RESOURCE PAPER IN TOMOGRAPHY

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Abstract

This resource paper provides a presentation of tomographic methods and a guide to the literature concerned with tomography both in classical and quantum physics. In this work, the aims and methods of tomography are clearly identified and presented. Also comments on the variety of applications that these methods have nowadays are given. (Applications include Material Sciences, Medicine and Neutrino Tomography of Vesuvius.) A historical account is given to explain the origins of the approach in its classical setting ,starting with Radon Transform up to the modern setting within quantum physics. It is known that tomography has evolved into an alternate picture of quantum mechanics, therefore it may be applied in all fields of physics (most notably quantum optics, quantum computation and quantum information and also quantum cosmology). The probabilistic contents of the quantum tomography makes it specially useful for statistical methods and for the comparison and the transition from quantum mechanics to classical mechanics. The tomographic picture of quantum mechanics is compared with the standard picture in terms of Hilbert spaces and with the algebraic approach arised from Heisenberg picture. - E

There exist different formulations of both classical mechanics and quantum mechanics.

In classical mechanics, the Newton equations play an important role as well as Hamiltonian and Lagrangian formalism. An alternative geometrical formalism in classical mechanics is the other important aspect of presentation of classical mechanics.

In quantum mechanics, there exist also different formulations like the standard Schrödinger representation with the wave function evolution equation, the Heisenberg picture, the Feynmann path integral formulation, as well as the Moyal formulation of quantum mechanics in a classical-like form.

In the second part of the last century, the attempts appeared to find formulations of both classical and quantum mechanics which are similar and provide the possibility to see in clear form the classical-quantum relation. The mathematical basis of such attempts called the tomographic picture of quantum (also classical) mechanics is the application of integral Radon transform. This transform is used in any tomographic procedure, e.g., in medical tomography. The last decades, tomography of large scale objects like the Vesuvio vulcan, the Globo itself, oil geophysical lakes, etc. are under discussion. This tomography, instead of electromagnetic beams of tomography used in medicine, plan to employ the neutrino beams. The generalization of the Radon transform appropriate for tomography of media were suggested recently.

The tomographic approach was also applied to cosmological problems. In fact, in quantum cosmology the notion of the Universe state can be given in terms of the tomographic probability distribution which is an alternative to the wave function of the Universe. Within the probability framework, the classical picture of the cosmological processes and quantum picture of the cosmological processes can be treated in view of a unified formalism. We point out that the tomographic-probability representation of quantum mechanics makes more clear the phase-space representation of quantum states. In fact, since the measurable tomographic-probability distribution is considered as a primary notion of quantum states containing the complete information on the state, the experiment to measure the photon states do not need the procedure of quasidistribution reconstruction since all physical characteristics are extracted from the optical tomograms.

The **aim** of this lecture is to present a review and literature sources on new formalism of classical and mainly quantum mechanics where the probability distributions play a primary role in the description of both classical and quantum states.

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Historical review of the tomographic probability representation of quantum states

The notion of state for a classical physical system has been developed during centuries of observations and experiments. It is commonly accepted as a quite natural and clear notion due to every-day experience with all surrounding events. First of all, for a system, say for a particle, it is assumed that position q in space and velocity \dot{q} at the given position are enough to fully characterize a state. If the particle possesses a mass m, the velocity of the massive particle provides the particle with a momentum $p = m\dot{q}$. Of course, intuitively we do not even question that the position in space can change with time t, an this gives rise to the velocity \dot{q} . So, the concept of space and position q in it, the notion of time tproviding the evolution of the particle's positions, i.e., the velocity \dot{q} , and the notion of momentum p are standard and intuitively accepted characteristics which we understand as characteristics of the particle state in a classical description.

The Kepler and Newton laws of the particle motion after some accurate discussion provide us with the understanding that the classical particle state can be identified with two numerical characteristics – position q and momentum p – identifying a point in the phase space, and the evolution of the state in time is simply a trajectory in the phase space, i.e., q(t) and p(t) as functions of time. These functions, called the motion of the particle, can be obtained by solving Newton laws. The situation changes if one considers the notion of state for a particle not moving in empty space (say a vacuum) but in a medium where the constituents of the medium collide with the particle chaotically, changing its position and velocity many times. In view of this, the particle can be found, in fact, in an interval $(\Delta q, \Delta p)$ around the point (q, p) in the phase space, and different points (q, p) in the phase space, in principle, do not behave in the same manner for the particle. Intervals around different points carry different densities, encoded in a function f(q, p). This function is the probability density function, which is nonnegative and normalized, i.e., its integral taken all over phase space is one.

Thus, to introduce an intuitively clear concept of state in the presence of position and velocity (momentum) fluctuations, we consider the notion of probability density f(q, p) in phase space. This is a basic concept of the state in classical statistical mechanics. So in the classical picture, to identify the state of particle, we use either two numbers q and p when we disregard fluctuations in position and momentum, this corresponds to classical mechanics, or we use a probability distribution in the phase space, say f(q, p), when we want to take into account fluctuations, this corresponds to classical statistical mechanics.

The notion of state changes drastically in quantum mechanics. Instead of the discussed nonnegative probability distributions used in the classical domain, a complex wave function ψ defined on the configuration space and time, is used to describe a quantum state. Thus, the state of quantum particle in the standard formulation of quantum mechanics is associated with a complex wave function $|\psi(x)|e^{i\phi(x)}$ where the modulus squared of the function has an accepted physical interpretation inherited from experiments.

Namely, $|\psi(x)|^2$ is interpreted as a probability density, which determines the probability to find the particle in an interval Δx around the point x, but the phase factor $e^{i\phi(x)}$ in the wave function has not an intuitively clear physical meaning, even though it is considered relevant for the description of interference phenomena. In any case, this factor is needed to properly describe a quantum state. There exist quite different states with the same modulus of the wave function and different phase factors which can be distinguished experimentally. Also it is worth noting that the wave function $\psi(x)$ cannot be as a whole associated with some physical property. It is used as a formal mathematical tool to calculate physical observable values, which can be measured in the experiments, like the particle energy or the particle momentum.

In quantum statistical mechanics, the notion of state was again generalized. The notion of density operator $\hat{\rho}$ or density matrix $\rho(x,x')$ has been introduced by Landau and von Neumann. The density matrix is, in general, a function of two variables, and only the diagonal $\rho(x,x)$ of the density matrix has an intuitively clear interpretation as a probability density of the quantum-particle position. Off-diagonal elements do not have an intuitively clear interpretation inherited from the laboratory experience.

From the very beginning of quantum mechanics, there were attempts to find alternative formulations for describing the particle quantum state with a probability distribution similar to the classical situation. For a pure quantum state described by the wave function $\psi(x)$, the density matrix $\rho_{\psi}(x, x') = \psi(x)\psi^*(x')$ contains the relevant information on the state as contained in the wave function. Only a position-independent phase factor can be added to the wave function, that does not change the expectation values of physical observables. In fact, the correspondence $\psi(x) \rightarrow \rho_{\psi}(x, x')$ may be considered as a "change of variables," and it is invertible only modulo the mentioned phase factor. The quadratic dependence of ρ_{ψ} on $\psi(x)$ requires now additional care to be able to deal with the description of interference phenomena.

In 1932, Wigner suggested another change of variables where the wave function $\psi(x)$ is associated with the so-called Wigner function W(q,p). This change of variables has the form of Fourier transform of the density matrix $\rho(x, x')$ and reads

$$W(q,p) = \int \rho \left(q + u/2, q - u/2 \right) e^{-ipu} du.$$
 (1)

We set here the Planck's constant $\hbar = 1$ in (1). The Fourier transform is invertible, i.e.,

$$\rho(x, x') = \frac{1}{2\pi} \int W\left(\frac{x+x'}{2}, p\right) e^{ip(x-x')} dp.$$
 (2)

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Thus, the information in the quantum state encoded by the density matrix is the same as the information contained in the Wigner function. The aim for the introduction of the Wigner function W(q, p) was exactly to bring the notion of quantum state closer to the classical notion. This was only partially realized since the Wigner function has some properties very similar to the properties of the classical probability distribution f(q, p), but not all of them. In fact, this function is real (not complex!). The marginals obtained from the Wigner function provide the probability distributions of the positions and momenta, respectively, exactly as the marginals obtained from the classical probability densities. Also calculations of all the highest moments of the particle's position or all the highest moments of the particle' momentum are calculated using the Wigner function exactly as in the corresponding statistical characteristics of the classical particle obtained by means of the probability distribution on the phase space.

The Wigner function is normalized, and the normalization condition looks exactly as the normalization condition for the classical probability distribution (modulo factor 2π). The behavior of the Wigner function mimicks the behavior of the classical probability distribution, but there is one important difference. While the classical probability distribution cannot take negative values, the Wigner function can take also negative values and must take them in generic situation due to the indetermination relations. In view of this, the Wigner function is not a probability distribution and it is called quasiprobability distribution or simply quasidistribution. Later on, other quasidistributions were introduced for quantum states like Husimi–Kano Q-function and Sudarshan $\varphi(z), z = q + ip)$ or Glauber P-function. All these functions are functions of q and p. They are related one to the other by integral transform with different Gaussian kernels. They are not probability distributions of the position and momentum. The existence of such joint probability distributions is forbidden by the uncertainty relations which follow from the impossibility to measure position and momentum simultaneously.

The impossibility to introduce a joint probability distribution on the phase space of a quantum particle has created the belief that it is impossible to find a fair probability description of the quantum state, description that would replace the wave function and the density matrix in the conventional quantum mechanics. In relatively recent times this situation has been improved by introducing the tomographic approach to quantum mechanics. In this approach it is possible to describe quantum states by means of fair probability distributions. In fact, the problem of measuring quantum states, for example, to obtain the Wigner function as an outcome of experiments was under discussion due to the relation between the Wigner function and the optical tomogram found in which is the Radon integral transform of the guasidistribution.

The important property of the Radon transform of the Wigner function is the fact that this transform provides the probability-distribution function $w(X, \theta)$ of real variable X called the homodyne guadrature. In addition, this function called the optical tomogram depends on an extra parameter θ which is called the local oscillator phase. The names were motivated from the measuring problem of the photon quantum states, where the notion of quadrature components is used instead of the position operator \hat{q} and the momentum operator \hat{p} , which are real and imaginary parts of the photon annihilation operator $\hat{a} = \frac{1}{\sqrt{2}} \left(\hat{q} + i \hat{p} \right)$. The two photon guadrature operators \hat{q} and \hat{p} do not have the physical meaning of position and momentum as for the mechanical oscillator, but the mathematical formalism used to describe these observables for photons is the same.

The optical tomogram $w(X, \theta)$ can be measured by homodyne detector, and it was considered as a technical tool to find the quantum state which is associated with the Wigner function. In all experiments on measuring photon quantum states by homodyne detector, the direct outputs of the experiment (optical tomograms) were not interpreted as providing the quantum state but were transformed by means of the inverse Radon integral into the Wigner function (procedure called the reconstruction of Wigner function) which was interpreted as quantum state. Experience with measuring the photon quantum state and the reconstruction of the Wigner function paved the way for the development of other tomographic approaches like symplectic tomography. Also it was realized that tomograms (optical and symplectic) can be considered as primary objects in quantum mechanics, and these objects contain complete information of quantum states. In fact, the optical tomogram is a measurable probability distribution, and one can get all physical characteristics if the tomogram is given exactly in the same sense that one can get all physical characteristics of a system if its wave function (or density matrix) is given. In this sense, tomograms are conceptual alternatives to the wave function (or density operator).

What is quite important with tomograms as primary objects defining the quantum states is the fact that the tomogram can be introduced also for classical systems as the Radon transform of the probability distribution on the phase space. In this sense, the optical tomogram describes the states in both classical and quantum mechanics. This property permits to consider the quantum-to-classical transition using the same carrier space of states, both classical and guantum. The difference between the classical notion, which is the probability distribution f(q, p) on the phase space, and the quantum notion, which is the complex wave function $\psi(x)$, is tremendous and provides with a lot of difficulties in studying the quantum-to-classical transition limit. Of course, the difficulties within the tomographic framework to study this limit also exist, but at least one deals with the same space of functions for both systems, classical and quantum.

Then the question arises – what is the difference between the classical and quantum states if they are described by one (and the same) tomographic-probability distribution (optical or symplectic tomogram)? The answer is contained in the fact that the set of tomograms corresponding to classical states and the set of tomograms corresponding to quantum states are different ,even though they are defined on the same space. The tomograms turn out to obey some integral constraints which are different for the quantum and classical domains. Also the calculation of physical quantities using the tomograms in the classical and quantum domains is different and this difference appears explicitly within the rules for the star-product for the observables and also for the Wigner functions and the Weyl symbols associated with the quantum observables. The appearance of the same carrier space for classical and quantum systems suggests to consider also the classical setting in terms of Hilbert spaces and operators following very closely the old proposal by Koopman adapted to the tomographic representation.

