On Non-commutative Corrections of Gravitational Energy in Teleparallel Gravity

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- Peierls: Landau problem.
- First formalism: Snyder (1947).

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- Non-commutative geometry: application in non-abelian theories, gravitation, in standard model and in Hall effect.
- Certainly the discover that the dynamics of an open string can be explained by non-commutative gauge theories at specific limits has contributed to this renewed interest of the scientific community in the topic.

Introduction

- Non-commutative geometry: application in non-abelian theories, gravitation, in standard model and in Hall effect.
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Introduction and Motivation

- Teleparallel gravity allows for the definition of quantities that are physically of interest, such as the gravitational energy-momentum and angular momentum tensors which are well behaved when compared to attempts made in the context of general relativity. In addition, none of such expressions obtained in the realm of general relativity are dependant on the reference frame, which is certainly not a desirable feature for energy, momentum and angular momentum.

Introduction and Motivation

- Therefore we have two successful theories described above and a natural forward step is combine both of them. Then here our aim is study the Teleparallelism Equivalent to General Relativity in the non-commutative space-time context.
- Notation: space-time indices $\mu, \nu, ...$ and SO(3,1) indices $a, b, ...$ run from 0 to 3. Time and space indices are indicated according to $\mu = 0, i, \ \ a = (0), (i)$. The tetrad field is denoted by $e^a{}_\mu$ and the determinant of the tetrad field is represented by $e = \det(e^a{}_\mu)$.

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Non-Commutative Geometry

- $$[\widehat{x}^{\mu}, \widehat{x}^{\nu}] = i\theta^{\mu\nu}\,,$$

  where $\theta^{\mu\nu}$ is an anti-symmetric constant tensor.

- $$\Delta\widehat{x}^{\mu}\Delta\widehat{x}^{\nu} \geq \frac{1}{2}|\theta^{\mu\nu}|\,,$$

  which suggest that effects due to the non-commutativity into spacetime turn out to be relevant at scales of the order of $\sqrt{|\theta^{\mu\nu}|}$.

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Non-Commutative Geometry

Usually the non-commutativity is introduced by means the use of the Moyal product defined as

$$\begin{aligned} f(x)\star g(x) &\equiv \exp\left(\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\right) f(x)g(y)|_{y\to x} \\ &= f(x)g(x)+\frac{i}{2}\theta^{\mu\nu}\partial_\mu f\partial_\nu g \\ &+ \frac{1}{2!}\left(\frac{i}{2}\right)^2\theta^{\mu_1\nu_1}\theta^{\mu_2\nu_2}(\partial_{\mu_1}\partial_{\mu_2}f)(\partial_{\nu_1}\partial_{\nu_2}g)+\cdots \end{aligned} \quad (1)$$

with a constant $\theta^{\mu\nu}$.

Teleparallel Gravity

The Teleparallelism Equivalent to General Relativity (TEGR) is constructed out of tetrad fields (instead of a metric tensor) in the Weitzenböck (or Cartan) space-time, in which it is possible to have distant (or absolute) parallelism. The tetrad field and metric tensor are related by

$$\begin{aligned} g^{\mu\nu} &= e^{a\mu} e_a{}^{\nu} ; \\ \eta^{ab} &= e^{a\mu} e^{b}{}_{\mu} , \end{aligned} \tag{2}$$

where $\eta^{ab} = diag(-+++)$ is the metric tensor of Minkowski space-time.

Teleparallel Gravity

Let us start with a manifold endowed with a Cartan connection, $\Gamma_{\mu\lambda\nu} = e^{a}{}_{\mu}\partial_{\lambda}e_{a\nu}$, which can be written as

$$\Gamma_{\mu\lambda\nu} = {}^{0}\Gamma_{\mu\lambda\nu} + K_{\mu\lambda\nu}\,, \tag{3}$$

where ${}^{0}\Gamma_{\mu\lambda\nu}$ are the Christoffel symbols and $K_{\mu\lambda\nu}$ is given by

$$K_{\mu\lambda\nu} \;=\; \frac{1}{2}(T_{\lambda\mu\nu} + T_{\nu\lambda\mu} + T_{\mu\lambda\nu})\,. \tag{4}$$

$K_{\mu\lambda\nu}$ is the contortion tensor defined in terms of the torsion tensor constructed from the Cartan connection.

