Non-Conventional Optic Response in Bilayered Arrays with Metamaterials

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1. Introduction

1.1. Anderson Localization

As known, the transport via any 1D disordered structure obeys the ***Anderson localization***. Its principal concept is ***single-parameter scaling***: all transport characteristics depend only on the ratio L/L_{loc} between the system length L and the single scaling parameter L_{loc}, which is called ***localization length*** (LL). Such a ***universal dependence*** manifests itself, for example, in the self-averaging logarithm of the transmittance,

$$\langle \ln T \rangle = -2L/L_{loc}\,.$$

Thus, the ***localization length*** is the ***key quantity*** that controls the transport in 1D geometry**.** That is why the ***knowledge of*** L_{loc} ***is crucial in the study of 1D disordered systems***.

1.2. Localization Length in Optics

The LL is determined by specific properties of a structure and by nature of a disorder. For continuous optical media with any kind of weak disorder, its inverse value, or the Lyapunov exponent (LE), reads

$$L_{loc}^{-1}(k) = \lambda = k^2\sigma^2 K(2k)/8\varepsilon_0^2(\omega)$$

The limit law : the LE obeys the quadratic frequency dependence at the bottom of the spectrum

$$L_{loc}^{-1} = \lambda \propto \omega^2, \qquad \omega \to 0.$$

This law remains valid for wide class of discrete, periodic-on-average, arrays composed of bilayer* (*RH-RH, LH-LH, or RH-LH*) *unit cells with weak positional or compositional uncorrelated disorder.

1.3. Suppression of Localization in Disordered Metamaterials

To great surprise, recently in the articles

A.A. Asatryan, L.C. Botten, M.A. Byrne, V.D. Freilikher, S.A. Gredeskul, I.V. Shadrivov, R.C. McPhedran, Y.S. Kivshar, Phys. Rev. Lett. **99**, 193902 (2007); Phys. Rev. B **81**, 075124 (2010).

it was ***numerically*** shown that in ***array of matched combination of two alternating RH and LH layers***, the LL changes dramatically, displaying ***enormously fast divergence*** as the wave frequency vanishes.

***The mysterious limit law*:** $L_{loc}^{-1} \propto \lambda \propto \omega^{6} \div \omega^{8.78}, \qquad \omega \to 0$!!!

The power increases with total number of layers N , the latter reaches for $N \approx 10^{12}$.

The model under consideration was a kind of invisible system:

(1) The ***unperturbed*** unit cell consists of two ***matched*** slabs (free space and ideal metamaterial),

$$\varepsilon_{a,b} = \pm 1, \quad \mu_{a,b} = \pm 1, \quad n_{a,b} = \pm 1, \quad Z_{a,b} = 1.$$

(2) The layers have the same thicknesses – ***balanced structure***, $d_a = d_b$.

(3) The dielectric permittivity is perturbed by uncorrelated disorder – ***white-noise compositional disorder***.

In this report we explain the origin and give the analytical description of such abnormal localization.

Two systems are compared*:** the ***homogeneous stack when both slabs composing the unit cell, are made of RH optic materials, and ***mixed stack*** with alternating RH and LH layers.

We show that the predicted ***effect emerges in a very specific structures*** (***not only due to LH inclusions***). It originates from non-uniform distribution of wave-phase, and cannot be found in the standard second-order perturbation theory.

2. Bilayer Array

We consider a ***1D dielectric array of two alternating* (RH-RH or RH-LH) *layers***,

$$(a_1,b_1),\ldots(a_n,b_n),\ldots(a_N,b_N)\,.$$

Every a_n and b_n ***layer*** is specified by its thickness $d_{a,b}$, the dielectric permittivity $\varepsilon_{a,b}$, magnetic permeability $\mu_{a,b}$, refractive index $n_{a,b}$, impedance $Z_{a,b}$ and wave number $k_{a,b}$,

$$n_{a,b}=\sqrt{\varepsilon_{a,b}\mu_{a,b}}\,,\qquad Z_{a,b}=\mu_{a,b}/n_{a,b}\,,\qquad k_{a,b}=\omega n_{a,b}/c\,.$$

Without disorder, or on average, the stack of bilayers is periodic with the period $d=d_a+d_b$.