Thus, in the tomographic approach, one can formulate quantum mechanics using only classical-like ingredients, such as probability distributions and functions defining the physical observables. Viceversa by changing the view point one can formulate classical mechanics using purely quantum-like ingredients as Hilbert-space vectors and operators (observables) acting in the Hilbert space. The other important advantage of tomographic approach is the possibility to use all known concepts of probability theory like Shannon entropy and Shannon information, Rényi entropy and adopt known results for these entropies in the form of entropic inequalities and use these results for the tomographic entropies.

We may summarize and formalize our discussion, we may say that minimal features that any physical system should possess are identified by a space of states Σ and a space of observables Θ along with a pairing μ taking values in the space of probability measures on the real line \mathbb{R} . We have $\mu: \Sigma \times \Theta \to \{\text{probability}\}$ measures on the real line \mathbb{R} , i.e., with a state ρ and an observable A, we have $\mu_{A,\rho}$ represents the probability distribution for the values we obtain when we measure A in the state ρ . When $E \subset \mathbb{R}, \mu_{A,o}(E) \in \mathbb{R}$ is the probability that we will obtain a value of A contained in E if the system is known to be in the state ρ . We outline here some mathematical aspects of these minimal features when we deal with classical and quantum systems ,subsequently we shall take up these aspects in a language that is much more familiar to the average reader.

A classical mechanical system is usually described by a finite-dimensional smooth manifold M called the phase space, a symplectic structure ω on M, that is a closed two-form $(d\omega = 0)$ which is not degenerate. A Hamiltonian function defines the evolution by means of the flow, a one-parameter group of canonical transformations associated with the Hamilton equations. From ω , one can construct the Liouville measure associated with the symplectic volume form $\Omega = \omega \wedge \omega \wedge \cdots \omega \wedge$. Considerations from statistical mechanics lead to the following:

- a) states are probability measures on M, say ν ;
- b) observables are measurable real valued functions on M;

c) the pairing map μ is given by $\mu_{\nu,A}(E) = \nu(A^{-1}(E))$, where $E \subset \mathbb{R}, A: M \to \mathbb{R}$, and ν a measure on M.

Statistical considerations suggest to consider states as represented by measures rather than points of M; in this way, we take into account that we may have only a statistical knowledge of the true state (fluctuations). The space of states is a convex set, therefore, we may identify its extreme states as pure states.

Thus, pure states are point measures and therefore in one-to-one correspondence with points of M. Every observable A is sharp in a pure state, i.e., the corresponding measure $\mu_{\nu,A}$ on \mathbb{R} is a point measure. This means that there is no dispersion when we measure an observable in a pure state. As noted by Koopman, previous picture can be translated into the Hilbert-space language. We denote by \mathbb{H} the Hilbert space of square integrable complex valued functions on M with respect to the Liouville measure. Each $\psi \in \mathbb{H}$ is associated with a probability measure $\nu_{\psi} = |\psi|^2 \Omega$ when $||\psi|| = 1$. If A is an observable, its expectation value will be

$$e_A(\psi) = \int_M A |\psi|^2 d\Omega = \langle \psi | A \psi \rangle,$$

where A is considered to be a multiplication operator on \mathbb{H} .

The map $\psi \to \nu_{\psi}$ from \mathbb{H} to ρ is many-to-one because $\nu_{\psi} = \nu_{\psi'}$ if $\psi' = \psi e^{i\alpha}$ and $\alpha \in F(M)$ is any real valued function on M. If the operators are required to commute with the multiplication by a phase, i.e., they are depending only on states, they must be multiplication operators. The Hamiltonian dynamics associated with the Hamiltonian function H will preserve the Liouville measure and therefore will be unitary. Dynamics will preserve the statistical interpretation because the probability is preserved.

In quantum mechanics, there are no dispersion-free states (we recall that a dispersion-free state implies that $\mu_{A,\phi}$ is a point measure for any observable A). In quantum mechanics, states are given by rays of a complex Hilbert space \mathbb{H} . One arrives at this identification by taking into account interference phenomena and probabilistic and statistical aspects. Mixed states will be convex combinations of these pure states. The identification of states out of normalized vectors in \mathbb{H} is now given by $\psi' \simeq \psi e^{i\alpha}$ with $\alpha \in \mathbb{R}$, not a function any more. We assume that expressions like $|\langle \psi | \phi \rangle|^2$ represent transition probabilities and may be measured in the laboratory.

Consider now an observable A. For each $E \in \mathbb{R}$, we have $\mu_{A,\psi}(E)$ which represents the probability of obtaining a value in E when measuring A in the state ψ . One arrives at the identification of a projection operator P_E^A on \mathbb{H} such that

$$\mu_{A,\psi}(E) = \langle \psi | P_E^A \psi \rangle.$$

Various theorems are available to associate with any self-adjoint operator a spectral decomposition and a spectral measure. Expectation value functions will be given by

$$e_A(\psi) = \frac{\langle \psi | A\psi \rangle}{\langle \psi | \psi \rangle}.$$

In view of Dirac's notation, pure states will be $\rho = \frac{|\psi\rangle\langle\psi\rangle}{\langle\psi|\psi\rangle}$, and mixed states are all possible convex combinations. In this picture, observables are self-adjoint operators on \mathbb{H} , states are normalized positive linear functionals on observables and $\mu_{A\rho}(E) = \rho \left(P_E^A\right)$, P_E^A being the spectral projections of A. Pure states are identified with rank-one projectors obtained from normalized vectors of \mathbb{H} .

This view point may be further generalized by considering the observables as a primary object and identified as the real elements of a \mathbb{C}^* -algebra.

In the classical situation, the algebra is the algebra of functions on the phase space, an associative and commutative algebra with the property that the support of the product of two functions is contained in the intersection of the support of the factors. Again, states are identified as normalized positive linear functionals on this algebra.

The quantum situation is distinguished by having a non-commutative algebra. We recall that any \mathbb{C}^* -algebra which is commutative, by Gel'fand-Naimark theorem, must be isomorphic to an algebra of functions on a space identified by the spectrum of the algebra itself. In the quantum case, we start with a \mathbb{C}^* -algebra \mathcal{A} (for instance, all bounded operators on some Hilbert space \mathbb{H}). Observables are real elements of A. The states are the normalized positive linear functionals on \mathcal{A} . If ρ is the state, $\rho(A)$ is the expectation of A in the state ρ . The Gel'fand–Naimark–Segal (GNS) construction plays a fundamental role. This construction shows that there is a representation of \mathcal{A} by means of bounded operators on a Hilbert space \mathbb{H} , say $\Pi_{a}: \mathcal{A} \to \mathcal{L}(\mathbb{H}, \mathbb{H})$ such that

$$\rho(\mathcal{A}) = \langle \psi | \Pi_{\rho}(\mathcal{A}) | \psi \rangle$$

for all $A \in \mathcal{A}$ and ψ a unit cyclic vector in \mathbb{H} . The triple $(\mathbb{H}, \Pi_{\rho}, \phi)$ is unique up to unitary equivalence.

In this way, we may construct our probability measures $\mu_{A,\rho}$. Again we have associated with a physical system a space of states, a space of observables and a pairing with values probability measures on \mathbb{R} , this association holds true for both classical and quantum systems. The GNS construction essentially enables one to recover the Hilbert space formalism from the abstract \mathbb{C}^* -algebra formalism. In the general formalism, one may characterize pure states as those for which Π_{ρ} is irreducible. In this formalism, the uncertainty relations acquire the form

$$\sigma_{\rho}(A)\sigma_{\rho}(B) \ge \frac{1}{2}\rho(C),$$

where C = i(AB - BA) and $\sigma_{\rho}(A)$ is the variance of the probability distribution $\mu_{A,\rho}$, $\sigma_{\rho}(A)^2 = \rho(A^2) - (\rho(A))^2 = \rho\left((A - \rho(A)\mathbb{I})^2\right).$ In the following sections, we shall take up explicitly the construction of the tomographic picture by using as carrier space a phase space which is a vector space. This phase space is also an Abelian vector group. This group property will play a relevant role for it makes available all tools coming from Fourier analysis. Also on a group there exist always two products on functions defined on it, the point-wise product and the convolution product. This latter defines a \mathbb{C}^* -algebra structure on functions which may be used for the GNS construction.

In the coming sections we shall develop more closely the tomographic point of view which makes much more clear the probabilistic and statistical aspect of our picture.

Classical mechanics within the tomographic framework Before introducing the probability representation in quantum mechanics, first we show how the tomographic representation can be introduced in classical statistical mechanics.

Let us consider the Radon transform of the probability-distribution function f(q,p) on the phase space of a classical particle. We denote the transform as $w(X,\mu,\nu)$ where the arguments are X,μ,ν and take real values. By definition,we have the association between these functions called symplectic tomogram

$$w(X,\mu,\nu) = \int f(q,p)\delta\left(X - \mu q - \nu p\right) \, dq \, dp.$$
(3)

Since the Dirac delta-function is understood as a high peaked "function it is positive and the normalized distribution is also positive, the result of integration is the positive function, $w(X, \mu, \nu) \ge 0$. In view of the property of the delta-function

$$\int \delta \left(X - a \right) \, dX = 1,\tag{4}$$

the tomogram is normalized

$$\int w(X,\mu,\nu) \, dX = 1. \tag{5}$$

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The tomogram can be rewritten in the form

$$w(X,\mu,\nu) = \frac{1}{2\pi} \int f(q,p) e^{ik(X-\mu q-\nu p)} dk \, dq \, dp,$$
 (6)

where Fourier decomposition of the Dirac delta-function was used

$$\delta(y) = \frac{1}{2\pi} \int e^{iky} dk.$$
 (7)

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Also the delta-function is homogeneous $\delta(\lambda y) = |\lambda|^{-1}\delta(y)$, and this property provides the homogeneity property of the symplectic tomogram, $w(\lambda X, \lambda \mu, \lambda \nu) = |\lambda|^{-1}w(X, \mu, \nu)$.

Being the probability distribution of random variable X, the tomogram determines the probability distribution f(q,p) in view of the inverse Radon transform

$$f(q,p) = \frac{1}{4\pi^2} \int w(X,\mu,\nu) e^{i(X-\mu q-\nu p)} dX \, d\mu \, d\nu, \qquad (8)$$

which is a particular Fourier transform of the tomogram. Formulae (3) and (8) provide bijective map of the probability density f(q,p) and the tomographic probability $w(X,\mu,\nu)$. Thus, all physical observables F = F(q,p) can be evaluated using the probability density f(q,p), for example, its mean value

$$\langle F \rangle = \langle F(q,p) \rangle = \int f(q,p)F(q,p) \, dq \, dp.$$
 (9)

In the tomographic-probability representation of the classical state, this formula for mean value $\langle F\rangle$ can be written as follows:

$$\langle F \rangle = \langle w_F^d(X,\mu,\nu) \rangle = \int w_F^d(X,\mu,\nu) w(X,\mu,\nu) \, dX \, d\mu \, d\nu,$$
 (10)

where

$$w_F^d(X,\mu,\nu) = \frac{1}{4\pi^2} \int F(q,p) e^{i(X-\mu q-\nu p)} dq \, dp.$$
(11)

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All the highest moments of the observable can be expressed in terms of the tomographic-probability distribution, using the characteristic function

$$\xi(k) = \langle e^{ikF} \rangle = \int e^{ikF(q,p)} f(q,p) \, dq \, dp, \tag{12}$$

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which can be given in the tomographic representation as follows:

The classical observables form an associative and commutative algebra, i.e., the rule of multiplication of the observables is the standard point-wise product

$$C(q,p) = A(q,p)B(q,p).$$
(14)

In the tomographic-probability representation, the functions $w^d_A(X,\mu,\nu)$ and $w^d_B(X,\mu,\nu)$ which provide the mean value $\langle C\rangle$, using

$$w_C^d(X,\mu,\nu) = \frac{1}{4\pi^2} \int A(q,p) B(q,p) e^{i(X-\mu q-\nu p)} dq \, dp, \quad (15)$$

are multiplied according to the formulae

$$A(q,p) = \int w_A^d(X_1,\mu_1,\nu_1)\delta(X_1-\mu_1q-\nu_1p)\,dX_1\,d\mu_1\,d\nu_1,$$
(16)

$$B(q,p) = \int w_B^d(X_2,\mu_2,\nu_2)\delta(X_2-\mu_2q-\nu_2p)\,dX_2\,d\mu_2\,d\nu_2.$$

These formulae provide the following relationship:

$$w_{C}^{d}(X,\mu,\nu) = \int w_{A}^{d}(X_{1},\mu_{1},\nu_{1})w_{B}^{d}(X_{2},\mu_{2},\nu_{2})$$

× $K(X_{1},\mu_{1},\nu_{1},X_{2},\mu_{2},\nu_{2},X,\mu,\nu) dX_{1} d\mu_{1} d\nu_{1} dX_{2} d\mu_{2} d\nu_{2},$
(17)

where the kernel of this nonlocal commutative and associative product reads

$$K(X_{1}, \mu_{1}, \nu_{1}, X_{2}, \mu_{2}, \nu_{2}, X, \mu, \nu) = \frac{1}{4\pi^{2}} \int \delta(X_{1} - \mu_{1}q - \nu_{1}p) \,\delta(X_{2} - \mu_{2}q - \nu_{2}p) \,e^{i(X - \mu q - \nu p)} dq \,dp$$

$$= \frac{1}{4\pi^{2}} \frac{1}{|\nu_{2}\mu_{1} - \nu_{1}\mu_{2}|} \exp\left[i\left(X - \mu \frac{\nu_{1}X_{2} - \nu_{2}X_{1}}{\mu_{2}\nu_{1} - \mu_{1}\nu_{2}} + \nu \frac{\mu_{1}X_{2} - \mu_{2}X_{1}}{\mu_{2}\nu_{1} - \mu_{1}\nu_{2}}\right)\right].$$

(18)

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Thus in the tomographic picture of classical statistical mechanics, the states are associated with tomographic-probability distributions $w(X, \mu, \nu)$ and the observables F – functions F(q, p) in the standard phase-space picture – are associated with the functions $w_F^d(X, \mu, \nu)$. The product of the observables is the commutative star-product of these functions with the kernel given by (18).

