Stability of Gapped Phases

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Joint work with Justyna Pytel - Oregon State University.

Benasque 2012.

The challenge...

Prove that gapped quantum phases are stable at zero temperature.

What does it mean to be a stable phase?

Opening the gap.

Splitting groundstate degeneracy.

Example

1. Consider the $N \times N$ **Ising Hamiltonian** H_N and a **perturbation** Δ_N:

$$H_N = -\sum_{p \sim q} \sigma_p^z \otimes \sigma_q^z, \quad \Delta_N = \delta_N \sum_p \sigma_p^z, \quad \delta_N \sim 1/N^2.$$

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4. **Good classical memory, bad quantum memory:** Encoded state $\big| + \big\rangle = \big|000 \ldots 0\big\rangle + \big|111 \ldots 1\big\rangle$ flips to $\big| - \big\rangle = \big|000 \ldots 0\big\rangle - \big|111 \ldots 1\big\rangle$, since $e^{itH_N'}\big| + \big\rangle \sim \big|000 \ldots 0\big\rangle + e^{it}\big|111 \ldots 1\big\rangle$.

So... how do we protect against splitting of the groundstates?

Topological Quantum Order.

Macroscopic indistinguishability of groundstates.

TQO: No two groundstates can be distinguished locally:

$$\langle\psi_0| \, O_{local} \, |\psi_0\rangle = \langle\phi_0| \, O_{local} \, |\phi_0\rangle = c(O_{local}). \qquad (1)$$

Note: Every Hamiltonian with a **unique groundstate** satisfies the above condition trivially - **nothing to distinguish**...

Examples of Topological Quantum Order and its absence.

Ising vs. Kitaev: 0-1.

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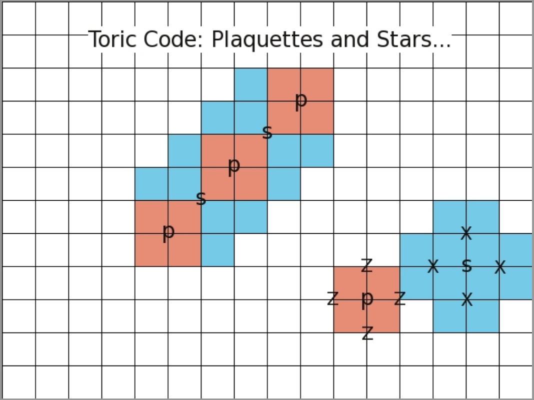
4. **Kitaev's Toric Code**, a four-fold degenerate groundstate subspace, **satisfies the TQO condition**.

So... is the toric code stable?

The toric code...



Plaquettes have $B_p = \prod_{j \in \mathrm{edges(p)}} \sigma_j^z$ and **stars** have $A_s = \prod_{j \in \mathrm{star(s)}} \sigma_j^x$.
Note that $[\mathbf{A_s}, \mathbf{B_p}] = \mathbf{0}$ for all s and p.

Kitaev's Toric Code

The toric code is a stable topological phase.

Example

The standard toric code model is defined by the Hamiltonian:

$$H_{tc} = -\sum_p B_p - \sum_s A_s,$$

where qubits live on the edges of a lattice on a **torus**.

- Lowest-energy subspace P_0 (toric code) has $B_p = 1$, $A_s = 1$ for all p and s. That is, for any ground state $\left|\Psi_0\right\rangle$ we have:

$$B_p\left|\Psi_0\right\rangle = A_s\left|\Psi_0\right\rangle = \left|\Psi_0\right\rangle. \quad \textbf{stabilizing property} \tag{2}$$

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- There are 4 such groundstates on the **torus**, distinguished only through **macroscopic string operators**.

Closing the spectral gap.

Breaking the Toric Code

Example

$$H_0 = -\sum_s A_s - \sum_{p \sim p'} B_p \cdot B_{p'} - B_0.$$

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- **Perturb** H_0 by adding the vanishing **B-field**: $\delta_\Lambda \sum_p B_p$, with $\delta_\Lambda \sim 1/|\Lambda|$ and Λ the lattice (torus) on which H_0 is defined.
- The subspace $A_s = 1, \mathbf{B_p = -1}$, becomes **new groundstate space**.

Distinguishability implies instability!

Hamiltonians are **unstable** because **local order parameters** can act as perturbations to **split the groundstate subspace**, or **close the gap** between **groundstates** and **local, low-energy excitations**.

But, can we prove that?

Yes! For an important class of gapped Hamiltonians!

Frustration-free Hamiltonians.

