume

$$f_n = f_n^1 + f_n^2,$$

with

$$f_n^1 \overset{*}{\rightharpoonup} f^1 \quad \text{in } \mathfrak{M}\big(0,T;L^2(\Omega)\big)$$

$$f_n^2 \overset{*}{\rightharpoonup} f^2 \quad \text{in } BV\big(0,T;H^{-1}(\Omega)\big)$$

$$\int_{t_1}^{t_2} f_n^2\,ds \to \int_{t_1}^{t_2} f^2\,ds \ \text{ in } H^{-1}(\Omega), \qquad 0 \le t_1 < t_2 \le T.$$

Remark: The existence and uniqueness of solution for $(\mathcal{P}_n)$ is due to

A. Arosio 1984 for $\rho_n = 1$ (L. De Simon, G. Torelli 1974 $f_n = 0$). Our little contribution to the existence result was to consider ρ_n in $BV\big(0,T;L^\infty(\Omega)\big)$ and uniformly elliptic.

Indeed these result hold in an abstract setting where $L^2(\Omega)$ and $H_0^1(\Omega)$ are replaced by abstract Hilbert spaces H, V,

V continuously imbedded in H.

Idea of the proof of the homogenization result.

For $t \in (0,T),\ h \in (t, T-t)$ the function

$$\bar{u}_n(x) = \int_t^{t+h} u_n(s,x)\,ds$$

satisfies the elliptic problem

$$-\mathrm{div}(A_n(t,x)\nabla\bar{u}_n(x))$$

$$= \rho_n(t+h,x)\partial_t u_n(t+h,x) - \rho_n(t,x)\partial_t u_n(t,x) + \int_t^{t+h} f_n(s)ds$$

$$+\mathrm{div}\left(\int_t^{t+h}\big(A_n(s,x)-A_n(t,x)\big)\nabla u_n(s,x)ds\right) \quad \text{in } \Omega$$

where in the right-hand side, for t outside a countable set, the last term is small for h small and the other terms converge strongly in $H^{-1}(\Omega)$.

Remark: (a first corrector result) If u is smooth enough, we have

$$\int_{t_1}^{t_2}\nabla u_n(s,x)ds - \int_{t_1}^{t_2} P_n(s,x)\nabla u(s,x)ds \to 0 \;\; in\ L^2(\Omega)^N,$$

for $0 \le t_1 < t_2 \le T$.

Here, for $s \in (0,T)$, $P_n(s,x)$ is the sequence of matrices giving the elliptic corrector for $A_n(s,x)$.

Question: Is the elliptic corrector a corrector for the wave problem?

Theorem (S. Brahim-Otsmane, G. Francfort, F. Murat 1992):

Assume ρ_n, A_n independent of t, $A_n \overset{H}{\rightharpoonup} A$, $\rho_n \overset{*}{\rightharpoonup} \rho$ in $L^\infty(\Omega)$,

$f_n = f \in L^2\big((0,T)\times\Omega\big)$, $u_n^0 \rightharpoonup u^0$ in $H_0^1(\Omega)$, $\vartheta_n \rightharpoonup \vartheta$ in $L^2(\Omega)$.

We define u_n, u by

$$(\mathcal{P}_n)\quad \begin{cases} \partial_t(\rho_n(x)\partial_t u_n) - \mathrm{div}_x(A_n(x)\nabla_x u_n) = f \text{ in } (0,T)\times\Omega \\ u_n(t,x) = 0 \text{ on } (0,T)\times\partial\Omega \\ u_n(0,x) = u_n^0(x),\ \ \rho_n(t,x)\partial_t u_n(0,x) = \vartheta_n(x) \quad \text{in } \Omega, \end{cases}$$

$$(\mathcal{P})\quad \begin{cases} \partial_t(\rho(x)\partial_t u) - \mathrm{div}(A(x)\nabla u) = f \text{ in } (0,T)\times\Omega \\ u(t,x) = 0 \text{ on } (0,T)\times\partial\Omega \\ u(0,x) = u^0(x),\ \ \rho(t,x)\partial_t u(0,x) = \vartheta(x) \quad \text{in } \Omega. \end{cases}$$