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The evolution equation of the classical probability distribution f(q, p, t) is given by the Liouville equation

$$\frac{\partial f(q, p, t)}{\partial t} + p \frac{\partial f(q, p, t)}{\partial q} - \frac{\partial U(q)}{\partial q} \frac{\partial f(q, p, t)}{\partial p} = 0,$$
(19)

where we use the Hamiltonian

$$H = \frac{p^2}{2} + U(q),$$
 (20)

with the particle's mass m=1 and the potential energy U(q) can be transformed into the tomographic form

$$\frac{\partial w(X,\mu,\nu,t)}{\partial t} - \mu \frac{\partial w(X,\mu,\nu,t)}{\partial \nu} - \frac{\partial U}{\partial q} \left(q \to -\left(\frac{\partial}{\partial X}\right)^{-1} \frac{\partial}{\partial \mu} \right) \nu \frac{\partial w(X,\mu,\nu,t)}{\partial X} = 0.$$
(21)

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Like in the Heisenberg picture of quantum mechanics, one can consider the evolution equation for the observables by considering the state-probability distributions, either f(q,p) or $w(X,\mu,\nu)$ as being independent on time but ascribing the time dependence to the phase-space observables F(q,p,t) or the tomographic observables $w_F^d(X,\mu,\nu,t)$. The resulting equation for the phase-space observables reads

$$\frac{\partial F(q, p, t)}{\partial t} - p \frac{\partial F(q, p, t)}{\partial q} + \frac{\partial U}{\partial q} \frac{\partial F(q, p, t)}{\partial p} = 0.$$
(22)

For the tomographic observables, the evolution equation is

$$\frac{\partial w_F^d(X,\mu,\nu,t)}{\partial t} + \mu \frac{\partial w_F^d(X,\mu,\nu,t)}{\partial \nu} + \frac{\partial U}{\partial q} \left(q \to -\left(\frac{\partial}{\partial X}\right)^{-1} \frac{\partial}{\partial \mu} \right) \nu \frac{\partial w_F^d(X,\mu,\nu,t)}{\partial X} = 0.$$
(23)

We conclude that in classical statistical mechanics the state can be associated either with the probability distribution on the phase space or with tomographic-probability distribution $w(X, \mu, \nu)$. In classical statistical mechanics, the observables can be associated either with functions F(q, p) on the phase space and point-wise product multiplication rule or with the functions $w_F^d(X, \mu, \nu)$ which are related to the functions F(q, p) by inverse Radon transform and the star-product of these functions is commutative but not point-wise with the kernel given by (18). Also the evolution equation of the states and observables in the both formulations of classical statistical mechanics can be given in the form of evolution equation either for the f(q, p, t) distribution or for the tomogram $w(X, \mu, \nu, t)$ which is the Radon transform of the phase-space distribution density. Alternatively, the evolution of a classical system can be associated with the evolution of observables F(q, p, t) and $w_F^d(X, \mu, \nu, t)$. The observables are connected by the Radon transform too with its inverse (dual kernel).

The tomographic description of classical statistical mechanics described is appropriate for introducing the tomographic-probability representation of quantum mechanics.

States in quantum mechanics

As we already pointed out, the states in quantum mechanics are associated with the wave function $\psi(x)$ or density matrix $\rho(x, x')$. These notions can be also replaced in a more geometrical picture by vectors $|\psi\rangle$ and density operators $\hat{\rho}$ – we call them density states $\hat{\rho}$ - in a Hilbert space. Then the wave function is the scalar product $\psi(x) = \langle x | \psi \rangle$ and the density matrix is the matrix element of the density operator $\rho(x, x') = \langle x | \hat{\rho} | x' \rangle$. Here we understand the vector $|x\rangle$ as improper eigenvector of the position operator \hat{q} which acts on the wave function $\hat{q}\psi(x) = x\psi(x)$. As we see, the quantum notion of state and observables like the position are very different in comparison with the ones discussed in classical statistical mechanics. As we show below, one can transform these notions to make them very close to the ones used in classical statistical mechanics.

We start now not from the standard definition of the states by means of the density operator but use the axiom that the quantum state is identified with the probability distribution function $w(X, \mu, \nu)$ which has the properties of nonnegativity and normalization, as well as homogeneity, which exactly coincide with the properties of the classical tomographic-probability distribution. Then the question arises – where is the density operator $\hat{\rho}$ in this picture? To answer this question, we must go back to the classical picture. Also we will show that the density operators $\hat{\rho}$ and the vectors in the Hilbert space $|\psi\rangle$ can be easily introduced in classical statistical mechanics following the spirit of the old Koopman paper but from the tomographic point of view. The idea is simply to use the standard formulae of Weyl symbols in the phase-space representation. The first one provides the operator $\hat{\rho}$ from the probability density f(q, p) as follows:

$$\hat{\rho} = \int f(q,p) \left| q - u/2 \right\rangle \langle q + u/2 \left| e^{-ipu} du. \right.$$
(24)

For normalized nonnegative probability density, this operator is Hermitian and satisfies the normalization condition $\operatorname{Tr} \hat{\rho} = 1$. The state $|q + u/2\rangle$ in (24) is improper eigenvector of the operator \hat{q} acting in the Hilbert space as the position operator. Thus, in classical mechanics the distribution function f(q, p) is mapped onto the density operator $\hat{\rho}$. It is easy to see that this formula can be inverted and, as a result, the state distribution function f(q, p) is written in terms of the density operator $\hat{\rho}$ as follows:

$$f(q,p) = \frac{1}{2\pi} \int \operatorname{Tr}\left(\hat{\rho} \left| q - u/2 \right\rangle \langle q + u/2 \right| e^{-ipu} du\right).$$
 (25)

Also for the observable F, i.e., the function F(q, p) in classical statistical mechanics, one can introduce the corresponding operator

$$\hat{F} = \int F(q,p) \left| q - u/2 \right\rangle \langle q + u/2 \left| e^{-ipu} du. \right\rangle$$
(26)

The formula for the mean value of the observable in classical statistical mechanics takes the form

$$\langle F \rangle = \int F(q,p) f(q,p) \, dq \, dp = \operatorname{Tr} \hat{\rho} \hat{F}.$$
 (27)

Analogously, we could start from the classical tomographic-probability distribution $w(X,\mu,\nu)$ and introduce the density operator in classical statistical mechanics as

$$\hat{\rho} = \frac{1}{2\pi} \int w(X,\mu,\nu) e^{i(X-\mu\hat{q}-\nu\hat{p})} dX \, d\mu \, d\nu.$$
(28)

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In order to introduce the observable \hat{F} , which provides the formula for the classical mean value $\langle F \rangle$ (27), one needs to introduce the operator \hat{F} using dual expression, i.e.,

$$\hat{F} = \int w_F^d(X,\mu,\nu)\delta(X-\mu\hat{q}-\nu\hat{p})\,dX\,d\mu\,d\nu.$$
(29)

In this case,

$$\operatorname{Tr}\hat{\rho}\hat{F} = \int w_F^d(X,\mu,\nu)w(X,\mu,\nu)\,dX\,d\mu\,d\nu.$$
(30)

In classical statistical mechanics, the state operators and the observable operators are introduced in different ways in the phase space and in the tomographic picture, and this is related to the fact that the star-product schemes in the both pictures are different. The star-product in the phase-space picture is based on formulae in terms of Weyl symbols and is self-dual, but the tomographic star-product formula is not self-dual (we explain the details in the following section). The operators obtained in view of this procedure do not contain all the operators but only the ones which have symmetrized form in the position and momentum.

Now we are starting to introduce state and observables in quantum mechanics using the same procedure.

We take the quantum tomogram of a state, i.e., the probability distribution $w(X, \mu, \nu)$ which is nonnegative, normalized and homogeneous. We define the state density operator $\hat{\rho}$ as follows:

$$\hat{\rho} = \frac{1}{2\pi} \int w(X,\mu,\nu) e^{i(X-\mu\hat{q}-\nu\hat{p})} dX \, d\mu \, d\nu.$$
(31)

We impose an extra condition which was not used for the state density operator in classical statistical mechanic, namely, the nonnegativity condition, i.e., we consider as a state only such tomographic-probability distribution for which

$$\langle \psi \left| \int w(X,\mu,\nu) e^{i(X-\mu\hat{q}-\nu\hat{p})} dX \, d\mu \, d\nu \right| \psi \rangle \ge 0 \tag{32}$$

for any vector in the Hilbert space. This is a difference between the quantum and classical state expressed in terms of tomogram $w(X, \mu, \nu)$.

Let us point out that in classical statistical mechanics the tomograms are such that for some of them one has inequality

$$\langle \psi \left| \int w(X,\mu,\nu) e^{i(X-\mu\hat{q}-\nu\hat{p})} dX \, d\mu \, d\nu \right| \psi \rangle < 0, \qquad (33)$$

i.e., we have inequality (33) for some vectors $|\psi\rangle$ in the Hilbert space.

On the other hand, classical tomograms must satisfy the condition of nonnegativity of Fourier integral

$$\int w(X,\mu,\nu)e^{i(X-\mu\hat{q}-\nu\hat{p})}dX\,d\mu\,d\nu \ge 0.$$
(34)

The quantum-state tomograms satisfying (32) can relax the condition (34). Thus, introducing the classical and quantum states starting from the tomographic-probability distributions $w(X, \mu, \nu)$, we can introduce the density operator for the classical state and the density operator for the quantum state using the same formula. Nevertheless, we impose different constraints onto these operators.

In the classical case, the density operator being Hermitian can be either positive or negative.

In the quantum case, the density operator being Hermitian is mandatory nonnegative.

These conditions provide different constraints on the classical and quantum tomograms.

If the quantum state is determined by a nonnegative density operator $\hat{\rho},$ its tomogram $w(X,\mu,\nu)$ reads

$$w(X,\mu,\nu) = \operatorname{Tr} \hat{\rho}\delta(X-\mu\hat{q}-\nu\hat{p}).$$
(35)

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In the Hilbert space, other Hermitian operators may act which are not given in the form of series of symmetrized polynomials in position and momentum. For these nonclassical observables \hat{F} , one has the dual tomographic symbols

$$w_F^d(X,\mu,\nu) = \frac{1}{2\pi} \text{Tr}\,\hat{F}e^{i(X-\mu\hat{q}-\nu\hat{p})}.$$
(36)

The product of the observables is not commutative and that reflects the noncommutativity of Weyl symbols of quantum observables given by twisted classical kernel (Grönewold kernel)

$$K(q_1, p_1, q_2, p_2, q_3, p_3) = \frac{1}{4\pi^2} \exp\left[2i(q_1p_2 - q_2p_1 + q_2p_3 - q_3p_2 + q_3p_1 - q_1p_3)\right] (37)$$

where under the exponent one has the expression in terms of symplectic area of the triangle associated with the three points in the phase space.

Star-product of functions and operators

In order to explain rules of multiplications of operators which provide the operator form of classical mechanics, in this section we discuss the star-product of functions or the rules of multiplications of the functions satisfying the associativity condition. Given function $F(\vec{X})$ where $\vec{X} = (X_1, X_2, \dots, X_N)$ contains components which may be either continuous variables X_i or discrete variables. Also one can consider the case where a part of the variables is continuous and the other part contains discrete variables. By definition, thhe associative product $(F_1 \star F_2)(\vec{X})$ of two functions $F_1(\vec{X})$ and $F_2(\vec{X})$ is associative if it satisfies the condition

$$(F_1 \star (F_2 \star F_3))(\vec{X}) = ((F_1 \star F_2) \star F_3)(\vec{X}).$$
(38)

This condition written in the form of constraints for the kernel, giving the product of two functions

$$(F_1 \star F_2)(\vec{X}) = \int K\left(\vec{X}_1, \vec{X}_2, \vec{X}\right) F_1(\vec{X}_1) F_2(\vec{X}_2) d\vec{X}_1 d\vec{X}_2,$$
(39)

provides one with the nonlinear equation for the kernel.

We point out that the integral over $\vec{X}_{1,2}$ in (39) means the integration over continuous components and the summation over discrete components of argument $\vec{X}_{1,2}$.

The product of the functions is commutative if the kernel is a symmetric function with respect to the permutation $\vec{X}_1 \leftrightarrow \vec{X}_2$. The standard point-wise product has the kernel

$$K_{\rm pw}(\vec{X}_1, \vec{X}_2, \vec{X}) = \delta(\vec{X}_1 - \vec{X})\delta(\vec{X}_2 - \vec{X}).$$
(40)

We make two comments.