Teleparallel Gravity

The torsion tensor is $T_{\mu\lambda\nu} = e_{a\mu} T^{a}{}_{\lambda\nu}$, with

$$T^{a}{}_{\lambda\nu} = \partial_\lambda e^{a}{}_{\nu} - \partial_\nu e^{a}{}_{\lambda}\,. \qquad (5)$$

The curvature tensor obtained from $\Gamma_{\mu\lambda\nu}$ is identically zero which, using (3), leads to

$$eR(e) \equiv -e(\frac{1}{4}T^{abc}T_{abc} + \frac{1}{2}T^{abc}T_{bac} - T^{a}T_{a}) + 2\partial_\mu(eT^\mu)\,, \qquad (6)$$

where $R(e)$ is the scalar curvature of a Riemannian manifold in terms of the tetrad field and $T^\mu = T^{b}{}_{b}{}^{\mu}$.

Teleparallel Gravity

Since the divergence term in eq. (6) does not contribute with the field equations, the Teleparallel Lagrangian density is

$$\begin{aligned} \mathfrak{L}(e_{a\mu}) &= -\kappa\, e\,(\frac{1}{4}T^{abc}T_{abc} + \frac{1}{2}T^{abc}T_{bac} - T^aT_a) - \mathfrak{L}_M \\ &\equiv -\kappa\, e\Sigma^{abc}T_{abc} - \mathfrak{L}_M \;, \end{aligned} \tag{7}$$

where $\kappa = 1/(16\pi)$.

Teleparallel Gravity

$\mathfrak{L}_M$ is the Lagrangian density of matter fields and Σ^{abc} is given by

$$\Sigma^{abc} = \frac{1}{4}(T^{abc} + T^{bac} - T^{cab}) + \frac{1}{2}(\eta^{ac}T^{b} - \eta^{ab}T^{c}) \, , \tag{8}$$

with $T^a = e^a{}_\mu T^\mu$.It is important to note that the Einstein-Hilbert Lagrangian density is equivalent to its teleparallel version given by eq. (7). Thus both theories share the same results concerning dynamics and, up to now, observational data.

Teleparallel Gravity

Performing a variational derivative of the Lagrangian density with respect to $e^{a\mu}$, which are the dynamical variables of the system, the field equations are

$$e_{a\lambda}e_{b\mu}\partial_\nu(e\Sigma^{b\lambda\nu}) - e(\Sigma^{b\nu}{}_a T_{b\nu\mu} - \frac{1}{4}e_{a\mu}T_{bcd}\Sigma^{bcd}) = \frac{1}{4\kappa}eT_{a\mu}\,, \quad (9)$$

where $T_{a\mu} = e_a{}^\lambda T_{\mu\lambda} = \frac{1}{e}\frac{\delta\mathcal{L}_M}{\delta e^{a\mu}}$ is the energy-momentum tensor of matter fields.

Teleparallel Gravity

The field equations can be rewritten as

$$\partial_\nu(e\Sigma^{a\lambda\nu}) = \frac{1}{4\kappa} e\, e^a{}_\mu (t^{\lambda\mu} + T^{\lambda\mu}) \,, \tag{10}$$

where $t^{\lambda\mu}$ is defined by

$$t^{\lambda\mu} = \kappa(4\Sigma^{bc\lambda} T_{bc}{}^{\mu} - g^{\lambda\mu}\Sigma^{bcd}T_{bcd}) \,. \tag{11}$$