We introduce $\hat{u}_n, \tilde{u}_n \in L^\infty\left(0,T;H_0^1(\Omega)\right)$ as the solutions of

$$\begin{cases} -\mathrm{div}\big(A_n(x)\nabla\hat{u}_n(t,x)\big) = -\mathrm{div}\big(A(x)\nabla u(t,x)\big) \text{ in } \Omega \\ \hat{u}_n(t,x) = 0 \text{ on } \partial\Omega. \end{cases} \quad t \in [0,T]$$

$$\begin{cases} \partial_t(\rho_n(x)\partial_t\tilde{u}_n) - \mathrm{div}(A_n(x)\nabla\tilde{u}_n) = 0 \text{ in } (0,T)\times\Omega \\ \tilde{u}_n(t,x) = 0 \text{ on } (0,T)\times\partial\Omega \\ \tilde{u}_n(0,x) = u_n^0(x) - \hat{u}_n(t,x), \;\; \rho_n(x)\partial_t\tilde{u}_n = \vartheta_n(x) - \dfrac{\rho_n(x)}{\rho(x)}\vartheta(x) \quad \text{in } \Omega. \end{cases}$$

Then $$\begin{aligned} &\nabla u_n - \nabla\hat{u}_n - \nabla\tilde{u}_n \to 0 \text{ in } L^\infty(0,T;L^2(\Omega)^N) \\ &\partial_t u_n - \partial_t u - \partial_t\tilde{u}_n \to 0 \text{ in } L^\infty\big(0,T;L^2(\Omega)\big). \end{aligned}$$

Remark: If u is smooth enough, the first equality can be written as

$$\nabla u_n - P_n(x)\nabla u - \nabla\tilde{u}_n \to 0 \text{ in } L^\infty(0,T;L^2(\Omega)^N).$$

$\tilde{u}_n$ only depends on the initial data

It only gives a *true* corrector if $\tilde{u}_n \to 0$ in $L^2\left(0,T;H_0^1(\Omega)\right)$.

Then $$\begin{aligned} &\nabla u_n - \nabla\hat{u}_n \to 0 \text{ in } L^\infty(0,T;L^2(\Omega)^N) \\ &\partial_t u_n - \partial_t u \to 0 \text{ in } L^\infty\big(0,T;L^2(\Omega)\big). \end{aligned}$$

The convergence $\tilde{u}_n \to 0$ in $L^2\left(0,T;H_0^1(\Omega)\right)$

only holds if the initial data is *well prepared*, in the following sense

$$-\mathrm{div}\big(A_n(x)\nabla u_n^0(x)\big) \;\; \text{compact in } H^{-1}(\Omega)$$

$$\frac{\vartheta_n}{\rho_n} - \frac{\vartheta}{\rho} \to 0 \;\text{ in } L^2(\Omega)$$

(second condition is equivalent to $\partial_t u_n(0) - \partial_t u(0) \to 0$ in $L^2(\Omega)$)

In another case, the elliptic corrector is not a corrector for ∇u_n and $\partial_t u_n$

does not converges strongly to $\partial_t u$.

Since

$$\int_{t_1}^{t_2} \big(\nabla u_n(s,x) - \nabla \hat{u}_n(s,x)\big) ds \to 0 \;\text{ in } L^2(\Omega)^N$$

this means that in general (although the coefficients do not depend on t) ∇u_n, $\partial_t u$ oscillate in t, which cannot be seen just approximating u_n by the solution of an elliptic problem.

We have generalized this result for coefficients depending on t.

We need a lot more of regularity than in the homogenization result.

Namely: We asume $A_n \in C^1([0,T];L^\infty(\Omega)^{N\times N})$, $\rho_n \in C^1\big([0,T];L^\infty(\Omega)\big)$

with the equicontinuity property

$$\lim_{h\to 0}\max_{|t_2-t_1|\leq h}\|\partial_t A_n(t_2,.)-\partial_t A_n(t_1,.)\|_{L^\infty(\Omega)^{N\times N}}=0$$

$$\lim_{h\to 0}\max_{|t_2-t_1|\leq h}\|\partial_t \rho_n(t_2,.)-\partial_t \rho_n(t_1,.)\|_{L^\infty(\Omega)}=0,$$

uniformly in n.