Any vector can be considered as a function of one variable. Also any matrix element can be considered as as a function of two variables, and the matrix itself can be considered as a column vector. From these observations follows the understanding that the star-product can also be introduced for vectors and operators which, in a chosen basis, are mapped onto the matrices. The matrix elements are functions of column and row indices, and one can introduce any kind of star-product for these functions which induces the star-product for the operators.

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The star-product for the operators can differ from the standard operator product which corresponds to the standard product of matrices given by rule raw by column product. We employ this freedom for choosing and constructing different products of operators, in particular, to construct the product of operators – classical observables.

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We use the following notation for two operators: The first operator which we call dequantizer reads

$$\hat{U}(\vec{X}) \equiv \hat{U}(q,p) = 2\hat{\mathcal{D}}(2\alpha)\hat{\mathcal{P}}, \qquad \vec{X} = (q,p) \in R,$$
(41)

where

$$\hat{\mathcal{D}} = \exp\left(\gamma \hat{a}^{\dagger} - \gamma^* \hat{a}\right), \qquad \hat{a} = (\hat{q} + i\hat{p})/\sqrt{2},$$
(42)

and $\hat{\mathcal{P}}$ is the parity operator. In another form, operator $\hat{U}(q,p)$ used in equations (25) and (26) is

$$\hat{U}(q,p) = \int |q+u/2\rangle \langle q-u/2| e^{-ipu} du.$$
(43)

The second operator called quantizer reads

$$\hat{D}(\vec{X}) = \hat{D}(q,p) = \frac{1}{2\pi}\hat{U}(q,p).$$
 (44)

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One can check that

$$\operatorname{Tr} \hat{D}(\vec{X})\hat{U}(\vec{X}') = \delta(X - X'), \tag{45}$$

i.e.,

$$\operatorname{Tr} \hat{D}(q, p) \hat{U}(q', p') = \delta(q - q') \delta(p - p').$$
(46)

These properties provide the following relationships for any given function ${\cal F}(q,p),$ namely,

$$\hat{F} = \int F(q,p)\hat{D}(q,p)\,dq\,dp = \int F(\vec{X})\hat{D}(\vec{X})\,d\vec{X} \qquad (47)$$

and

$$F(q,p) = \operatorname{Tr} \hat{U}(q,p)\hat{F}.$$
(48)

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Thus, given any two functions $F_1(q, p)$ and $F_2(q, p)$, one has two operators, given in view of Eq. (47), as follows:

$$\hat{F}_1 = \int F_1(q, p) \hat{D}(q, p) \, dq \, dp, \qquad \hat{F}_2 = \int F_2(q, p) \hat{D}(q, p) \, dq \, dp.$$
(49)

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The question arises.

If the product of functions $F_1(\vec{X})$ and $F_2(\vec{X})$ is defined as a point-wise product, which corresponds to multiplication rule of classical observables, what kind of product is induced by this multiplication rule of functions $F_1(q, p)$ and $F_2(q, p)$ for the constructed operators?

Replying to this question, we arrive at the result which we first formulate within the general framework, namely, given a pair of operators – quantizer $\hat{D}(\vec{X})$ and diquantizer $\hat{U}(\vec{X})$ satisfying (45). Given two functions $F_1(\vec{X})$ and $F_2(\vec{X})$ and their star-product with the kernel providing (39). Let us construct two operators

$$\hat{F}_j = \int F_j(\vec{X}) \hat{D}(\vec{X}) \, d\vec{X}, \qquad j = 1, 2,$$
 (50)

what is the product rule for operators \hat{F}_j (we call star-product) such that

$$\hat{F}_1 \star \hat{F}_2 \leftrightarrow (F_1 \star F_2)(\vec{X})? \tag{51}$$

In fact, we must construct the kernel for multiplication of matrix elements of the operators \hat{F}_1 and \hat{F}_2 , if the kernel for multiplication of the functions $F_1(\vec{X})$ and $F_2(\vec{X})$ is given.

Let us have a basis $|n\rangle$ in the Hilbert space where $\hat{D}(\vec{X})$ and $\hat{U}(\vec{X})$ act. In this basis, which we consider as complete and orthonormal set of vectors in the Hilbert space, our operators have the matrix elements

$$\hat{D}(\vec{X})_{nm} = \langle n | \hat{D}(\vec{X}) | m \rangle = \operatorname{Tr} \hat{D}(\vec{X}) | m \rangle \langle n |,$$

$$\hat{U}(\vec{X})_{nm} = \langle n | \hat{U}(\vec{X}) | m \rangle = \operatorname{Tr} \hat{U}(\vec{X}) | m \rangle \langle n |,$$
(52)

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i.e.,

$$\hat{D}(\vec{X}) = \sum_{nm} \hat{D}(\vec{X})_{nm} |m\rangle \langle n|, \qquad \hat{U}(\vec{X}) = \sum_{nm} \hat{U}(\vec{X})_{nm} |m\rangle \langle n|.$$
(53)

The star-product of operators \hat{F}_1 and \hat{F}_2 reads

$$\hat{F}_{1} \star \hat{F}_{2} = \int d\vec{X}_{1} d\vec{X}_{2} d\vec{X} \sum_{abcdnm} K(\vec{X}_{1}, \vec{X}_{2}, \vec{X}) \langle b|\hat{U}(\vec{X}_{1})|a\rangle \\
\times \langle d|\hat{U}(\vec{X}_{2})|c\rangle \langle m|\hat{D}(\vec{X}_{2})|n\rangle \langle a|\hat{F}_{1}|b\rangle \langle c|\hat{F}_{2}|d\rangle|n\rangle \langle m|.$$
(54)

This formula means that the kernel of star-product of functions $F_1(\vec{X})$ and $F_2(\vec{X})$ induces the star-product of matrix elements of the corresponding operators $(\hat{F}_1)_{ab}$ and $(\hat{F}_2)_{cd}$, providing the star-product of the operators. It is given by the kernel

$$k(a, b, c, d, m, n) = \int d\vec{X}_1 \, d\vec{X}_2 \, d\vec{X} \, K(\vec{X}_1, \vec{X}_2, \vec{X}) \hat{U}(X_1)_{ba} \hat{U}(X_2)_{dc} \hat{D}(X)_{nm}.$$
(55)

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Thus, the star-product of the matrix elements of operators \hat{F}_1 and \hat{F}_2 reads

$$(\hat{F}_1 \times \hat{F}_2)_{nm} = \sum_{abcd} k(a, b, c, d, m, n) (\hat{F}_1)_{ab} (\hat{F}_2)_{cd}.$$
 (56)

If the product of functions is point-wise and given by the kernel (40), the kernel of the product of matrix elements reads

$$k_{\rm pw}(a,b,c,d,m,n) = \int d\vec{X} \, \hat{U}(\vec{X})_{ba} \hat{U}(\vec{X})_{dc} \hat{D}(\vec{X})_{mn}.$$
 (57)

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Also in the case where

$$K(\vec{X}_{1}, \vec{X}_{2}, \vec{X}) = \mathsf{Tr}\left(\hat{D}(\vec{X}_{1})\hat{D}(\vec{X}_{3})\hat{U}(\vec{X})\right),$$
(58)

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the star-product of the operators is the usual operator product, i.e., the kernel of the product of matrices gives a standard row-column rule of the multiplication of matrices. If the product of functions is commutative, i.e., the kernel $K(\vec{X}_1, \vec{X}_2, \vec{X})$ is symmetric with respect to permutation $1 \leftrightarrow 2$, the star-product of the operators is also commutative, i.e., $\hat{F}_1 \star \hat{F}_2 = \hat{F}_2 \star \hat{F}_1$, that follows from the corresponding permutation symmetry of kernel (51).

Thus, the Grönewold kernel of star-product of Weyl symbols just satisfies the condition which is obtained using dequantizer (43) and quantizer (44) in view of formula (54). This means that the product of observables – operators corresponding to the functions on the phase space which are Weyl symbols of the operators is just the standard product of the operators, but the commutative kernel for the product of functions on the phase space induces the kernel for the star-product of the operators – observables in the formalism of Hilbert space and operators for classical statistical mechanics.

The evolution equation for quantum tomograms

The Schrödinger equation for the state vector $|\psi,t\rangle$ for the system with the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2} + U(\hat{q})$$
 (59)

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reads

$$i\frac{\partial}{\partial t}|\psi,t\rangle = \hat{H}|\psi,t\rangle \qquad (\hbar = 1).$$
 (60)

In the coordinate representation, the equation has the form of differential equation for the wave function $\psi(x,t),$ i.e.,

$$i\frac{\partial\psi(x,t)}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi(x,t)}{\partial x^2} + U(x)\psi(x,t).$$
(61)

The von Neumann equation for the density matrix of pure state $\rho(x, x', t) = \psi(x, t)\psi^*(x', t)$ can easily be derived from Eq. (61) and it has the form

$$i\frac{\partial\rho(x,x',t)}{\partial t} = -\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x'^2}\right)\rho(x,x',t) + \left(U(x) - U(x')\right)\psi(x,x',t)$$
(62)

This equation is also valid for any convex sum of the density matrices of pure states, i.e., for mixed states.

The evolution equation can be transformed into the Moyal equation for the Wigner function W(q, p, t) using the change of variables induced by Fourier transform of the density matrix providing the Wigner function. The Moyal equation reads

$$\frac{\partial W(q, p, t)}{\partial t} + p \frac{\partial W(q, p, t)}{\partial q} + \frac{1}{i} \left[U \left(q - \frac{i}{2} \frac{\partial}{\partial p} \right) - \text{c.c.} \right] W(q, p, t) = 0.$$
(63)

In operator form, this equation for the quantum state associated with the density operator $\hat{\rho}(t)$ is

$$\frac{\partial \hat{\rho}(t)}{\partial t} + i \left[\hat{H}, \hat{\rho}(t) \right] = 0.$$
(64)

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This means that the density operator is an integral of the motion.

Thus, we have the quantum evolution equation for the system's state written in the three different forms (62)–(64). The tomographic form of the evolution equation can easily be obtained applying the Radon integral transform to the Moyal equation, and the result is written in as follows:

$$\frac{\partial w(X,\mu,\nu,t)}{\partial t} - \mu \frac{\partial w(X,\mu,\nu,t)}{\partial \nu} - \frac{1}{i} \left[U \left(-\left(\frac{\partial}{\partial X}\right)^{-1} \frac{\partial}{\partial \mu} + \frac{i\nu}{2} \frac{\partial}{\partial X} \right) - \text{c.c.} \right] w(X,\mu,\nu,t) = 0.(65)$$

Making change of variables $\mu = \cos \theta$ and $\nu = \sin \theta$ which provides the optical tomogram $w(X, \mu, \nu, t) \rightarrow w(X, \theta, t)$ yields the evolution equation for the optical tomogram

$$\frac{\partial}{\partial t}w(X,\theta,t) = \left[\cos^2\theta \frac{\partial}{\partial \theta} - \frac{1}{2}\sin 2\theta \left\{1 + X\frac{\partial}{\partial X}\right\}\right]w(X,\theta,t)$$
$$+ 2\left[\operatorname{Im} U\left\{\sin\theta \frac{\partial}{\partial \theta} \left[\frac{\partial}{\partial X}\right]^{-1} + X\cos\theta + i\frac{\sin\theta}{2}\frac{\partial}{\partial X}\right\}\right]w(X,\theta,t) (66)$$

Energy level equations for tomograms

For stationary states, the energy level equations are obtained by solving the Schrödinger equation for the wave function

$$\hat{H}\psi_E(x) = -\frac{1}{2}\frac{\partial^2}{\partial x^2}\psi_E(x) + U(x)\psi_E(x) = E\psi_E(x).$$
(67)

This equation can also be transformed into the tomographic form as well as into the Moyal form. The Moyal equation for the energy levels is

$$EW_{E}(q,p) = -\frac{1}{4} \left[\left(\frac{1}{2} \frac{\partial}{\partial q} + ip \right)^{2} + \left(\frac{1}{2} \frac{\partial}{\partial q} - ip \right)^{2} \right] W_{E}(q,p) + \frac{1}{2} \left[U \left(q + \frac{i}{2} \frac{\partial}{\partial p} \right) + U \left(q - \frac{i}{2} \frac{\partial}{\partial p} \right) \right] W_{E}(q,p).$$
(68)

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For the symplectic tomogram the energy-level equation reads

$$Ew_{E}(X,\mu,\nu) = -\frac{1}{4} \left[\left(\frac{1}{2} \mu \frac{\partial}{\partial X} - i \frac{\partial}{\partial \nu} \left(\frac{\partial}{\partial X} \right)^{-1} \right)^{2} + c.c. \right] w_{E}(X,\mu,\nu) + \frac{1}{2} \left[U \left(-\frac{\partial}{\partial \mu} \left(\frac{\partial}{\partial X} \right)^{-1} + \frac{i}{2} \nu \frac{\partial}{\partial X} \right) + c.c. \right] w_{E}(X,\mu,\nu).$$
(69)

For the optical tomogram, the energy-level equation has the form

$$Ew_{E}(X,\theta) = \left[\left\{ \frac{\cos^{2}\theta}{2} \left[\frac{\partial}{\partial X} \right]^{-2} \left(\frac{\partial^{2}}{\partial \theta^{2}} + 1 \right) - \frac{X}{2} \left[\frac{\partial}{\partial X} \right]^{-1} \right. \\ \left(\cos^{2}\theta + \sin 2\theta \frac{\partial}{\partial \theta} \right) + \frac{X^{2}}{2} \sin^{2}\theta - \frac{\cos^{2}\theta}{8} \frac{\partial^{2}}{\partial X^{2}} \right\} \right] w_{E}(\vec{X},\vec{\theta}) \\ \left. + \left[\operatorname{Re} V \left\{ \sin \theta \frac{\partial}{\partial \theta} \left[\frac{\partial}{\partial X} \right]^{-1} + X \cos \theta + i \frac{\sin \theta}{2} \frac{\partial}{\partial X} \right\} \right] w_{E}(\vec{X},\vec{\theta})$$

$$(70)$$
The solutions of the energy-level equation in the symplectic form for the harmonic oscillator reads

$$w_n(X,\mu,\nu) = \frac{e^{-\frac{X^2}{\mu^2 + \nu^2}}}{\sqrt{\pi(\mu^2 + \nu^2)}} \frac{1}{n!2^n} H_n^2\left(\frac{X}{\sqrt{\mu^2 + \nu^2}}\right).$$
 (71)

The solutions of the energy-level equation in the optical form for the harmonic oscillator reads

$$w_n(X,\theta) = \frac{e^{-X^2}}{\sqrt{\pi}} \frac{1}{n!2^n} H_n^2(X).$$
 (72)

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One can see that the optical tomogram of the Fock state $|n\rangle$ does not depend on the local oscillator phase θ .