Inside the two basic slabs of every unit (a,b) cell, the electric field obeys the ***Helmholtz equation with two boundary conditions*** at the interfaces between slabs,

$$E''_{a,b}(x)+k^2_{a,b}E_{a,b}(x)=0\,,\qquad E_a(x_i)=E_b(x_i)\,,\qquad \mu_a^{-1}E'_a(x_i)=\mu_b^{-1}E'_b(x_i)\,.$$

Its solution gives rise to the ***recurrent relations for normalized electric*** Q_n ***and magnetic*** P_n ***fields*** at the left-hand edges of successive nth and $(n+1)$th unit (a,b) cells,

$$Q_{n+1}=A_nQ_n+B_nP_n\,,\qquad P_{n+1}=-C_nQ_n+D_nP_n\,.$$

Remarkably: ***these relations can be treated as the Hamiltonian map of trajectories in the phase space (Q,P) with discrete time*** n ***for a linear oscillator with time-dependent parametric force.***

3. Hamiltonian Map Approach

It is conventional to pass from coordinate Q_n and momentum P_n to polar coordinates R_n, θ_n ,

$$Q_n = R_n \cos\theta_n \,, \qquad P_n = R_n \sin\theta_n \,.$$

The ***Hamiltonian map*** in the ***radius-angle presentation*** gets the form

$$\left(\frac{R_{n+1}}{R_n}\right)^{-2} = \frac{d\theta_{n+1}}{d\theta_n}, \qquad \theta_{n+1} = \arctan\left(\frac{-C_n + D_n \tan\theta_n}{A_n + B_n \tan\theta_n}\right)$$

The localization length L_{loc} (LL) is derived according to its definition via the Lyapunov exponent λ (LE),

$$\frac{d}{L_{loc}} \equiv \lambda \equiv \frac{1}{2}\langle \ln\left(\frac{R_{n+1}}{R_n}\right)^2 \rangle = -\frac{1}{2}\langle \ln\frac{d\theta_{n+1}}{d\theta_n}\rangle .$$

Remarkably: ***the*** $\theta -$ ***map is the unique necessary equation to be treated.***

The random factors

$$A_n = \cos\tilde{\varphi}_a \cos\tilde{\varphi}_b - Z_a^{-1} Z_b \sin\tilde{\varphi}_a \sin\tilde{\varphi}_b \qquad B_n = Z_a \sin\tilde{\varphi}_a \cos\tilde{\varphi}_b + Z_b \cos\tilde{\varphi}_a \sin\tilde{\varphi}_b$$

$$C_n = Z_a^{-1} \sin\tilde{\varphi}_a \cos\tilde{\varphi}_b + Z_b^{-1} \cos\tilde{\varphi}_a \sin\tilde{\varphi}_b \qquad D_n = \cos\tilde{\varphi}_a \cos\tilde{\varphi}_b - Z_a Z_b^{-1} \sin\tilde{\varphi}_a \sin\tilde{\varphi}_b$$