Since $\Sigma^{a\lambda\nu}$ is skew-symmetric in the last two indices, it follows that

$$\partial_\lambda\partial_\nu(e\Sigma^{a\lambda\nu}) \equiv 0 \,. \tag{12}$$

Teleparallel Gravity

Thus we get

$$\partial_\lambda(et^{a\lambda} + eT^{a\lambda}) = 0 \tag{13}$$

which yields the continuity equation

$$\frac{d}{dt}\int_V d^3x\, e\, e^a{}_\mu(t^{0\mu} + T^{0\mu}) = -\oint_S dS_j\, \left[e\, e^a{}_\mu(t^{j\mu} + T^{j\mu})\right] .$$

It should be noted that the above expression works as a conservation law for the sum of energy-momentum tensor of matter fields and for the quantity $t^{\lambda\mu}$. Thus $t^{\lambda\mu}$ is interpreted as the energy-momentum tensor of the gravitational field.

Teleparallel Gravity

Therefore, one can write the total energy-momentum contained in a three-dimensional volume V of space as

$$P^a = \int_V d^3x \, e \, e^a{}_\mu (t^{0\mu} + T^{0\mu}) \,. \tag{14}$$

It is worth to note that the above expression is invariant under coordinate transformation and transforms like a vector under Lorentz transformations. Such features are desirable and expected for a true energy-momentum vector.

Non-commutative Corrections

In this section we shall start with Schwarzschild space-time, described by the following line element

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \,, \tag{15}$$

where M is the mass of font.

Non-commutative Corrections

Then a tetrad field adapted to an observer at rest at spatial infinity, which yields the above metric, is

$$e^a{}_\mu = \begin{bmatrix} \sqrt{-g_{00}} & 0 & 0 & 0 \\ 0 & \sqrt{g_{11}}\sin\theta\cos\phi & \sqrt{g_{22}}\cos\theta\cos\phi & -\sqrt{g_{33}}\sin\phi \\ 0 & \sqrt{g_{11}}\sin\theta\sin\phi & \sqrt{g_{22}}\cos\theta\sin\phi & \sqrt{g_{33}}\cos\phi \\ 0 & \sqrt{g_{11}}\cos\theta & -\sqrt{g_{22}}\sin\theta & 0 \end{bmatrix} . \tag{16}$$

Non-commutative Corrections

The non-commutativity is introduced by means of the Moyal product, defined in eq. (1), between two tetrad field, given by

$$\tilde{g}_{\mu\nu} = \frac{1}{2}\left(e^{a}{}_{\mu} \star e_{a\nu} + e^{a}{}_{\nu} \star e_{a\mu}\right) .$$

Such a procedure could seems ad-hoc, however this is well established.

Non-commutative Corrections

Thus the new components of the metric tensor due to the non-commutativity of space-time are written in terms of the old ones, up to second order in $\theta^{\mu\nu}$, as

$$\begin{aligned}
\tilde{g}_{00} &= g_{00}\,, \\
\tilde{g}_{11} &= g_{11} + \frac{1}{4}\theta^{23} \\
&\times \left[\theta^{23} g_{11}\cos(2\theta) + \theta^{13}\sin(2\theta)\left(\frac{dg_{11}}{dr}\right)\right] \\
&+\frac{1}{8}\left[(\theta^{13}\sin\theta)^2 + (\theta^{12})^2\right]\left(\frac{d^2 g_{11}}{dr^2}\right),
\end{aligned}$$

Non-commutative Corrections

$$\begin{aligned}\tilde{g}_{22} &= g_{22} - \frac{1}{4}\theta^{23}\left[\theta^{23} g_{22}\cos(2\theta) + \theta^{13}\sin(2\theta)\left(\frac{dg_{22}}{dr}\right)\right] \\ &\quad + \frac{1}{8}\left[(\theta^{13}\cos\theta)^2 + (\theta^{12})^2\right]\left(\frac{d^2 g_{22}}{dr^2}\right),\end{aligned}$$