Integrals of motion in the probability representation In classical mechanics of Hamiltonian systems, the integrals of motion are functions I(q, p, t) on the phase space satisfying the condition of conservation on the trajectory

$$\frac{dI(q, p, t)}{dt} = 0 \tag{73}$$

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given by solving the Hamiltonian equations

$$\dot{q} = \frac{\partial H(q, p, t)}{\partial p}, \qquad \dot{p} = -\frac{\partial H(q, p, t)}{\partial q}.$$
 (74)

The condition (73) can be rewritten in the form

$$\frac{\partial I(q, p, t)}{\partial t} + p \frac{\partial I(q, p, t)}{\partial q} - \frac{\partial U(q)}{\partial q} \frac{\partial I(q, p, t)}{\partial p} = 0.$$
(75)

Equation (75) can be presented in the tomographic form, e.g., for symplectic tomographic symbol (dual one) of the integral of motion. Equation (75) coincides with the Liouville equation for the probability distribution (19). This means that the probability distribution f(q, p, t) is the constant of motion. The equation for the tomographic symbol of the integral of motion $w_I^d(X, \mu, \nu, t)$ is identical to (23).

In quantum domain, the integrals of motion are observables associated with the operators $\hat{I}(\hat{q},\hat{p},t).$ The operator satisfies the equation in the Schrödinger representation

$$\frac{dI(\hat{q},\hat{p},t)}{dt} = 0, \tag{76}$$

and this equation is rewritten as

$$\frac{\partial I(\hat{q},\hat{p},t)}{\partial p} + i\left[\hat{H}(\hat{q},\hat{p},t),\hat{I}(\hat{q},\hat{p},t)\right] = 0 \qquad (\hbar = 1).$$
(77)

This equation coincides with the von Neumann equation for the density operator (see equation (62) written in the position representation). This means that the density operator $\hat{\rho}(t)$ is the integral of motion containing the time dependence explicitly. The commutator of the integral of motion is not equal to zero, but this commutator is compensated by the partial time derivative. The equation of dual tomographic symbol of the quantum integral of motion is identical to equation (62) where the term with the time derivatives must be taken with negative sign.

Quantum inequalities for continuous variables For continuous variables, the wave function $\psi(x)$ provides the probability-distribution density

$$P(x) = |\psi(x)|^2.$$
 (78)

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The corresponding Shannon entropy reads

$$S_x = -\int |\psi(x)|^2 \ln |\psi(x)|^2 \, dx.$$
(79)

In the momentum representation, one has the wave function

$$\widetilde{\psi}(p) = \frac{1}{\sqrt{2\pi}} \int \psi(x) e^{-ipx} \, dx \qquad (\hbar = 1).$$
(80)

The corresponding Shannon entropy related to the momentum-probability density $|\widetilde{\psi}(p)|^2$ reads

$$S_p = -\int |\widetilde{\psi}(p)|^2 \ln |\widetilde{\psi}(p)|^2 \, dp.$$
(81)

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There exists the correlation of entropies S_x and S_p , since the function $\psi(x)$ determines the Fourier component $\tilde{\psi}(p)$. This means that the entropies S_x and S_p have to obey some constrains. These constrains are entropic uncertainty relations. For the one-mode system, the inequality reads

$$S_x + S_p \ge \ln(\pi e). \tag{82}$$

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One has the optical-tomogram expression in terms of the wave function

$$w(X,\theta) = \left| \int \psi(y) \exp\left[\frac{i}{2} \left(\cot\theta \ (y^2 + X^2) - \frac{2X}{\sin\theta} \ y\right)\right] \frac{dy}{\sqrt{2\pi i \sin\theta}} \right|^2$$
(83)

On the other hand, this tomogram formally equals to

$$w(X,\theta) = |\psi(X,\theta)|^2,$$
(84)

where the wave function reads

$$\psi(X,\theta) = \frac{1}{\sqrt{2\pi i \sin \theta}} \int \exp\left[\frac{i}{2}\left(\cot \theta \ (y^2 + X^2) - \frac{2X}{\sin \theta} y\right)\right] \psi(y) \, dy,$$
(85)

being the fractional Fourier transform of the wave function $\psi(y)$. This wave function corresponds to the wave function of a harmonic oscillator with $\hbar = m = \omega = 1$ taken at the time moment θ provided the wave function at the initial time moment $\theta = 0$ equals to $\psi(y)$. In view of expressions of tomogram in terms of the wave function (84) and (85), one has the entropic uncertainty relation in the form

$$S(\theta) + S(\theta + \pi/2) \ge \ln \pi e.$$
(86)

Here $S(\theta)$ is the tomographic Shannon entropy associated with optical tomogram (83) which is measured by homodyne detector. We illustrate the entropic inequality (86) by the example of the harmonic oscillator's ground state with the wave function

$$\psi_0(x) = \pi^{-1/4} e^{-x^2/2} \tag{87}$$

written in dimensionless variables. Using (85), we obtain the tomogram

$$w(X,\theta) = \pi^{-1/2} e^{-X^2}.$$
(88)

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The ground-state tomogram does not depend on the angle θ . In view of this, the entropy is $S(\theta)=S(\theta+\pi/2)=\frac{1}{2}\ln\pi e$. The sum of these two entropies saturates inequality (86). In recent paper, the new uncertainty relation was obtained for Rényi entropy related to the probability distributions for position and momentum of quantum state with density operator $\hat{\rho}$. The uncertainty relation reads

$$\frac{1}{1-\alpha} \ln\left(\int_{-\infty}^{\infty} dp \left[\rho(p,p)\right]^{\alpha}\right) + \frac{1}{1-\beta} \ln\left(\int_{-\infty}^{\infty} dx \left[\rho(x,x)\right]^{\beta}\right)$$
$$\geq -\frac{1}{2(1-\alpha)} \ln\frac{\alpha}{\pi} - \frac{1}{2(1-\beta)} \ln\frac{\beta}{\pi}, \tag{89}$$

where positive parameters α and β satisfy the constrain

$$(1/\alpha) + (1/\beta) = 2.$$
 (90)

Rényi entropies R_{α} and R_{β} related to the momentum and position distributions, respectively, are just two terms on the left-hand side of (89). For $\alpha, \beta \longrightarrow 1$, these entropies become Shannon entropies S_p and S_x .

We illustrate this inequality by the example of the harmonic oscillator's ground state. In this case, one has the Rényi entropies

$$R_{\alpha} = \frac{\ln \pi}{2} - \frac{1}{2} \frac{\ln \alpha}{1 - \alpha}, \qquad R_{\beta} = \frac{\ln \pi}{2} - \frac{1}{2} \frac{\ln \beta}{1 - \beta}$$

which, in the limit $\alpha \to 1$ and $\beta \to 1$, go to $(\ln \pi)/2$. Also the sum of the entropies reads

$$R_{\alpha} + R_{\beta} = \ln \pi - \frac{1}{2} \frac{\ln \alpha}{1 - \alpha} - \frac{1}{2} \frac{\ln \beta}{1 - \beta}$$

In view of (90) this sum equals to the right-hand side of inequality (89). Thus, the harmonic oscillator's ground state saturates this inequality.

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Using the same argument that we employed to obtain inequality (86) for Shannon entropies, we arrive at the condition for optical tomogram

$$(q-1) \ln \left\{ \int_{-\infty}^{\infty} dX \left[w \left(X, \theta + \pi/2 \right) \right]^{1/(1-q)} \right\} + (q+1) \ln \left\{ \int_{-\infty}^{\infty} dX \left[w(X, \theta) \right]^{1/(1+q)} \right\} \ge (1/2) \left\{ (q-1) \ln \left[\pi(1-q) \right] + (q+1) \ln \left[\pi(1+q) \right] \right\}$$
(91)

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where the parameter q is defined by $\alpha = (1 - q)^{-1}$. This inequality can also be checked experimentally.

Checking position–momentum uncertainty relations In view of the physical meaning of optical tomogram, one can calculate higher moments of the probability distribution

$$\langle X^n \rangle (\mu, \nu) = \int X^u M(X, \mu, \nu) \, dX, \qquad n = 1, 2, \dots$$
 (92)

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for any value of the parameters μ and ν ; in particular, for any given phase of the local oscillator θ . This provides the possibility to check the inequalities for the quantum uncertainty relations.

The Heisenberg uncertainty relation connects position and momentum variances σ_{QQ} and σ_{PP} by means of an inequality. In the tomographic-probability representation, the Heisenberg relation reads:

$$\sigma_{PP}\sigma_{QQ} = \left(\int X^2 M(X,0,1) \, dX - \left[\int X M(X,0,1) \, dX\right]^2\right) \\ \times \left(\int X^2 M(X,1,0) \, dX - \left[\int X M(X,1,0) \, dX\right]^2\right) \ge \frac{1}{4}.$$
(93)

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The Schrödinger–Robertson uncertainty relation contains the contribution of the position–momentum covariance σ_{OP} and reads

$$\sigma_{QQ}\sigma_{PP} - \sigma_{QP}^2 \ge \frac{1}{4}.$$
(94)

In view of Eq. (92), the variance σ_{XX} of the homodyne quadrature X in terms of the parameters μ , ν , and the quadratures variances and covariance is

$$\sigma_{XX}(\mu,\nu) = \mu^2 \sigma_{QQ} + \nu^2 \sigma_{PP} + 2\,\mu\nu\sigma_{QP}.$$
(95)

The above formula is obtained using the definition of homodyne-quadrature-component operator

$$\hat{X} = \mu \hat{q} + \nu \hat{p}.$$
(96)

Thus, one has

$$\hat{X}^{2} = \mu^{2} \hat{q}^{2} + \nu^{2} \hat{p}^{2} + 2\mu \nu \frac{\hat{q}\hat{p} + \hat{p}\hat{q}}{2}.$$
(97)

Taking average for any state in (96), one has the equality for the mean value $% \left(\frac{1}{2} \right) = 0$

$$\langle \hat{X} \rangle = \mu \langle \hat{q} \rangle + \nu \langle \hat{p} \rangle.$$
 (98)

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Averaging (97), we obtain

$$\langle \hat{X}^2 \rangle = \mu^2 \langle \hat{q}^2 \rangle + \nu^2 \langle \hat{p}^2 \rangle + 2\mu\nu \langle \frac{\hat{q}\hat{p} + \hat{p}\hat{q}}{2} \rangle.$$
(99)

Thus

$$\sigma_{XX}(\mu,\nu) = \langle \hat{X}^2 \rangle - \langle \hat{X} \rangle^2 = \mu^2 \left(\langle \hat{q}^2 \rangle - \langle \hat{q} \rangle^2 \right) + \nu^2 \left(\langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2 \right) + 2\mu\nu \left(\langle \frac{\hat{q}\hat{p} + \hat{p}\hat{q}}{2} \rangle - \langle \hat{q} \rangle \langle \hat{p} \rangle \right).$$
(100)

While deriving (100), we used that

$$\langle \hat{X} \rangle^2 = \mu^2 \langle \hat{q} \rangle^2 + \nu^2 \langle \hat{p} \rangle^2 + 2\mu \nu \langle \hat{q} \rangle \langle \hat{p} \rangle.$$