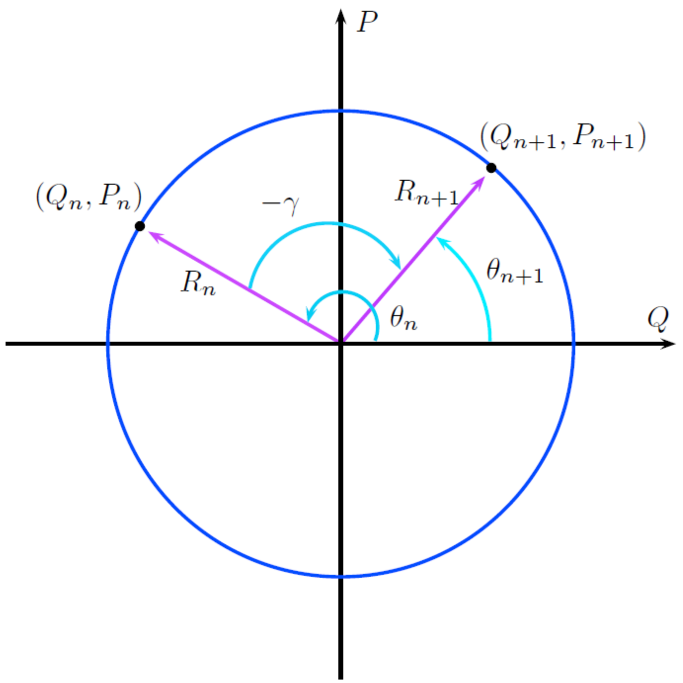
depend, in general, on the ***cell index*** n due to randomized phase shifts $\tilde{\varphi}_{a,b} = k_{a,b} d_{a,b}$ and impedances.

4. Unperturbed Structure: Quarter Stack with Matched Slabs

$$n_a d_a = |n_b d_b| \rightarrow \varphi_a = \pm\varphi_b; \qquad Z_a = Z_b.$$

Without disorder, the factors A, B, C, D do not depend on the cell index n. As a result, the trajectory in phase space (Q,P) obeys the relations

$$R_{n+1} = R_n, \qquad \theta_{n+1} = \theta_n - \gamma, \qquad \gamma = \varphi_a + \varphi_b.$$



Here γ is the wave-phase shift over a unit (a,b) cell (***Bloch phase***)

Homogeneous RH-RH array : $\varphi_a = \varphi_b = \varphi$

The Bloch phase is non-zero, $\gamma = 2\varphi$.
The unperturbed map is the ***circle*** with uniform phase shift –
the trajectory conserves its radius, while its phase θ ***changes by the Bloch phase in one step of "time"*** *n*.

Mixed RH-LH stack : $\varphi_a = -\varphi_b = \varphi$

The phase shift gained in any RH *a*-layer is canceled by the subsequent negative shift in the next LH *b*-layer.
The Bloch phase vanishes after passing every unit (*a*,*b*) cell, $\gamma = 0$!!! -- a kind of invisible system !!!
The unperturbed trajectory degenerates into a ***single point*** !!!

Remarkably: ***Even in zero order in disorder the topology of the problem drastically depends on the type of structure* (conventional or metamaterial) *. This defines the specific properties of the LL.***

5. White-Noise Compositional Disorder

Consider a case of ***weak*** disorder incorporated into the ***refractive indices of both basic slabs***,

$$n_{a,b}(n) = n_{a,b}[1+\eta_{a,b}(n)], \qquad \sigma^2 << 1, \qquad (\sigma\varphi)^2 << 1, \qquad \langle \eta_a(n)\eta_b(n')\rangle = \sigma^2 \delta_{ab}\delta_{nn'}.$$

Hamiltonian map within quadratic approximation

$$\theta_{n+1} - \theta_n = -\gamma - \eta_a(n)U(\theta_n) - \eta_b(n)U(\theta_n - \gamma/2) - \sigma^2 W(\theta_n).$$

Via reducing this map to the stochastic Ito equation, we come to the stationary ***Fokker-Plank equation***,

$$\frac{d^2}{d\theta^2}[U^2(\theta) + U^2(\theta-\gamma/2)]\rho(\theta) + 2\frac{d}{d\theta}[\frac{\gamma}{\sigma^2} + W(\theta)]\rho(\theta) = 0.$$

It should be complemented by the ***normalization condition*** and the ***condition of periodicity***.