Non-commutative Corrections

$$
\begin{aligned}
\tilde{g}_{33} \quad = \quad & g_{33} + \frac{1}{8}\left(\theta^{13}\frac{\partial}{\partial r} + \theta^{23}\frac{\partial}{\partial \theta}\right)^2 g_{33} \\
& + \frac{(\theta^{12})^2}{32 g_{33}^2}\left[\left(\frac{\partial^2 g_{33}}{\partial \theta^2}\right)\left(\frac{\partial g_{33}}{\partial r}\right)^2\right. \\
& + 2\, g_{33}\left(\frac{\partial^2 g_{33}}{\partial \theta \partial r}\right)^2 + \left(\frac{\partial g_{33}}{\partial \theta}\right)^2\left(\frac{\partial^2 g_{33}}{\partial r^2}\right) \\
& - 2\, g_{33}\left(\frac{\partial^2 g_{33}}{\partial \theta^2}\right)\left(\frac{\partial^2 g_{33}}{\partial r^2}\right) \\
& \left. - 2\left(\frac{\partial g_{33}}{\partial \theta}\right)\left(\frac{\partial g_{33}}{\partial r}\right)\left(\frac{\partial^2 g_{33}}{\partial \theta \partial r}\right)\right],
\end{aligned}
$$

Non-commutative Corrections

$$\begin{aligned}\tilde{g}_{12} \quad = \quad & \left(\frac{\sin(2\theta)}{32\, g_{22}^{3/2} g_{11}^{3/2}}\right)\left\{-\frac{1}{2}(\theta^{13})^2\left[g_{22}\left(\frac{dg_{11}}{dr}\right) - g_{11}\left(\frac{dg_{22}}{dr}\right)\right]^2\right. \\ & +(\theta^{13})^2 g_{11} g_{22}\left[g_{22}\left(\frac{d^2 g_{11}}{dr^2}\right) + g_{11}\left(\frac{d^2 g_{22}}{dr^2}\right)\right] - 8(\theta^{23})^2 g_{11}^2 g_{22}^2 \\ & \left. +4\,\theta^{13}\theta^{23} g_{11} g_{22} \cot(2\theta)\left[\frac{d}{dr}\left(g_{22} g_{11}\right)\right]\right\},\end{aligned}$$

Non-commutative Corrections

$$
\begin{aligned}
\tilde{g}_{13} = & \left(\frac{\theta^{12}}{16 g_{33}^{3/2} g_{11}^{1/2}}\right) \\
\times \quad & \left\{\theta^{23}\left[\cos\theta\left(\left(\frac{dg_{11}}{dr}\right)\left(\frac{\partial g_{33}}{\partial\theta}\right) g_{33} - 2g_{11}\left(\frac{\partial^2 g_{33}}{\partial\theta\partial r}\right) g_{33} + g_{11}\left(\frac{\partial g_{33}}{\partial\theta}\right)\left(\frac{\partial g_{33}}{\partial r}\right)\right) + \right.\right. \\
+ \quad & \left.\sin\theta\left(-\frac{1}{2}\left(\frac{dg_{11}}{dr}\right)\left(\frac{\partial g_{33}}{\partial\theta}\right)^2 + g_{33}\left(\frac{dg_{11}}{dr}\right)\left(\frac{\partial^2 g_{33}}{\partial\theta^2}\right) + 2g_{11}g_{33}\left(\frac{\partial g_{33}}{\partial r}\right)\right)\right] + \\
+ \quad & \theta^{13}\left[\cos\theta\left(-g_{33}\left(\frac{dg_{11}}{dr}\right)\left(\frac{\partial g_{33}}{\partial r}\right) + g_{11}\left(\frac{\partial g_{33}}{\partial r}\right)^2 - 2g_{11}g_{33}\left(\frac{\partial^2 g_{33}}{\partial r^2}\right)\right)\right. \\
+ \quad & \sin\theta\left(-\frac{1}{2}\left(\frac{dg_{11}}{dr}\right)\left(\frac{\partial g_{33}}{\partial\theta}\right)\left(\frac{\partial g_{33}}{\partial r}\right) + g_{33}\frac{\partial}{\partial r}\left(\frac{\partial g_{33}}{\partial\theta}\cdot\frac{dg_{11}}{dr}\right)\right. \\
- \quad & \left.\left.\left.\frac{1}{2}\left(\frac{g_{33}}{g_{11}}\right)\left(\frac{\partial g_{33}}{\partial\theta}\right)\left(\frac{dg_{11}}{dr}\right)^2\right)\right]\right\}, \qquad (17)
\end{aligned}
$$