For the right-hand side, we just assume $f_n = f_n^1 + f_n^2$ with

$$f_n^1 \rightharpoonup f^1 \ \text{ in } L^1\big(0,T;L^2(\Omega)\big), \qquad \lim_{h\to 0}\sup_n \int_0^{T-h}\|f_n^1(t+h,.)-f_n^1(t,.)\|_{L^2(\Omega)}=0$$

$$f_n^2 \to f^2 \ \text{ in } W^{1,1}\big(0,T;H^{-1}(\Omega)\big).$$

Idea of the proof:

Assume that the coefficients and the right-hand side are sufficiently smooth in the time variable and that the initial data satisfies (in particular it is *well prepared*)

$$-\mathrm{div}(A_n(0,x)\nabla u_n^0) \text{ bounded in } L^2(\Omega)^N$$

$$\frac{\vartheta_n(x)}{\rho_n(0,x)} \text{ bounded in } H_0^1(\Omega).$$

Then, we can derive in $(\mathcal{P}_n)$ with respect to t, to obtain

$$\partial_t u_n \text{ bounded in } L^\infty\left(0,T;H_0^1(\Omega)\right)$$

$$\partial_{tt}^2 u_n \text{ bounded in } L^\infty\left(0,T;L^2(\Omega)\right)$$

and then $\partial_t u_n$ is compact in $L^\infty\left(0,T;L^2(\Omega)\right)$.

Theorem: The above corrector result does not hold if we just assume

$$A_n \text{ bounded in } C^1([0,T]; L^\infty(\Omega)^{N\times N})$$

$$\rho_n \text{ bounded in } C^1\big([0,T]; L^\infty(\Omega)\big).$$

Counterexample

$$\begin{cases} \partial_{tt}^2 u_n - \partial_x\left(\left(1+\dfrac{1}{n}\sin(nt)\cos(nx)\cos x\right)\partial_x u_n\right) = 0 \text{ in } (0,T)\times(0,\pi) \\ u_n(t,0) = u_n(t,\pi) = 0, \text{ a.e. } t\in(0,T) \\ u_n(0,x) = 0, \ \ \partial_t u_n(0,x) = \sin x\,, \ \text{ a.e. } x\in(0,\pi). \end{cases}$$

Here $A_n(t,x) = 1 + \frac{1}{n}\sin(nt)\cos(nx)\cos x$ is bounded in $C^1([0,T]\times\overline{\Omega})$

and converges uniformly to 1

$$\partial_x(A_n(0,x)\partial_x u_n^0) = 0, \quad \frac{\vartheta_n}{\rho_n} = \sin x$$

Thus $u_n \overset{*}{\rightharpoonup} u = \sin t \sin x$ in $L^\infty\left(0,T;H_0^1(0,\pi)\right) \cap W^{1,\infty}\left(0,T;L^2(0,\pi)\right)$

solution of
$$\begin{cases} \partial_{tt}^2 u - \partial_{xx}^2 u = 0 \text{ in } (0,T)\times(0,\pi) \\ u(t,0) = u(t,\pi) = 0, \text{ a.e. } t \in (0,T) \\ u(0,x) = 0, \ \ \partial_t u(0,x) = \sin x\,, \ \ \text{a.e. } x \in (0,\pi). \end{cases}$$

However

$$\partial_x u_n \nrightarrow \partial_x u, \ \ \partial_t u_n \nrightarrow \partial_t u \quad \text{in } L^2\big((0,T)\times(0,\pi)\big).$$

Periodic case

We consider the case $\Omega = \mathbb{R}^N$.

$$\begin{cases} \partial_t\big(\rho_\varepsilon(t,x)\partial_t u_\varepsilon\big) - \mathrm{div_x}\big(A_\varepsilon(t,x)\nabla_x u_\varepsilon\big) = f_\varepsilon \ \text{ in } (0,T) \times \mathbb{R}^N \\ \quad u_\varepsilon(0,x) = u_\varepsilon^0(x), \ \ \rho_\varepsilon(t,x)\partial_t u_\varepsilon(0,x) = \vartheta_\varepsilon(x) \quad \text{in } \mathbb{R}^N, \end{cases}$$

$$\rho_\varepsilon \equiv \rho_0\left(\frac{x}{\varepsilon}\right) + \varepsilon\rho_1\left(t,x,\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right), \qquad A_\varepsilon \equiv A_0\left(\frac{x}{\varepsilon}\right) + \varepsilon A_1\left(t,x,\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right)$$

$$f_\varepsilon \equiv f\left(t,x,\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right), \ \ u_\varepsilon^0 \equiv u^0(x) + \varepsilon u^1\left(x,\frac{x}{\varepsilon}\right), \ \ \vartheta_\varepsilon \equiv \vartheta\left(x,\frac{x}{\varepsilon}\right).$$

The functions are periodic of period $Y = (0,1)^N$ in x/ε

and Bohr almost-periodic in t/ε.