Definition

1. We say $\mathbf{H_0} = \sum_{\mathbf{u}\in\mathbf{\Lambda}} \mathbf{Q_u}$ is **frustration-free**, if the groundstate subspace P_0 satisfies:

$$\mathbf{Q_u P_0} = \lambda_{\mathbf{u}} \mathbf{P_0}, \quad \forall \mathbf{u} \in \mathbf{\Lambda} \subset \mathbb{Z}^{\mathbf{d}}$$

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2. NOT all COMMUTING Hamiltonians are FRUSTRATION-FREE!
   Take 3 **qubits on the vertices** $\{u, v, w\}$ **of a triangle**, with Ising Hamiltonian $\mathbf{H}_{\triangle} = \sigma^{\mathbf{z}}_{\mathbf{u}} \otimes \sigma^{\mathbf{z}}_{\mathbf{v}} + \sigma^{\mathbf{z}}_{\mathbf{v}} \otimes \sigma^{\mathbf{z}}_{\mathbf{w}} + \sigma^{\mathbf{z}}_{\mathbf{u}} \otimes \sigma^{\mathbf{z}}_{\mathbf{w}}$.
   Since $\sigma^z_u \otimes \sigma^z_w = (\sigma^z_u \otimes \sigma^z_v) \cdot (\sigma^z_v \otimes \sigma^z_w)$, it is **impossible to have common groundstate** for all three terms.

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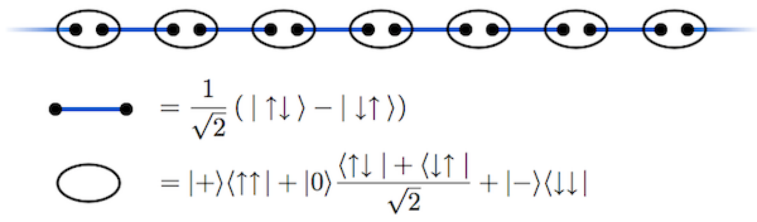
3. NOT all FRUSTRATION-FREE Hamiltonians are COMMUTING!
   Parent Hamiltonians of MPS and PEPS (e.g. AKLT).

The AKLT model.

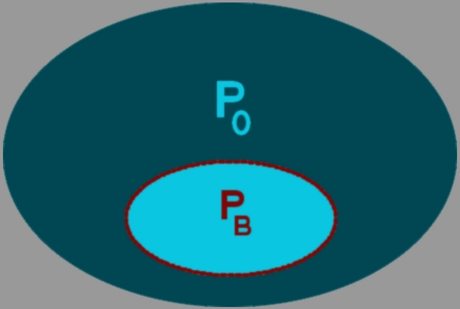


The groundstate of the spin-1 **AKLT** Hamiltonian $H = \sum_j Q^{(2)}_{j,j+1}$.
Each Hamiltonian term $Q^{(2)}_{j,j+1}$ projects onto the spin-2 subspace of two spin-1s.

- Any state with reduced density matrix in the spin-0, or spin-1 state of two neighboring spins, will be a groundstate.
- For periodic chains, the above state is the **unique groundstate**. The total spin of two neighboring particles cannot be larger than 1, since the singlet connecting them has total spin 0.

TQO is not enough for stability. Now what?

Focus on Local Groundstates



Every frustration-free Hamiltonian H_0 on $\Lambda \subset \mathbb{Z}^d$ is the extension of another frustration-free Hamiltonian H_B on $B \subset \Lambda$. This implies that the **local** groundstate projector P_B contains the **global** P_0; that is, $\mathbf{P_B P_0 = P_0}$.

Stability needs...

Local Groundstate Indistinguishability.

Local-TQO: $\mathbf{H_0}$ satisfies **Local-TQO**, if there exists a **rapidly-decaying function** $\Delta_0(\ell)$, such that:

$$\|\rho_A^1 - \rho_A^2\|_1 \leq \Delta_0(\ell), \tag{3}$$

where $\rho_A^i = \mathrm{Tr}_{A^c} |\psi^i_{B(\ell)}\rangle\langle\psi^i_{B(\ell)}|$, $i = 1, 2$ and

$$P_{B(\ell)}\big|\psi^i_{B(\ell)}\big\rangle = \big|\psi^i_{B(\ell)}\big\rangle.$$

Here, $B = B_u(r)$, $r \leq L^*$ and $B(\ell) := B_u(r+\ell)$.

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- So focus on parent Hamiltonians **constructed using only local information of the global groundstate** subspace. Just like parent Hamiltonians of MPS and PEPS.
- Parent Hamiltonians of MPS, such as the AKLT model, satisfy **Local-TQO** with **exponentially-decaying** Δ_0.

Local Gaps.