$$ \qquad (18) $$

Non-commutative Corrections

$$\begin{aligned}
\tilde{g}_{23} &= \left(\frac{\theta^{12}}{32 g_{33}^{3/2} g_{22}^{3/2}}\right)\Bigg[4\,\theta^{13} g_{22}^{2} g_{33}\sin\theta\left(\frac{\partial^2 g_{33}}{\partial r^2}\right) - \theta^{13} g_{33}\cos\theta\left(\frac{\partial g_{33}}{\partial\theta}\right)\left(\frac{dg_{22}}{dr}\right)^2 + \\
&+ 2\,\theta^{13} g_{33} g_{22}\cos\theta\left(\frac{\partial g_{33}}{\partial\theta}\right)\left(\frac{d^2 g_{22}}{dr^2}\right) + 4\,\theta^{23} g_{22}^{2} g_{33}\cos\theta\left(\frac{\partial g_{33}}{\partial r}\right) \\
&- 2\,\theta^{13} g_{22}^{2}\sin\theta\left(\frac{\partial g_{33}}{\partial r}\right)^2 - 2\,\theta^{23} g_{22}^{2}\sin\theta\left(\frac{\partial g_{33}}{\partial\theta}\right)\left(\frac{\partial g_{33}}{\partial r}\right) \\
&- 2\,\theta^{23} g_{33} g_{22}\sin\theta\left(\frac{dg_{22}}{dr}\right)\left(\frac{\partial g_{33}}{\partial\theta}\right) - \theta^{23} g_{22}\cos\theta\left(\frac{dg_{22}}{dr}\right) \\
&\times \left(\frac{\partial g_{33}}{\partial\theta}\right)^2 + 2\,\theta^{23} g_{33} g_{22}\cos\theta\left(\frac{dg_{22}}{dr}\right)\left(\frac{\partial^2 g_{33}}{\partial\theta^2}\right) + 2\,\theta^{13} g_{33} g_{22}\sin\theta\left(\frac{dg_{22}}{dr}\right)\left(\frac{\partial g_{33}}{\partial r}\right) - \\
&- \theta^{13} g_{22}\cos\theta\left(\frac{dg_{22}}{dr}\right)\left(\frac{\partial g_{33}}{\partial\theta}\right)\left(\frac{\partial g_{33}}{\partial r}\right) + 2\,\theta^{13} g_{22} g_{33}\cos\theta\left(\frac{dg_{22}}{dr}\right)\left(\frac{\partial^2 g_{33}}{\partial\theta\partial r}\right) + \\
&+ 4\,\theta^{23} g_{22}^{2} g_{33}\sin\theta\left(\frac{\partial^2 g_{33}}{\partial\theta\partial r}\right)\Bigg].
\end{aligned}$$

Non-commutative Corrections

We assume the following correspondence $\tilde{g}_{\mu\nu} = \tilde{e}^{a}{}_{\mu}\tilde{e}_{a\nu}$, where $\tilde{e}^{a}{}_{\mu}$ is given by

$$\tilde{e}^{a}{}_{\mu} = \begin{bmatrix} A & 0 & 0 & 0 \\ 0 & B\sin\theta\cos\phi + Er\cos\theta\cos\phi - Fr\sin\theta\sin\phi & Cr\cos\theta\cos\phi - Gr\sin\theta\sin\phi & -Hr\sin\theta\sin\phi \\ 0 & B\sin\theta\sin\phi + Er\cos\theta\sin\phi + Fr\sin\theta\cos\phi & Cr\cos\theta\sin\phi + Gr\sin\theta\cos\phi & Hr\sin\theta\cos\phi \\ 0 & B\cos\theta - Er\sin\theta & -Cr\sin\theta & 0 \end{bmatrix},$$