Since

$$\sigma_{QP} = \langle \frac{\hat{q}\hat{p} + \hat{p}\hat{q}}{2} \rangle - \langle \hat{q} \rangle \langle \hat{p} \rangle,$$

one can get the expression of the covariance σ_{QP} in terms of the tomographic characteristics of the state. Taking $\mu = \nu = \sqrt{2}/2$ corresponding to the local oscillator phase $\theta = \pi/4$, one has

$$\sigma_{QP} = \sigma_{XX} \left(\theta = \frac{\pi}{4} \right) - \frac{1}{2} (\sigma_{QQ} + \sigma_{PP}), \tag{101}$$

where σ_{PP} and σ_{QQ} are the factors appearing on the left-hand side of Eq. (93), respectively. The term $\sigma_{XX}(\theta = \pi/4)$ is given by Eq. (92) as

$$\sigma_{XX}\left(\theta = \frac{\pi}{4}\right) = \left\langle X^2 \right\rangle \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) - \left[\left\langle X \right\rangle \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)\right]^2.$$
(102)

The check of Schrödinger–Robertson uncertainty relations requires extra elaboration of the available experimentally obtained optical tomogram of the photon quantum state. We express this procedure as the following inequality for optical tomogram. Let us calculate the function $F(\theta)$ which we call the tomographic uncertainty function

$$F(\theta) = \left(\int X^2 w(X,\theta) dX - \left[\int X w(X,\theta) dX\right]^2\right)$$

$$\times \left(\int X^2 w\left(X,\theta + \frac{\pi}{2}\right) dX - \left[\int X w\left(X,\theta + \frac{\pi}{2}\right) dX\right]^2\right)$$

$$- \left\{\int X^2 w\left(X,\theta + \frac{\pi}{4}\right) dX - \left[\int X w\left(X,\theta + \frac{\pi}{4}\right) dX\right]^2$$

$$- \frac{1}{2} \left[\int X^2 w(X,\theta) dX - \left[\int X w(X,\theta) dX\right]^2$$

$$+ \int X^2 w\left(X,\theta + \frac{\pi}{2}\right) dX - \left[\int X w\left(X,\theta + \frac{\pi}{2}\right) dX\right]^2\right] \right\}^2 - \frac{1}{4}$$

The tomographic uncertainty function must be nonnegative

$$F(\theta) \ge 0 \tag{104}$$

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for all the values of the local oscillator phase angle $0 \le \theta \le 2\pi$. The previous equation (103) for $\theta = 0$ yields Eq. (94).

Measuring highest moments of quadratures by homodyne detector

Let us discuss first how to measure highest moments for one-mode-light quadrature $\hat{X}(\mu,\nu) = \mu Q + \nu P$, where Q and P are operators of the one-mode photon quadratures. Mean values, variances and covariances can be given in terms of the optical tomogram $\mathcal{W}(X,\theta)$. Let us construct cubic moments. Then one has to find the moments of the operators P^3, P^2Q, PQ^2, Q^3 because the remaining may be expressed by commutators as

$$PQP = PPQ + P[Q, P] = P^{2}Q + iP$$

$$\Rightarrow \langle PQP \rangle = \langle P^{2}Q \rangle + i \langle P \rangle, \qquad (105)$$

$$QPP = PQP + [Q, P] P = P^{2}Q + 2iP$$

$$\Rightarrow \langle QP^{2} \rangle = \langle P^{2}Q \rangle + 2i \langle P \rangle,$$

and analogously

$$\langle QPQ \rangle = \langle PQ^2 \rangle + i \langle Q \rangle ,$$

$$\langle Q^2P \rangle = \langle PQ^2 \rangle + 2i \langle Q \rangle .$$

$$(106)$$

The cubic power $\hat{X}^{3}\left(\mu,\nu\right)$ reads

$$\hat{X}^{3}(\mu,\nu) = \mu^{3}Q^{3} + \nu^{3}P^{3} + \mu^{2}\nu \left(Q^{2}P + QPQ + PQ^{2}\right) + \mu\nu^{2} \left(P^{2}Q + PQP + QP^{2}\right)$$
(107)

so that

$$\left\langle \hat{X}^{3} \right\rangle (\mu, \nu) = \mu^{3} \left\langle Q^{3} \right\rangle + \nu^{3} \left\langle P \right\rangle^{3} + 3\mu^{2}\nu \left(\left\langle PQ^{2} \right\rangle + i \left\langle Q \right\rangle \right) + 3\mu\nu^{2} \left(\left\langle P^{2}Q \right\rangle + i \left\langle P \right\rangle \right).$$
 (108)

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The means of the quadratures read

$$\langle \hat{X} \rangle (1,0) = \langle Q \rangle , \langle \hat{X} \rangle (0,1) = \langle P \rangle.$$
 (109)

Besides, one has

$$\left\langle \hat{X}^{3} \right\rangle(1,0) = \left\langle Q^{3} \right\rangle, \left\langle \hat{X}^{3} \right\rangle(0,1) = \left\langle P^{3} \right\rangle,$$
 (110)

 $\quad \text{and} \quad$

$$\left\langle \hat{X}^{3} \right\rangle (\mu, \nu) = \mu^{3} \left\langle \hat{X}^{3} \right\rangle (1, 0) + \nu^{3} \left\langle \hat{X}^{3} \right\rangle (0, 1) + 3\mu^{2}\nu \left(\left\langle PQ^{2} \right\rangle + i \left\langle \hat{X} \right\rangle (1, 0) \right) + 3\mu\nu^{2} \left(\left\langle P^{2}Q \right\rangle + i \left\langle \hat{X} \right\rangle (0, 1) \right).$$
(111)

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Introducing the function

$$A(\mu,\nu) := \left\langle \hat{X}^{3} \right\rangle(\mu,\nu) - \mu^{3} \left\langle \hat{X}^{3} \right\rangle(1,0) - \nu^{3} \left\langle \hat{X}^{3} \right\rangle(0,1) -3\mu^{2}\nu i \left\langle \hat{X} \right\rangle(1,0) - 3\mu\nu^{2} i \left\langle \hat{X} \right\rangle(0,1)$$
(112)

we obtain two linear equations for the remaining two moments:

$$A(\mu_{\alpha},\nu_{\alpha}) = 3\mu_{\alpha}^{2}\nu_{\alpha} \langle PQ^{2} \rangle + 3\mu_{\alpha}\nu_{\alpha}^{2} \langle P^{2}Q \rangle; \quad (113)$$

$$A(\mu_{\beta},\nu_{\beta}) = 3\mu_{\beta}^{2}\nu_{\beta} \langle PQ^{2} \rangle + 3\mu_{\beta}\nu_{\beta}^{2} \langle P^{2}Q \rangle;$$

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which can be readily solved in terms of the homodyne quadratures given by the tomogram $\mathcal{W}(X,\theta)$ only.

The previous construction of the solutions

$$\langle PQ^2 \rangle = \frac{1}{\Delta} \det \begin{pmatrix} A(\mu_{\alpha}, \nu_{\alpha}) & 3\mu_{\alpha}\nu_{\alpha}^2 \\ A(\mu_{\beta}, \nu_{\beta}) & 3\mu_{\beta}\nu_{\beta}^2 \end{pmatrix}; \langle P^2Q \rangle = \frac{1}{\Delta} \det \begin{pmatrix} 3\mu_{\alpha}^2\nu_{\alpha} & A(\mu_{\alpha}, \nu_{\alpha}) \\ 3\mu_{\beta}^2\nu_{\beta} & A(\mu_{\beta}, \nu_{\beta}) \end{pmatrix},$$
(114)

with

$$\Delta = \det \begin{pmatrix} 3\mu_{\alpha}^{2}\nu_{\alpha} & 3\mu_{\alpha}\nu_{\alpha}^{2} \\ 3\mu_{\beta}^{2}\nu_{\beta} & 3\mu_{\beta}\nu_{\beta}^{2} \end{pmatrix},$$
(115)

shows that the same procedure can be applied to get all the highest moments $\langle P^nQ^m\rangle$ and $\langle P^mQ^n\rangle~(n,m=0,1,\ldots)$ in terms of the tomogram $\mathcal{W}\left(X,\theta\right)$ only. It provides the tool to check all the known high moments quantum uncertainty relations in fact both in one mode and multimode case. As an example we derive simple uncertainty relations for cubic moments.

Let us consider the linear forms:

$$\hat{f} = y_1 Q + y_2 P^2 ; \ \hat{f}^{\dagger} = y_1^* Q + y_2^* P^2 .$$
 (116)

The obvious inequality for the mean value

$$\left\langle \hat{f}\hat{f}^{\dagger}\right\rangle \geq 0$$
 (117)

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gives a condition of nonnegativity for the quadratic form

$$y_1 y_1^* \langle Q^2 \rangle + y_1 y_2^* \langle QP^2 \rangle + y_2 y_1^* \langle P^2 Q \rangle + y_2 y_2^* \langle P^4 \rangle \ge 0.$$
 (118)

Thus the matrix of the quadratic form

$$M = \begin{pmatrix} \langle Q^2 \rangle & \langle QP^2 \rangle \\ \langle P^2 Q \rangle & \langle P^4 \rangle \end{pmatrix}$$
(119)

must be nonnegative, and this implies

$$\left\langle Q^{2}\right\rangle \left\langle P^{4}\right\rangle - \left\langle QP^{2}\right\rangle \left\langle P^{2}Q\right\rangle \geq 0.$$
 (120)

This inequality can be written in terms of tomograms as

$$\int X^{2} \mathcal{W}(X, \theta = 0) dX \int X^{4} \mathcal{W}(X, \theta = \frac{\pi}{2}) dX - \left[\left\langle QP^{2} \right\rangle \left\langle P^{2}Q \right\rangle \right]_{\theta_{\alpha}, \theta_{\beta}} \ge 0$$
(121)

where local oscillator phases, for instance $\theta_{\alpha} = \pi/3$, $\theta_{\beta} = 2\pi/3$, are taken in Eq. (114), so that the parameters $(\mu_{\alpha}, \nu_{\alpha})$ and $(\mu_{\beta}, \nu_{\beta})$ are $(\sqrt{3}/2, 1/2)$ and $(1/2, \sqrt{3}/2)$ respectively. Of course, one could use other suitable local oscillator phases, such that $\Delta \neq 0$ in Eq. (114). The above cubic-in-quadrature uncertainty relation must be satisfied by any of the six modes used in experiments.

In view of the generalization for Schrödinger–Robertson uncertainty relations, an analogous generalization can be proposed for the above highest order moments inequality, that can be written in covariant form, i.e. for all the local oscillator phases as:

$$\int X^{2} \mathcal{W}(X,\theta) dX \int X^{4} \mathcal{W}(X,\theta+\frac{\pi}{2}) dX - \left[\left\langle QP^{2} \right\rangle \left\langle P^{2} Q \right\rangle \right]_{\theta+\theta_{\alpha},\theta+\theta_{\beta}} \ge 0,$$
(122)

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where, as before, Eq. (114) has to be used with the new values of local oscillator phases, say $\theta + \pi/3$, $\theta + 2\pi/3$.

Group properties of tomograms in quantum mechanics As we have mentioned in the previous sections, it has been shown how to describe quantum states by using a standard positive probability distribution called a symplectic probability distribution or symplectic tomogram. We recall that the symplectic tomogram $\mathcal{W}(X,\mu,\nu)$ is a nonnegative function of the random position Xmeasured in reference frames in phase-space with rotated and scaled axes $q \to \mu q$, $p \to \nu p$ where $\mu = e^{\lambda} \cos \theta$, $\nu = e^{-\lambda} \sin \theta$, θ is the angle of rotation and e^{λ} is the scaling parameter.

The symplectic tomographic probability distribution $\mathcal{W}(X, \mu, \nu)$ contains complete information on quantum states in the sense that for a given wave function $\psi(x)$ or density operator $\hat{\rho}$ (determining the quantum state in the conventional formulation of quantum mechanics) the tomogram can be calculated.

On the other hand, for a given tomogram $\mathcal{W}(X, \mu, \nu)$ one can reconstruct explicitly the density operator $\hat{\rho}$. It means that for a given symplectic tomogram of a system with continuous variables all the properties of the quantum system can be obtained as well as for a given density operator $\hat{\rho}$.

Analogous complete information on the quantum states is contained in the Wigner function W(q, p) which is a real function on the phase space of the system. The Wigner function is related to the symplectic tomogram by means of an integral Radon transform, however the Wigner function is not definite in sign, it takes negative values for some quantum states and cannot be considered as a positive probability distribution on phase space. The necessary and sufficient conditions for a real function on the phase space to describe the Wigner function of a quantum state were found in where the corresponding properties of the function

under consideration were associated with the so called h-positivity condition of a function on the Abelian translation group on the phase space.

As we have shown elsewhere, in this description plays an important role the Weyl-Heisenberg group and its group of automorphisms, along with the Abelian vector group which arises as quotient group of Weyl-Heisenberg group by its central subgroup. In this section, we would like to consider the tomographic description of quantum mechanics as another picture, on the same footing as the Schroedinger, Heisenberg or Weyl-Wigner pictures. To this aim, we have to provide a characterization of symplectic tomograms which stands on its own, without relying on other pictures. In other terms, we need necessary and sufficient conditions for a function $f(X, \mu, \nu)$ to be the symplectic tomogram

 $\mathcal{W}(X,\mu,\nu)$ of a quantum state.