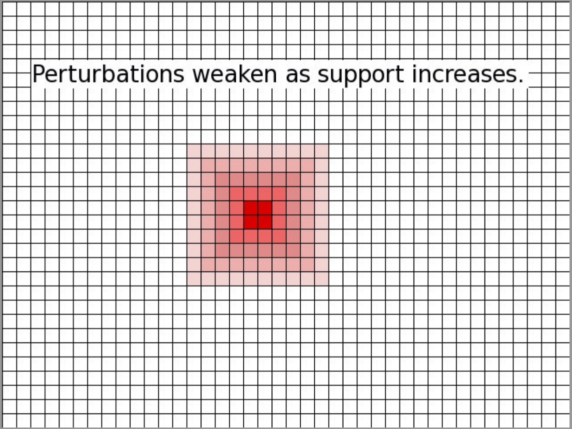
Definition

Local-Gap: We define $\mathbf{H_0}$ to be **locally gapped** w.r.t. a function $\gamma(r)$, if $\mathbf{H_B} \geq \gamma(\mathbf{r})(\mathbf{1} - \mathbf{P_B})$, where $B = B_u(r)$. If $\gamma(r)$ decays at most **polynomially**, we say that H_0 satisfies the **Local-Gap** condition.

Open Problem: Is the **Local-Gap** condition always satisfied if H_0 is a sum of local projections with frustration-free ground-state and a spectral gap?

Open Problem: Does **Local-TQO** $\Longrightarrow$ **Local-Gap** in this setting?

Decaying perturbations...



For each site $u \in \Lambda$, we allow perturbations supported on $B_u(r)$. As the radius of the support increases, the norm of the perturbation decreases rapidly.

The Perturbations: Local decomposition and strength.

Definition

We say that V **has strength** J **and rapid decay** f, if we can write

$$V = \sum_{u \in \Lambda} V_u, \quad V_u := \sum_{r \geq 0} V_u(r),$$

such that $V_u(r)$ has support on $B_u(r)$ and $\|V_u(r)\| \leq J\, f(r)$, $r \geq 0$.

The main result.

Theorem

- *Let H_0 be a* **frustration-free** *Hamiltonian satisfying* **Local-TQO** *and* **Local-Gap** *with decay given by $\Delta_0(r)$ and $\gamma(r)$, respectively.*

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- *Let V be a* **strength** J *perturbation, with* **decay** $f(r)$.
- *Then, $H_0 + V$ has* **spectral gap bounded below** *by*

$$(1 - c_0 J)\gamma - 2J\,\delta(L),$$

where $c_0 = \sum_{r=1}^{L} r^d \cdot \left(w(r)/\gamma(r)\right)$, $w(r)$ has rapid decay related to the decay rate of $f(r)$ and $\delta(L)$ has rapid decay related to $\Delta_0(L^*)$.

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- **Groundstate splitting** *is bounded by $J \cdot \delta(L)$.*

Questions?

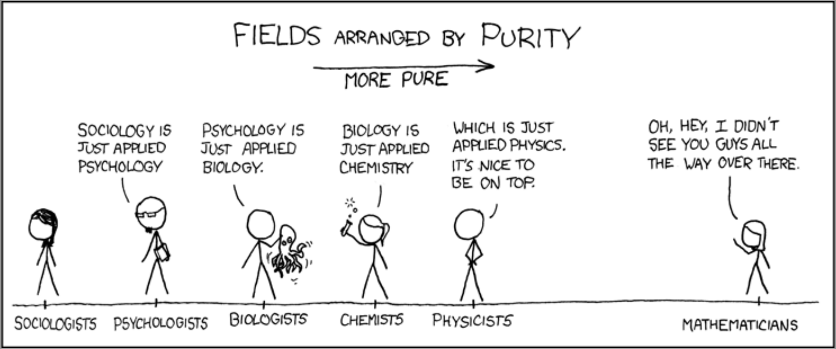

Figure: Xkcd wisdom.

Thank you!

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- **Decompose** $V'_u = W_u + \Delta_u + E_u \mathbb{1}$, where:
  - $\Delta_u = (V'_u - E_u)P_0$,
  - $W_u = (V'_u - E_u)(1 - P_0)$ and
  - E_u is a constant energy. (energy-shift)

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- But, $H_0 + \sum_u V'_u - E \cdot \mathbb{1}$ and $H_0 + sV$ have **equal spectral gap**! (unitary transformation + energy-shift)
- **Contradiction!** $H_0 + s^* V$ has gap at most $\gamma/2$, by assumption! So, $s^* = 1$, for $J \leq 1/(5c_0)$.