Non-commutative Corrections

$$\begin{aligned}
A &= \sqrt{-\tilde{g}_{00}}\,, \\
B &= \frac{\delta_1}{\delta}\,, \\
C &= \frac{\delta}{r\sqrt{\tilde{g}_{33}}}\,, \\
E &= \frac{\tilde{g}_{12}\tilde{g}_{33} - \tilde{g}_{23}\tilde{g}_{13}}{r\sqrt{\tilde{g}_{33}}\,\delta}\,, \\
F &= \frac{\tilde{g}_{13}}{\sqrt{\tilde{g}_{33}}\,r\sin\theta} \\
G &= \frac{\tilde{g}_{23}}{\sqrt{\tilde{g}_{33}}\,r\sin\theta}\,, \\
H &= \frac{\sqrt{\tilde{g}_{33}}}{r\sin\theta}\,.
\end{aligned}$$

Non-commutative Corrections

The quantities δ and δ_1 are defined by

$$\delta^2 = \tilde{g}_{22}\tilde{g}_{33} - \tilde{g}_{23}^2$$

and

$$\delta_1^2 = \tilde{g}_{11}\delta^2 - \tilde{g}_{22}\tilde{g}_{13}^2 - \tilde{g}_{33}\tilde{g}_{12}^2 + 2\tilde{g}_{12}\tilde{g}_{23}\tilde{g}_{13}\,.$$

Non-commutative Corrections

$$
\begin{aligned}
\tilde{T}_{112} &= -\frac{1}{2}\left(\frac{\partial\tilde{g}_{11}}{\partial\theta}\right)+\frac{1}{2\delta^2}\left[\alpha\left(\frac{\partial\tilde{g}_{22}}{\partial r}\right)+\beta\left(\frac{\tilde{g}_{23}}{\tilde{g}_{33}}\right)\left(\frac{\partial\tilde{g}_{33}}{\partial r}\right)-2\beta\left(\frac{\partial\tilde{g}_{23}}{\partial r}\right)\right], \\
\tilde{T}_{212} &= \frac{1}{2}\left(\frac{\partial\tilde{g}_{22}}{\partial r}\right)-\left(\frac{\partial\tilde{g}_{12}}{\partial\theta}\right)-\delta_1\tilde{g}_{33}^{-1/2}+\frac{1}{2\delta^2}\left[\alpha\left(\frac{\partial\tilde{g}_{22}}{\partial\theta}\right)+\beta\left(\frac{\tilde{g}_{23}}{\tilde{g}_{33}}\right)\left(\frac{\partial\tilde{g}_{33}}{\partial\theta}\right)\right. \\
&- \left. 2\beta\left(\frac{\partial\tilde{g}_{22}}{\partial\theta}\right)\right], \\
\tilde{T}_{312} &= \frac{1}{2\tilde{g}_{33}}\left[2\tilde{g}_{33}\left(\frac{\partial\tilde{g}_{23}}{\partial r}\right)-\tilde{g}_{23}\left(\frac{\partial\tilde{g}_{33}}{\partial r}\right)+\tilde{g}_{13}\left(\frac{\partial\tilde{g}_{33}}{\partial\theta}\right)-2\tilde{g}_{33}\left(\frac{\partial\tilde{g}_{13}}{\partial\theta}\right)\right], \\
\tilde{T}_{113} &= \frac{1}{2}\left(\frac{\tilde{g}_{13}}{\tilde{g}_{33}}\right)\left(\frac{\partial\tilde{g}_{33}}{\partial r}\right),
\end{aligned}
$$