The right-hand side and the initial data are small at infinity.

We define ($\mathcal{M}$ denotes mean)

$$\bar{\rho} = \mathcal{M}_y(\rho_0(y)),$$

$$A^h\xi = \mathcal{M}_y\left(A_0(y)\left(\xi + \nabla w_\xi(y)\right)\right), \qquad \forall\xi \in \mathbb{R}^N, \text{ with}$$

$$-\mathrm{div}\left(A_0(y)\left(\xi + \nabla w_\xi(y)\right)\right) = 0 \text{ in } \mathbb{R}^N, \qquad w_\xi \text{ periodic}$$

$$\bar{f}(t,x) = \mathcal{M}_{y,s}\big(f(t,x,s,y)\big), \quad \bar{\vartheta}(x) = \mathcal{M}_y(\vartheta(x,y)).$$

Then, $u_\varepsilon \overset{*}{\rightharpoonup} u_0$ in $L^\infty\big(0,T;H^1(\mathbb{R}^N)\big) \cap W^{1,\infty}\big(0,T;L^2(\mathbb{R}^N)\big)$, solution of

$$\begin{cases} \bar{\rho}\partial_{tt}^2 u_0 - \mathrm{div}(A^h\nabla u_0) = \bar{f} \text{ in } (0,T) \times \mathbb{R}^N \\ u_0(0,x) = u^0(x), \ \bar{\rho}\partial_t u_0(0,x) = \bar{\vartheta}(x) \quad \text{in } \mathbb{R}^N, \end{cases}$$

Moreover, if u_0 is smooth enough, we have

$$u_\varepsilon - u_0 - \varepsilon u_1\left(t,x,\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right) \to 0 \text{ in } H^1(\mathbb{R}^N \times (0,T))$$

The function $u_1 = u_1(x,t,s,y)$ satisfies

$$\rho_0(y)\partial_{ss}^2 u_1 - \mathrm{div}_\mathrm{y}\left(A_0(y)\left(\nabla_x u_0 + \nabla_y u_1\right)\right) = 0 \;,$$

and it is periodic with respect to x and Besicovitch almost-periodic with respect to s.

It recalls the equation satisfied by the classical corrector in periodic homogenization. It does not suffices to determine u_1. Indeed, we have

$$u_1(x,t,s,y) = \hat{u}_1(x,t,y) + \tilde{u}_1(x,t,s,y)$$

With $\hat{u}_1$ the elliptic corrector, solution of

$$-\mathrm{div}_\mathrm{y}\left(A_0(y)\left(\nabla_x u_0 + \nabla_y \hat{u}_1\right)\right) = 0$$

and $\tilde{u}_1$ satisfying $\rho_0(y)\partial_{ss}^2\tilde{u}_1 - \mathrm{div}_\mathrm{y}\left(A_0(y)\nabla_y \tilde{u}_1\right) = 0$

which has a large number of solutions.

Namely, denoting by $\lambda_0 = 0 < \lambda_1 < \lambda_2 < \cdots$, the eigenvalues of

$$-\mathrm{div_y}\big(A_0(y)\nabla_y \Phi\big) = \lambda_i \rho_0(y)\Phi(y) \text{ in } \mathbb{R}^N$$

with periodic boundary conditions, we have

$$\tilde{u}_1(x,t,s,y) = \sum_{i=0}^{\infty}\left(\Phi_i(x,t,y)\cos\left(\sqrt{\lambda_i}s\right) + \psi_i(x,t,y)\sin\left(\sqrt{\lambda_i}s\right)\right)$$

With Φ_i, ψ_i eigenfunctions with respect to y.

It is necessary to add another equation to determine $\tilde{u}_1$ and then u_1.
Formally (it can be justified using two-scale convergence), it is obtained from the asymptotic expansion

$$u_\varepsilon(t,x) = u_0(t,x) + \varepsilon u_1\left(t,x,\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(t,x,\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right) + \cdots$$

We have:

$$2\rho_0(y)\partial^2_{ts}u_1 - \mathrm{div}_x\big(A_0(y)\nabla_y u_1\big) - \mathrm{div}_y\left(A_0(y)\nabla_x u_1\right)$$

$$+\partial_s\big(\rho_1(t,x,s,y)(\partial_t u_0 + \partial_s u_1)\big) - \mathrm{div}_y\left(A_1(y)\big(\nabla_x u_0 + \nabla_y u_1\big)\right)$$

$$+\rho_0(y)\partial^2_{ss}u_2 - \mathrm{div}_y\big(A_0(y)\nabla_y u_2\big) = f - \bar{f}$$

together to the initial conditions

$$u_1(0,x,0,y) - u^1(x,y) \text{ independent of } y$$

$$\rho_0(y)\big(\partial_t u_0(0,x) + \partial_s u_1(0,x,0,y)\big) = \vartheta(x,y)$$

Remark: The equation contains simultaneously the microscopic and macroscopic variables. The function u_1 depends non-locally of u_0.