Homogeneous RH-RH array **(** $\gamma = 2\varphi$ **).** – The term containing γ/σ^2 prevails over the others under the weak-disorder conditions: we arrive at the uniform phase distribution and the standard result for the LL,

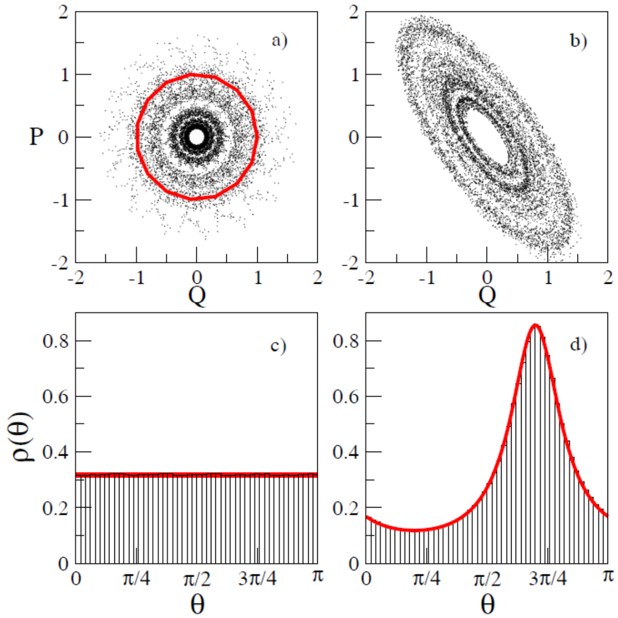
$$\rho(\theta) = 1/\pi \quad \rightarrow \quad d/L_{loc} \equiv \lambda = \sigma^2 \sin^2\varphi \quad \propto \sigma^2\omega^2 \quad \text{for} \;\; \varphi << 1.$$

Mixed RH-LH stack **(** $\gamma = 0$ **).** – The solution of the Fokker-Plank equation is highly ***non-uniform***,

$$\rho(\theta) = \sqrt{\varphi^2 - \sin\varphi^2} \Big/ \pi[\varphi + \sin\varphi\cos(2\theta-\varphi)] \quad \rightarrow \quad d/L_{loc} \equiv \lambda = \sigma^2 \times 0.$$

Bad Luck: ***The LL diverges – the standard second-order perturbation theory fails*** **!!!**
One has to develop the approach in the next, fourth-order, approximation.

5.1. Fourth-order Approximation



Left Figs show that for weak disorder the type of ***exact*** Hamiltonian map is univocally associated with the kind of phase distribution:

The circle ($\gamma = 2\varphi$) results in the uniform distribution.

The nonuniform distribution ($\gamma = 0$) corresponds to the ellipse.

If the random ellipse is transformed into the circle, the phase distribution for new $\theta -$ map should be expected as uniform.

The exact expressions for the Hamiltonian map is invariant with respect to such a ***canonical rotation-rescaling transformation.***

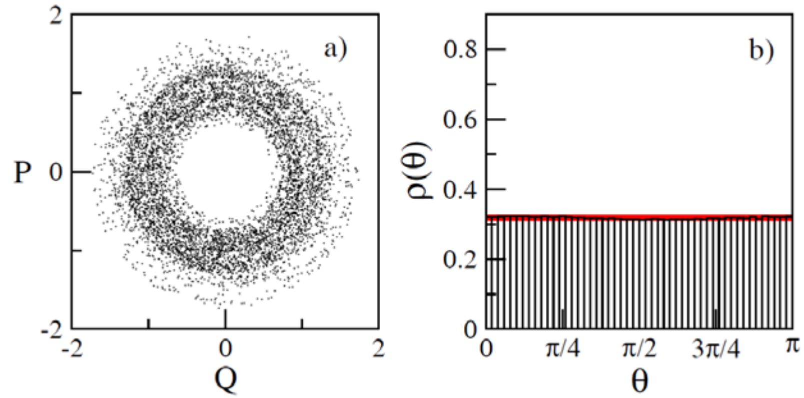
Only the random factors A, B, C, D are changed.

We have managed to realize this idea.

Right Figs confirm the excellent applicability of the method.