The strategy to find these conditions is based on Naimark's theorem that provides a characterization of positive operator-valued measures and that allows to characterize functions which are elements of matrices of group representations.

In particular, we use the result that a function $\varphi(g)$ on a group G, $g \in G$, which is a diagonal matrix element of a unitary representation of the group G, has the property of being positive definite in the sense that the matrix

$$M_{jk} = \varphi(g_j g_k^{-1}) \tag{123}$$

for any j, k = 1, 2, ..., N and arbitrary N, is positive definite.

Below we will show that symplectic tomograms can be associated with positive definite functions φ on the Weyl-Heisenberg group. Since Naimark's theorem for positive operator-valued measures allows to construct and determine uniquely a Hilbert space and a vector in it representing the function φ (using what today is called the Gelfand-Naimark-Segal (GNS) method) the connection established below of the symplectic tomograms with positive definite functions on the Weyl-Heisenberg group yields the necessary and sufficient condition which we are looking for. It is worthy to note that this condition can be also studied using the necessary and sufficient condition for a function to be a Wigner function, but we do not use here the connection of symplectic tomogram with the Wigner function and provide the condition for the tomogram independently of any other result concerning Wigner functions

Symplectic tomography

In this section, we again recall the construction of tomographic probability densities determining the quantum state of a particle in one degree of freedom. Generalizations to many degrees of freedom are also possible. Hereafter, we put $\hbar = 1$.

Given the density operator $\hat{\rho}$ of a particle quantum state, $\hat{\rho} = \hat{\rho}^{\dagger}$, $\operatorname{Tr}\hat{\rho} = 1$, and $\hat{\rho} \ge 0$, the symplectic tomogram of $\hat{\rho}$ is defined by:

$$\mathcal{W}(X,\mu,\nu) = \operatorname{Tr}[\hat{\rho}\,\delta(X\,\hat{1}-\mu\hat{Q}-\nu\hat{P})], \qquad X,\mu,\nu\in\mathbb{R}.$$
 (124)

Here \hat{Q} and \hat{P} are the position and momentum operators. The Dirac delta-function with operator arguments is defined by the standard Fourier integral,

$$\delta(X\,\hat{1} - \mu\hat{Q} - \nu\hat{P}) = \int e^{-ik(X\,\hat{1} - \mu\hat{Q} - \nu\hat{P})} \frac{dk}{2\pi}.$$

The symplectic tomogram $\mathcal{W}(X, \mu, \nu)$ has the properties which follow from its definition by using the known properties of delta-function, namely:

i. Nonnegativity:

$$\mathcal{W}(X,\mu,\nu) \ge 0 \tag{125}$$

(this holds by observing that the trace of the product of two positive operators is a positive number).

ii. Normalization:

$$\int \mathcal{W}(X,\mu,\nu)dX = 1.$$
 (126)

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iii. Homogeneity:

$$\mathcal{W}(\lambda X, \lambda \mu, \lambda \nu) = \frac{1}{|\lambda|} \mathcal{W}(X, \mu, \nu).$$
(127)
However, the three above properties are by no means sufficient to determine the quantum character of a tomographic function $f(X, \mu, \nu)$. For instance, consider

$$f(X,\mu,\nu) = \exp\left(-\frac{X^2}{2(\mu^2 + \nu^2)}\right) \frac{5(\mu^2 + \nu^2) - X^2}{\sqrt{2(\mu^2 + \nu^2)^3}}.$$
 (128)

Despite the uncertainty relations are satisfied by such a function, f is not a quantum tomogram because $\left\langle \hat{P}^2 \right\rangle = \left\langle \hat{Q}^2 \right\rangle = -1/2$, as it can be checked using

$$\left\langle \hat{P}^{2} \right\rangle = \int X^{2} \left[f(X,\mu,\nu) \right]_{\mu=0,\nu=1} dX$$
 (129)

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and analogously for $\left< \hat{Q}^2 \right>$.

On the other hand, it is easy to see that formula (124) has an inverse:

$$\hat{\rho} = \frac{1}{2\pi} \int \mathcal{W}(X,\mu,\nu) e^{i(X\,\hat{1}-\mu\hat{Q}-\nu\hat{P})} \, dX \, d\mu \, d\nu.$$
(130)

Thus the knowledge of the symplectic tomogram $\mathcal{W}(X, \mu, \nu)$ means that the density operator $\hat{\rho}$ is also known, more precisely, can be reconstructed. This correspondence between symplectic tomograms $\mathcal{W}(X, \mu, \nu)$ and density operators $\hat{\rho}$ gives the possibility to formulate the notion of quantum state using tomograms as the primary notion. However to make this idea precise, we need to formulate additional conditions to be satisfied by the function $\mathcal{W}(X,\mu,\nu)$ which are extra to the conditions (125)-(127) and which guarantee that by using the inversion formula (130) we get an operator with all the necessary properties of a density state. The general recipe to formulate these demands can be given by checking the nonnegativity condition of the integral:

$$\int \mathcal{W}(X,\mu,\nu)e^{i(X\hat{1}-\mu\hat{Q}-\nu\hat{P})}\,dX\,d\mu\,d\nu \ge 0. \tag{131}$$

It means that for a given function $\mathcal{W}(X, \mu, \nu)$ satisfying the conditions (125)-(127) one has to check the nonnegativity of the operator (131), thus if the inequality (131) holds the function $\mathcal{W}(X,\mu,\nu)$ is the symplectic tomogram of a quantum state, however it must be realized that this is not an operative procedure. Below we formulate the conditions for a function $\mathcal{W}(X, \mu, \nu)$ to be a tomogram of a quantum state avoiding the integrations in eq. (131). As anticipated in the introduction, to be able to use Naimark's results we have to deal with functions defined on a group. Thus, we have to show how symplectic tomograms may be associated with the Weyl-Heisenberg group. In doing this we can exploit results where the theorems on properties of diagonal matrix elements of unitary representations provide the key to construct tomograms which represent quantum states.

Tomographic probability measures

To get a mathematical formulation of the tomographic picture we invoke the spectral theory of Hermitian operators, which moreover will provide us with a probabilistic interpretation of the symplectic tomogram. We start rewriting the formal definition, eq. (124), for a quantum tomogram:

$$\mathcal{W}(X,\mu,\nu) = \operatorname{Tr}\left[\hat{\rho}\int e^{ik(X\,\hat{1}-\mu\hat{Q}-\nu\hat{P})}\frac{dk}{2\pi}\right] = \int e^{ikX}\operatorname{Tr}[\hat{\rho}e^{-ik(\mu\hat{Q}+\nu\hat{P})}]\frac{dk}{2\pi}$$
(132)

then we observe that

$$\mu \hat{Q} + \nu \hat{P} = S_{\mu\nu} \hat{Q} S^{\dagger}_{\mu\nu} \tag{133}$$

where

$$S_{\mu\nu} = \exp\left[\frac{i\lambda}{2}\left(\hat{Q}\hat{P} + \hat{P}\hat{Q}\right)\right] \exp\left[\frac{i\theta}{2}\left(\hat{Q}^2 + \hat{P}^2\right)\right], \quad (134)$$

with

$$\mu = e^{\lambda} \cos \theta , \nu = e^{-\lambda} \sin \theta . \tag{135}$$

In other words, by acting with the unitary operators $S_{\mu\nu}$ on the position operator \hat{Q} we get out the iso-spectral family of hermitian operators

$$X_{\mu\nu} = \mu \hat{Q} + \nu \hat{P}.$$

This family is a symplectic tomographic set.

To any operator of this family is associated a projector valued measure $\Pi_{\mu\nu}$ on the σ -algebra of Borel sets on the real line:

$$\mu \hat{Q} + \nu \hat{P} = \int \lambda \, d \, \Pi_{\mu\nu}(\lambda).$$

Given any density state $\hat{\rho}$, the projector valued measure $\Pi_{\mu\nu}$ yields a normalized probability measure $m_{\rho,\mu\nu}$ on the Borel sets $E \in Bo(\mathbb{R})$ of the real line:

$$m_{\rho,\mu\nu}(E) = \text{Tr}[\hat{\rho} \Pi_{\mu\nu}(E)]; \quad m_{\rho,\mu\nu}(\mathbb{R}) = 1.$$
 (136)

We recall that $m_{\rho,\mu\nu}(E)$ is the probability that a measure of the observable $\mu \hat{Q} + \nu \hat{P}$ in the state $\hat{\rho}$ is in E. All these measures $m_{\rho,\mu\nu}$ are absolutely continuous with respect to the Lebesgue measure dX on the real line, so that densities $V_\rho(X,\mu,\nu)$ may be introduced such that

$$m_{\rho,\mu\nu}(E) = \int_E V_{\rho}(X,\mu,\nu) \, dX.$$
 (137)

We can write

$$\operatorname{Tr}\left(\hat{\rho}e^{-i\lambda(\mu\hat{Q}+\nu\hat{P})}\right) = \operatorname{Tr}\left(\hat{\rho}S_{\mu\nu}e^{-i\lambda\hat{Q}}S_{\mu\nu}^{\dagger}\right) = \int e^{-i\lambda X} V_{\rho}(X,\mu,\nu) dX$$
(138)

so that

$$\mathcal{W}(X,\mu,\nu) = \int e^{ikX} \operatorname{Tr}[\hat{\rho}e^{-ik(\mu\hat{Q}+\nu\hat{P})}]\frac{dk}{2\pi}$$
(139)
$$= \int e^{ikX} e^{-ikX'}V_{\rho}(X',\mu,\nu)dX'\frac{dk}{2\pi}$$

$$= \int \delta(X-X')V_{\rho}(X',\mu,\nu)dX' = V_{\rho}(X,\mu,\nu).$$

In other words we have shown that the symplectic tomogram $\mathcal{W}(X,\mu,\nu)$ of a given state $\hat{\rho}$ is nothing but the density $V_{\rho}(X,\mu,\nu)$ of the probability measure associated to the state by means of the symplectic tomographic set. The tomographic character of the family of observables $X_{\mu\nu}$ is contained in the possibility of reconstructing any state out of the corresponding probability measures by means of the previous reconstructing formula. By using eqs. (138) and (130), we get

$$\hat{\rho} = \frac{1}{2\pi} \int \text{Tr}[\hat{\rho}e^{i(\mu\hat{Q}+\nu\hat{P})}] e^{-i(\mu\hat{Q}+\nu\hat{P})} d\mu \, d\nu, \qquad (140)$$

moreover

$$\frac{1}{2\pi} \int \text{Tr}[e^{i(\mu\hat{Q}+\nu\hat{P})}] e^{-i(\mu\hat{Q}+\nu\hat{P})} d\mu d\nu = \hat{1}.$$
 (141)

The presence of the Weyl operators $D(\mu, \nu) = e^{i(\mu \hat{Q} + \nu \hat{P})}$ suggests that we are dealing with projective representations of the Abelian vector group. We shall take up group theoretical aspects in next section.

A group theoretical description of quantum tomograms The probabilistic interpretation above allows to consider the tomographic description of quantum states as a picture of quantum mechanics on the same footing as other well known representations, like Schrödinger, Heisenberg and Wigner-Weyl for instance. Thus, to be an alternative picture of quantum mechanics we need criteria to recognize a function $f(X, \mu, \nu)$ as a tomogram of a quantum state. For this, the use of the reconstruction formula to check if the obtained operator is a density operator would be unsatisfactory, mainly because this check requires to switch from tomographic to Schrödinger picture. In other words, we would like to establish self-contained criteria for a function to be a quantum tomogram. More precisely, we have to address the following problem: given a tomogram-like function $f(X, \mu, \nu)$, that is a function with the above properties eqs. (125)-(127) of a tomogram, what are the necessary and sufficient conditions to recognize f as a quantum tomogram?

To this aim we begin to observe that in the characteristic tomographic function

$$\operatorname{Tr}[\hat{\rho}e^{i(\mu\hat{Q}+\nu\hat{P})}] = \operatorname{Tr}[\hat{\rho}D(\mu,\nu)]$$
(142)

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a projective representation of the translation group appears. This projective representation can be lifted to a true unitary representation of the Weyl-Heisenberg group by means of a central extension of the translation group. Such central extension defines the Weyl-Heisenberg group WH(2) whose elements are denoted by (μ, ν, t) and the group law reads:

$$\begin{split} (\mu,\nu,t)\circ(\mu',\nu',t')&=(\mu+\mu',\nu+\nu',t+t'+\frac{1}{2}\omega((\mu,\nu),(\mu',\nu'))),\\ (143) \end{split}$$
 where $\omega((\mu,\nu),(\mu',\nu'))&=\mu\nu'-\nu\mu' \text{ denotes the symplectic form on }\mathbb{R}^2. \end{split}$

The nontrivial unitary irreducible representations of the Weyl-Heisenberg group are provided by the expression:

$$U_{\gamma}(\mu,\nu,t) = D_{\gamma}(\mu,\nu)e^{i\gamma tI}.$$
(144)

where γ is a non-vanishing real number. In what follows we will set $\gamma=1.$ Hence we immediately observe that

$$\operatorname{Tr}[\hat{\rho}D(\mu,\nu)] = e^{-it}\operatorname{Tr}[\hat{\rho}U(\mu,\nu,t)]$$
(145)

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where the function $\operatorname{Tr}[\hat{\rho}U(\mu,\nu,t)]$ is of positive type.