Some particular cases:

$$\begin{cases} \partial_t\left(\rho_0\left(\frac{x}{\varepsilon}\right)\partial_t u_\varepsilon\right) - \mathrm{div_x}\left(A_0\left(\frac{x}{\varepsilon}\right)\nabla_x u_\varepsilon\right) = f\left(t,x,\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right) \ \text{ in } (0,T)\times\mathbb{R}^N \\ u_\varepsilon(0,x) = u_\varepsilon^0(x),\ \ \rho_\varepsilon(t,x)\partial_t u_\varepsilon(0,x) = \vartheta_\varepsilon(x) \ \text{ in } \mathbb{R}^N, \end{cases}$$

$$u_\varepsilon^0 \equiv u^0(x) + \varepsilon u^1\left(x,\frac{x}{\varepsilon}\right),\ \ \vartheta_\varepsilon \equiv \vartheta\left(x,\frac{x}{\varepsilon}\right).$$

Then $u_\varepsilon \sim u_0 + \varepsilon\hat{u}_1\left(t,x,\frac{x}{\varepsilon}\right) + \varepsilon\tilde{u}_1\left(t,x,\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right)$, with

$$\rho_0(y)\partial_{ss}^2\tilde{u}_1 - \mathrm{div_y}\left(A_0(y)\nabla_y\tilde{u}_1\right) = 0$$

$$2\rho_0(y)\partial_{ts}^2\tilde{u}_1 - \mathrm{div}_x\left(A_0(y)\nabla_y\tilde{u}_1\right) - \mathrm{div}_y\left(A_0(y)\nabla_x\tilde{u}_1\right)$$

$$+\rho_0(y)\partial_{ss}^2 u_2 - \mathrm{div}_y\left(A_0(y)\nabla_y u_2\right) = f - \bar{f}$$

$$\hat{u}_1(0,x,y) + \tilde{u}_1(0,x,0,y) - u^1(x,y) \text{ independent of } y$$

$$\rho_0(y)\left(\partial_t u_0(0,x) + \partial_s\tilde{u}_1(0,x,0,y)\right) = \vartheta(x,y)$$

It is coherent with Brahim-Otsmane, Francfort, Murat result

Example: A constant, a, b smooth with compact support.

$$\begin{cases} \partial_{tt}^2 u_\varepsilon - \mathrm{div}_x(A\nabla_x u_\varepsilon) = 0 \ \text{ in } (0,T)\times\mathbb{R}^N \\ u_\varepsilon(0,x) = \varepsilon a(x) e^{\frac{2\pi i k\cdot x}{\varepsilon}},\ u_\varepsilon(0,x) = b(x) e^{\frac{2\pi i k\cdot x}{\varepsilon}} \ \text{ in } \mathbb{R}^N, \end{cases}$$

Then $u_\varepsilon(t,x) \sim$

$$\frac{\varepsilon}{2}\left(a\left(x - \frac{Ak}{\sqrt{Ak\cdot k}}t\right) + \frac{1}{2\pi i\sqrt{Ak\cdot k}} b\left(x - \frac{Ak}{\sqrt{Ak\cdot k}}t\right)\right) e^{\frac{2\pi i k\cdot x + \sqrt{Ak\cdot k}\,t}{\varepsilon}}$$

$$+\frac{\varepsilon}{2}\left(a\left(x + \frac{Ak}{\sqrt{Ak\cdot k}}t\right) - \frac{1}{2\pi i\sqrt{Ak\cdot k}} b\left(x + \frac{Ak}{\sqrt{Ak\cdot k}}t\right)\right) e^{\frac{2\pi i k\cdot x - \sqrt{Ak\cdot k}\,t}{\varepsilon}}$$

It was obtained by G. Francfort, F. Murat 1992 using geometrical optics.

Related results A. Bensoussan, J. Lions, G. Papanicolau 1978.