The corresponding LE reads

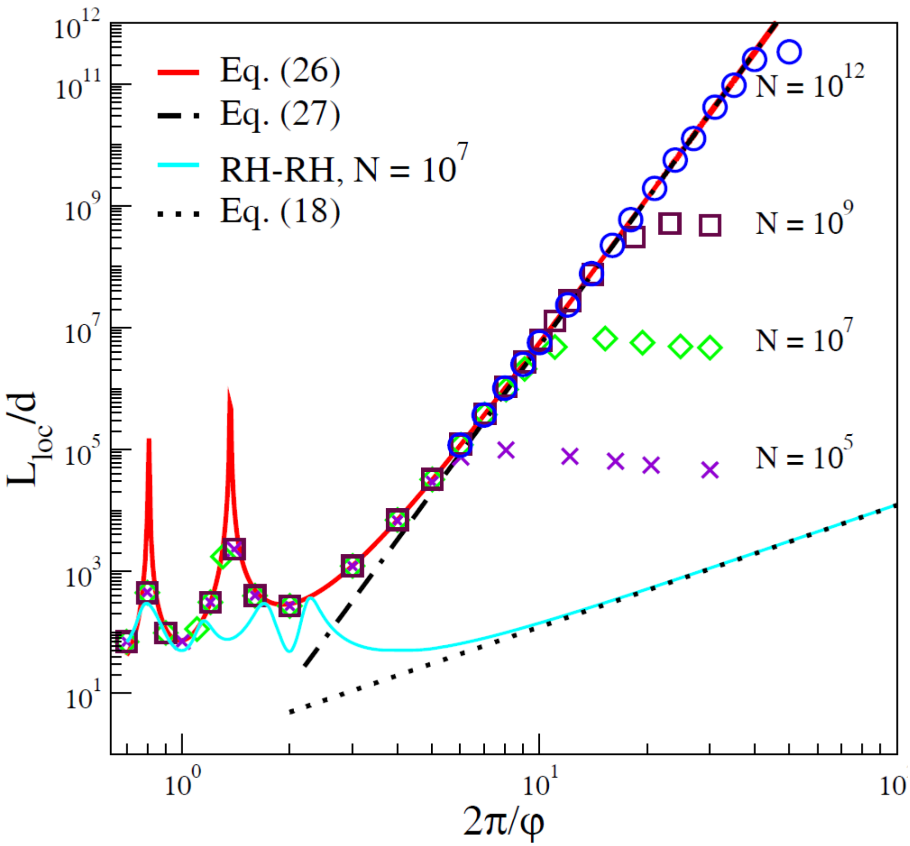
$$\frac{d}{L_{loc}} \equiv \lambda = \frac{\sigma^4}{5} \frac{[(2\varphi^2 - \sin^2\varphi)\cos\varphi - \varphi\sin\varphi]^2}{\varphi^2 - \sin^2\varphi}.$$