Non-commutative Corrections

$$
\begin{aligned}
\tilde{T}_{213} &= -\frac{1}{2\tilde{g}_{33}\delta}\left[2\tilde{g}_{33}\beta\cos\theta + 2\tilde{g}_{23}\tilde{g}_{33}^{1/2} - \tilde{g}_{23}\delta\left(\frac{\partial\tilde{g}_{33}}{\partial r}\right)\right] \,, \\
\tilde{T}_{313} &= \frac{1}{2}\left(\frac{\partial\tilde{g}_{33}}{\partial r}\right) - \frac{1}{\delta}\left(\alpha\cos\theta + \delta_1\tilde{g}_{33}^{1/2}\sin\theta\right) \,, \\
\tilde{T}_{123} &= -\frac{1}{2}\left(\frac{\tilde{g}_{13}}{\tilde{g}_{33}}\right)\left(\frac{\partial\tilde{g}_{33}}{\partial\theta}\right) - \frac{1}{\delta}\left(\beta\cos\theta + \delta_1\tilde{g}_{23}\tilde{g}_{33}^{-1/2}\sin\theta\right) \,, \\
\tilde{T}_{223} &= \frac{1}{2}\left(\frac{\tilde{g}_{23}}{\tilde{g}_{33}}\right)\left(\frac{\partial\tilde{g}_{33}}{\partial\theta}\right) \,, \\
\tilde{T}_{323} &= \frac{1}{2}\left(\frac{\partial\tilde{g}_{33}}{\partial\theta}\right) - \delta\cos\theta \,.
\end{aligned}
$$

Non-commutative Corrections

where

$$\alpha = \tilde{g}_{33}\tilde{g}_{12} - \tilde{g}_{23}\tilde{g}_{13}$$

and

$$\beta = \tilde{g}_{12}\tilde{g}_{23} - \tilde{g}_{22}\tilde{g}_{13} \, .$$

Non-commutative Corrections

$$
\begin{aligned}
\tilde{\Sigma}^{001} &= \frac{1}{2(-\tilde{g}_{00})\delta_1^2}\Big[-\tilde{T}_{112}\left(\delta^2\tilde{g}_{12}+\alpha\tilde{g}_{11}\right)-\tilde{T}_{212}\left(\delta^2\tilde{g}_{22}+\alpha\tilde{g}_{12}\right)-\tilde{T}_{312}\left(\delta^2\tilde{g}_{23}+\alpha\tilde{g}_{12}\right)-\\
&- \tilde{T}_{113}\left(\delta^2\tilde{g}_{13}-\beta\tilde{g}_{11}\right)-\tilde{T}_{213}\left(\delta^2\tilde{g}_{23}-\beta\tilde{g}_{12}\right)-\tilde{T}_{313}\left(\delta^2\tilde{g}_{33}-\beta\tilde{g}_{13}\right)+\tilde{T}_{123}(\alpha\tilde{g}_{13}+\\
&+ \beta\tilde{g}_{12})+\tilde{T}_{223}\left(\alpha\tilde{g}_{23}+\beta\tilde{g}_{22}\right)+\tilde{T}_{323}\left(\alpha\tilde{g}_{33}+\beta\tilde{g}_{23}\right)\Big],
\end{aligned}
$$

Non-commutative Corrections

$$
\begin{aligned}
4e\,\tilde{\Sigma}^{(0)01} \quad &= \quad \left(\frac{2}{\delta_1}\right)\left\{\tilde{g}_{33}^{1/2}\delta_1 + \left(\frac{\delta\delta_1}{\sqrt{\tilde{g}_{33}}}\right)\sin\theta - \frac{1}{2}\frac{\partial}{\partial r}\left(\tilde{g}_{33}\tilde{g}_{22}\right) + \frac{\partial}{\partial\theta}\left(\tilde{g}_{33}\tilde{g}_{12}\right)\right.\\
&+ \quad \left.\tilde{g}_{23}\left[\left(\frac{\partial\tilde{g}_{23}}{\partial r}\right) - \left(\frac{\partial\tilde{g}_{13}}{\partial\theta}\right)\right]\right\} +\\
&+ \quad \left(\frac{1}{\delta^2\delta_1}\right)\left\{\left(\tilde{g}_{23}\tilde{g}_{13} - \tilde{g}_{12}\tilde{g}_{33}\right)\left[\frac{\partial\left(\tilde{g}_{22}\tilde{g}_{33}\right)}{\partial\theta}\right] + 2\tilde{g}_{33}\left(\tilde{g}_{12}\tilde{g}_{23} - \tilde{g}_{22}\tilde{g}_{13}\right)\left(\frac{\partial\tilde{g}_{23}}{\partial\theta}\right)\right\}\,.
\end{aligned}
$$