For convenience we recall the definition of functions of positive type. Given a group G a function $\varphi(g)$ on G ($g \in G$) is of positive type, or definite positive, if for any *n*-tuple of group elements $(g_1, g_2, ..., g_n)$ the matrix

$$M_{jk} = \varphi(g_j g_k^{-1}) \qquad j, k = 1, 2, ..., n,$$
(146)

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is positive semi-definite for any $n \in \mathbb{N}$, or in other words, if for any finite family of elements $g_1, g_2, \ldots, g_n \in G$ and for any family of complex numbers ξ_1, \ldots, ξ_n , we have $\sum_{j,k=1}^n \bar{\xi_j} \xi_k \varphi(g_j g_k^{-1}) \ge 0$, for any n.

Moreover, a simple computation shows that given any unitary representation U(g) of G and a state ρ , $\operatorname{Tr}[\hat{\rho}U(g)]$ is a group function of positive type. *Viceversa* any positive type group function $\varphi(g)$ can be written in the form

$$\operatorname{Tr}[\hat{\rho}_{\xi}U(g)] = \langle \xi, U(g)\xi \rangle, \tag{147}$$

where U(g) is a unitary representation and $|\xi\rangle$ is a cyclic vector in a suitable Hilbert space, obtained for instance by means of a GNS construction.

So, the positivity condition on the matrix introduced in (146) is a way to affirm that φ is associated with a state without making recourse to a representation.

Thus we can state the required condition:

A tomogram–like function $f(X,\mu,\nu)$ is a quantum tomogram, i.e., there exists a quantum state $\hat{\rho}$ such that $f(X,\mu,\nu) = \mathrm{Tr}[\hat{\rho}\,\delta(X\,\hat{1}-\mu\hat{Q}-\nu\hat{P})]$, if and only if its Fourier

transform evaluated at 1 may be written in the form

$$\int f(X,\mu,\nu)e^{iX}dX = e^{-it}\varphi_f(\mu,\nu,t),$$
(148)

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where $\varphi_f(\mu, \nu, t)$ is a positive definite function on the Weyl-Heisenberg group.

In fact if \mathcal{W} is a quantum tomogram, then because of eqs.(132) and (145) we have,

$$\int \mathcal{W}(X,\mu,\nu)e^{iX} dX = \operatorname{Tr}[\hat{\rho}D(\mu,\nu)]$$
$$= e^{-it}\operatorname{Tr}[\hat{\rho}U(\mu,\nu,t)] = e^{-it}\varphi(\mu,\nu,t), \quad (149)$$

where $\varphi(\mu, \nu, t)$ is a positive definite function on the Weyl–Heisenberg group.

Moreover, if we define $\psi(\mu, \nu) = \text{Tr}(\hat{\rho}D(\mu, \nu))$, then $\psi(\mu, \nu)$ is a function on the translation group considered as a quotient of the Weyl–Heisenberg group by the central element. It means that we are dealing with a projective representation and not a unitary representation like in Naimark's theorem eq.(147).

Then, we could ask about the properties enjoyed by the matrix \hat{M}_{jk} constructed using ψ instead of φ . If we denote as above by ω the 2-cocycle defining the projective representation, then we will say that \tilde{M}_{jk} is of ω -positive type, i.e.

$$\tilde{M}_{jk} = \psi((\mu_j, \nu_j)^{-1} \circ (\mu_k, \nu_k)) e^{i\frac{1}{2}\omega((\mu_k, \nu_k), (\mu_j, \nu_j)}$$
(150)

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is positive semi-definite.

This yields the corresponding condition:

A tomogram–like function $f(X.\mu,\nu)$ is a quantum tomogram if and only if its Fourier transform evaluated at 1 may be written in the form

$$\int f(X,\mu,\nu)e^{iX}dX = \psi_f(\mu,\nu)$$
(151)

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where $\psi_f(\mu,\nu)$ is a function of the translation group of $\omega-{\rm positive}$ type.

We observe that $\psi_f(\mu, \nu)$ may be at same time of positive and ω -positive type on the translation group. Then by Bochner theorem $\psi_f(\mu, \nu)$ is the Fourier transform of a probability measure on the phase space. In other words $f(X, \mu, \nu)$ is the (classical) Radon transform of such a probability measure, i.e. a classical tomogram. The tomogram of the ground state of the harmonic oscillator provides an example of the above situation. In that case, the GNS construction yields a Hilbert space of square integrable functions on phase space with respect to the measure provided by the Bochner theorem.

To finish this analysis let us notice that if ψ is a function of ω -positive type on the translation group, then the function $\varphi(\mu,\nu,t)=e^{it}\psi(\mu,\nu)$ will be a positive definite function on the Weyl-Heisenberg group $\mathcal{WH}(2)$ and, by Naimark's theorem, there will exist a unitary representation U of $\mathcal{WH}(2)$ and a cyclic state vector $|\xi\rangle$ such that $\varphi(\mu,\nu,t)=\langle\xi,U(\mu,\nu,t)\xi\rangle.$

On the other hand, $\psi(\mu, \nu)$ is obtained by $f(X, \mu, \nu)$, which is a tomogram of a quantum state $\hat{\rho}$. Up to a unitary transformation $\hat{\rho}$ will coincide with $\hat{\rho}_{\xi}$ iff it is a pure state. Notably, the purity of $\hat{\rho}$ can be expressed, with $v = (\mu, \nu)$, as:

$$\operatorname{tr}\hat{\rho}^{2} = \frac{1}{2\pi} \int \mathcal{W}(X, v) \mathcal{W}(Y, -v) e^{i(X+Y)} dX dY dv = \frac{1}{2\pi} \int_{\mathbb{R}^{2}} \left| \psi(v) \right|^{2} dv$$
(152)

so that the above condition can be stated as:

$$\int_{\mathbb{R}^2} |\psi(v)|^2 \, dv = 1 \tag{153}$$

Thick quantum tomography

We now turn our attention to "thick" tomographic maps, which is a more realistic approach for practical applications, because instead of marginals defined over lines, as in the classical Radon transform or in the transform on guadratic curves, it involves a "thick" window function Ξ . This is convoluted with the tomographic map and concentrates the marginals around some given background curves (that can be lines or quadrics), without resorting to a singular delta function. For example, if the weight function Ξ is a step function, it defines marginals along thick lines or thick quadratic curves. In the quantum case this amounts to replacing in the definition of the dequantizer $\hat{U}(x)$ the Dirac delta function by the weight function Ξ . For the symplectic quantum tomography one has the dequantizer

$$\hat{U}(X,\mu,\nu) = \Xi \left(X - \mu \hat{q} - \nu \hat{p} \right).$$
 (154)

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The new tomogram reads

$$w_{\Xi}(X,\mu,\nu) = \operatorname{Tr} \hat{\rho} \Xi \left(X - \mu \hat{q} - \nu \hat{p} \right), \qquad (155)$$

Using the Weyl map one obtains a thick tomogram for the Wigner function

$$w_{\Xi}(X,\mu,\nu) = \frac{1}{2\pi} \int W(p,q) \,\Xi \left(X - \mu q - \nu p\right) \,dp \,dq.$$
 (156)

The interesting property of the above formula (156) is that it can be inverted in completely analogy with the classical thick tomography. The thick tomogram can be expressed in terms of standard symplectic tomograms via a convolution formula

$$w_{\Xi}(X,\mu,\nu) = \int w(X,\mu,\nu) \,\Xi \,(X-Y) \,\,dY,$$
(157)

which leads to the explicit construction of the inverse transform.

Indeed, the inverse transform is obtained by means of a Fourier transform of the convolution integral

$$W(p,q) = \frac{\mathcal{N}_{\Xi}}{2\pi} \int w_{\Xi}(Y,\mu,\nu) \,\mathrm{e}^{i(X-\mu q-\nu p)} \,dX \,d\mu \,d\nu, \qquad (158)$$

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where

$$\mathcal{N}_{\Xi} = \frac{1}{\widetilde{\Xi}(-1)}, \qquad \widetilde{\Xi}(-1) = \int \Xi(z) e^{iz} dz.$$

In invariant form the state reconstruction is achieved by

$$\hat{\rho} = \frac{\mathcal{N}_{\Xi}}{2\pi} \int w_{\Xi}(X,\mu,\nu) \,\mathrm{e}^{i(X\mathbb{I}-\mu\hat{q}-\nu\hat{p})} \,dX \,d\mu \,d\nu.$$

The quantizer operator in thick symplectic tomography is

$$\hat{D}(X,\mu,\nu) = \frac{\mathcal{N}_{\Xi}}{2\pi} e^{i(X\mathbb{I}-\mu\hat{q}-\nu\hat{p})}.$$
(159)

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Now we consider a particular example of thick tomogram to illustrate the potentialities of the new method. If the weight function is a gaussian function

$$\Xi(z) = \frac{1}{\sqrt{2\pi\sigma^2}} \mathrm{e}^{-z^2/2\sigma^2}$$

which tends to the delta distribution in the $\sigma \rightarrow 0$ limit,

$$\lim_{\sigma \to 0} \Xi(z) = \delta(z),$$

the thick tomogram of the coherent states $|\alpha\rangle\langle\alpha|$ read

$$w_{\sigma}^{\alpha}(X,\mu,\nu) = \frac{1}{\sqrt{\pi(\mu^2 + \nu^2 + \sigma^2)}} e^{-(X-\bar{X})^2/(\mu^2 + \nu^2 + \sigma^2)},$$
 (160)

where

$$\bar{X} = \sqrt{2}\,\mu\,\mathrm{Re}\,\alpha + \sqrt{2}\,\nu\,\mathrm{Im}\,\alpha. \tag{161}$$

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For the vacuum state $|0\rangle\langle 0|$ the tomogram reads

$$w_{\sigma}^{\text{vac}}(X,\mu,\nu) = \frac{1}{\sqrt{\pi(\mu^2 + \nu^2 + \sigma^2)}} e^{-X^2/(\mu^2 + \nu^2 + \sigma^2)}, \quad (162)$$

The quantizer reads

$$\hat{D}_{\sigma}(X,\mu,\nu) = \frac{1}{2\pi} \mathrm{e}^{-\sigma^2/[2+i(X\mathbb{I}-\mu\hat{q}-\nu\hat{p})]}$$

and the dequantizer is given by

$$\hat{U}_{\sigma}(X,\mu,\nu) = \frac{1}{\sqrt{2\pi\sigma^2}} \mathrm{e}^{-[(X\mathbb{I}-\mu\hat{q}-\nu\hat{p})^2]/2\sigma^2}$$

One interesting property, that is preserved by the smoothing of the tomogram, is that the marginals $w_{\Xi}(X, \mu, \nu)$ are also probability distributions. In the limit $\sigma \to 0$, $\Xi(z) \to \delta(z)$, $\widetilde{\Xi}(-1) = 1$, $\mathcal{N}_{\Xi} = 1$.

Conclusions and outlooks

To conclude we resume the main results of our work. The symplectic tomographic probability distribution, considered as the primary concept of a particle quantum state alternative to the wave function or density matrix, was shown to be associated with a unitary representation of the Weyl-Heisenberg group.

This connection was used to formulate an autonomous conditions for the symplectic tomogram to describe quantum states using the positivity properties of the matrix M_{jk} of eq.(146) and connected with the diagonal elements of the unitary representation (positive type function $\varphi(g)$ on the group).

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The function $f(X, \mu, \nu)$, satisfying the necessary properties of tomographic probability distribution, i.e. non-negativity, homogeneity and normalization, was shown to be a quantum tomogram iff its Fourier transform in the quadrature variable X can be written in the form of eq.(148) as the product of a positive type function on the Weyl-Heisenberg group and a phase factor associated with central elements of the group.

By using the quantum Radon anti-transform eq.(130), this condition guarantees that the function $f(X, \mu, \nu)$ provides a density state, so that f is the symplectic tomogram of a quantum state. The criterion, formulated in terms of positivity properties of a group function obtained from the tomographic function, is not easy to implement operatively. Nevertheless, it is simpler than the criterion based on checking the non negativity of the operator given by the quantum Radon anti transform.

Also, we have shown that the purity of the quantum state can be expressed as the square of the L_2 -norm of that positive group function, which is obtained by tomograms measured directly in optical experiments, without considering density matrices or Wigner functions.

As a spin-off we have shown that the notion of h-positivity may be subsumed under the notion of positivity for a centrally extended group.

We have considered tomograms associated with the Weyl-Heisenberg group. It can be shown how to deal with the tomographic picture for general Lie groups and for finite groups. In this connection, the C^* -algebraic approach to quantum mechanics and its counterpart in terms of tomograms can be elaborated.

To resume, quantum mechanics can be formulated using the fair probability distribution as a concept of the quantum state instead of the wave function and density operator. This provides the possibility to obtain the quantum evolution and energy-level equations for the fair probability distributions like the evolution equations in classical statistical mechanics.

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