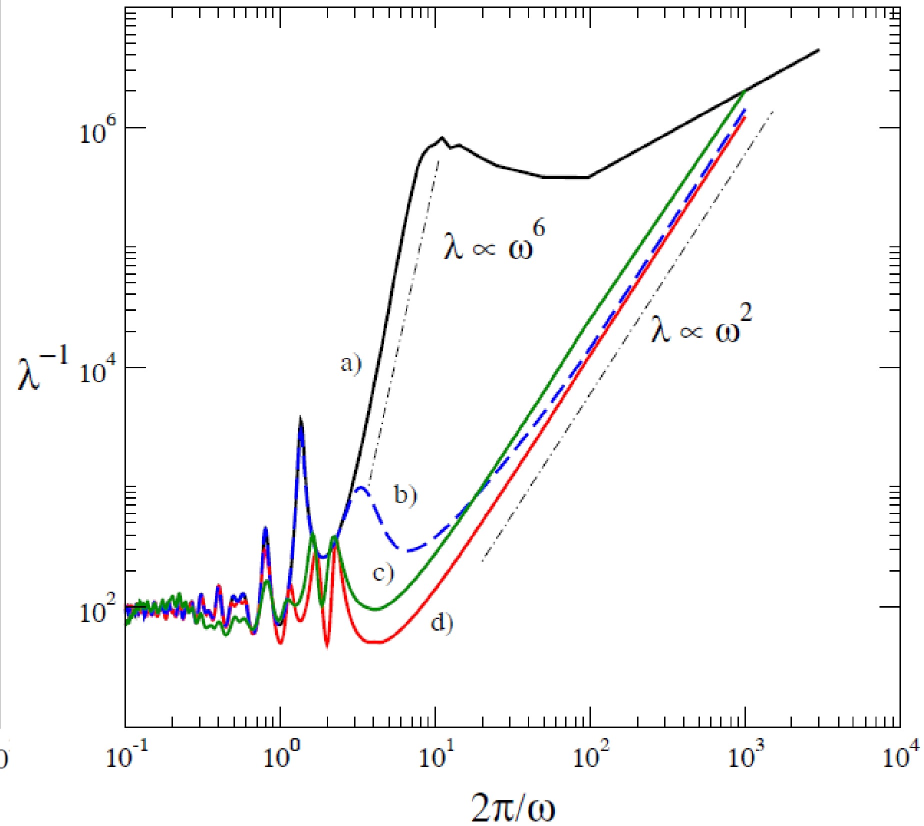


Good Luck: ***The LL provides quite surprising octal frequency dependence*** **!!!**

$$d / L_{loc} \equiv \lambda \propto \sigma^4 \omega^8 \quad \text{for} \quad \varphi << 1.$$

It is in contradiction with numerically found $\lambda \propto \omega^6$ that should be regarded as the intermediate one.





Left Fig displays a perfect agreement between the LL obtained numerically and that from analytical results.

5.2. From Octal to Quadratic Frequency Dependence

The ***crossover*** is governed by the ratio between relative thickness detuning and disorder variance:

$$L_{loc}^{-1} \equiv \lambda \propto \omega^2 \quad \text{for} \;\; \sigma^2 << |d_a - d_b|/d\,, \qquad L_{loc}^{-1} \equiv \lambda \propto \omega^8 \quad \text{for} \;\; |d_a - d_b|/d << \sigma^2 .$$

Right Fig clearly displays how the frequency dependence of the LL changes when the thicknesses of two basic layers differ from each other. The curves correspond:

$$(a): \, d_a = d_b, \;\; (b): \, d_a = 0.99 d_b, \;\; (c): \, d_a = 0.1 d_b \, .$$

6. Summary

❖ We have resolved the problem of Anderson localization in a 1D array of two alternating a and b layers with equal unperturbed impedances and optical widths. Two possible kinds of the system are realized: the ***homogeneous stack*** (both basic slabs are made from RH materials), and the ***mixed stack*** (alternating RH and LH layers). The uncorrelated weak disorder is incorporated into refractive indices (***compositional disorder***).

❖ We have shown that the inverse LL of ***homogeneous matched quarter stack*** is given by the simple expression and obeys the conventional ***quadratic frequency dependence*** at the bottom of its spectrum.

❖ However, the ***mixed matched quarter stack*** (**invisible structure!!!**) exhibits highly non-trivial properties originated from the ***nonuniform distribution of wave phase***. In particular, ***the LL diverges in the standard second-order perturbation theory***.

❖ We have developed a ***new method*** allowing us to construct the perturbation theory within the fourth order in disorder.

❖ We have derived the ***exact analytical expression*** for the LL in fourth-order approximation. For vanishing frequency it gives quite unexpected ***octal frequency dependence***.

❖ The crossover from octal to quadratic frequency dependence is governed by the ratio between the detuning of optical widths and the disorder variance. Therefore, for weakly disordered thicknesses of slabs the standard quadratic dependence recovers.

❖ Our numerical data manifest excellent agreement with the analytical results.