Non-commutative Corrections

Then taking the limit $r \to \infty$ it yields

$$4e\,\tilde{\Sigma}^{(0)01} = 4\,M\sin\theta - (\theta^{23})^2\left(\frac{\cos^2\theta}{\sin\theta}\right)\left[M\left(\frac{11}{8}+\cos^2\theta\right)+\lim_{r\to\infty}\frac{3}{2}r\left(\frac{1}{4}+\cos^2\theta\right)\right].$$

Non-commutative Corrections

In this sense, we obtain

$$\tilde{\eta}_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 + \frac{(\theta^{23})^2 \cos(2\theta)}{4} & -\frac{(\theta^{23})^2 r \sin(2\theta)}{4} & \frac{\theta^{12}\theta^{23}\left(\sin^2\theta - \cos\theta\right)}{4} \\ 0 & -\frac{(\theta^{23})^2 r \sin(2\theta)}{4} & \left[1 - \frac{(\theta^{23})^2 \cos(2\theta)}{4}\right] r^2 & 0 \\ 0 & \frac{\theta^{12}\theta^{23}\left(\sin^2\theta - \cos\theta\right)}{4} & 0 & \left[1 + \frac{(\theta^{23})^2 \cos(2\theta)}{4\sin^2\theta}\right] r^2 \sin^2\theta \end{pmatrix},$$

which generates non-vanishing components of the torsion tensor.

Non-commutative Corrections

Hence we get

$$\tilde{P}^{(0)} = M + \frac{1}{4}\left(\frac{49}{3} - \frac{15}{2}\ln 2\right)(\theta^{23})^2\, M\,,$$

thus the correction in the gravitational energy due to the non-commutativity of space-time is

$$\Delta P^{(0)} = \frac{1}{4}\left(\frac{49}{3} - \frac{15}{2}\ln 2\right)(\theta^{23})^2\, M\,.$$

Alternatively the need for a regularization procedure could be contoured by means the choice $\theta^{23} = 0$, in such a way there would be no correction for the Schwarzschild energy.

Conclusion

In this work we start with Schwarzschild space-time, then we give the corrections due to the non-commutativity of space-time. Here it is introduced by replacing the normal product between tetrads by the Moyal product, rather than applying such a procedure in lagrangian density. This approach is well known in the literature to predict some non-commutative corrections in the metric tensor.

Conclusion

The new metric tensor leads to a new tetrad field which is used to calculate the gravitational energy of space-time. It is well known that the energy of Schwarzschild space-time is equal to M, therefore we get a correction in the energy equal to $\Delta P^{(0)}$. We stress out that the expression for the gravitational field has been developed and tested over the years in the context of TEGR. Since the non-commutative parameter is arbitrary (it should be given by experimental data) we speculate that such a correction in the gravitational energy can be associated to quantum effects in the realm of gravitational field. If the correction represents the energy of gravitons, then it should be proportional to the Planck's constant.

Conclusion

On the other hand the correction is proportional to the mass of the font, which could mean a new kind of quantization associated to the mass of a black-hole or a star, for example. Therefore the gravitational energy turns out to be of fundamental importance (experimental purposes), since it can tell if the space-time is commutative or not. For future works we intend to investigate the corrections of the gravitational energy in the context of Kerr space-time on the outer event horizon. We also want to study the solutions of the non-commutative equations that come from the lagrangian density replaced by the Moyal product.

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