Using pulsars to locate objects inside the solar system

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SATELLITE MOTIONS AND RPS (I)

GPS AND GALILEO SATELLITE CONSTELLATIONS ARE SIMULATED.

- Satellite trajectories are assumed to be circumferences in the Schwarzschild space-time created by an ideal spherically symmetric Earth. Our almost inertial reference is asymptotic to Schwarzschild, and our approach is first order in $GM_{\oplus}/r$; then:
- The angular velocity is $\Omega = (GM_{\oplus}/R^3)^{1/2}$
- Coordinate and proper times are related as follows: $\gamma = \frac{dt}{d\tau} = \left(1 - \frac{3GM_{\oplus}}{R}\right)^{-1/2}$.
- angles θ and ϕ fixes the orbital plane (see N. Puchades & D. Sáez, Astrophys. Space Sci, 341, 2012, 631 for details), and the angle $\alpha_A(\tau) = \alpha_{A0} - \Omega\gamma\tau$ localizes the satellite on its trajectory

This simple model is good enough to describe a background configuration. Deviations with respect to the background satellite world lines will be necessary to develop our study about positioning accuracy (see below). We could take elliptical trajectories to define another background configuration, but the deviations with respect to these trajectories would not lead to new significant results in our study of positioning errors

SATELLITE MOTIONS AND RPS (II)

- Relativistic positioning requires four satellites of some Global Navigation Satellite System
- Given the angles θ and ϕ, the third angle $\alpha_A(\tau)$ may be calculated for any τ. From these three angles, the satellite inertial coordinates (x^1, x^2, x^3, x^4) may be easily found (see N. Puchades & D. Sáez, Astrophys. Space Sci, 341, 2012, 631) for any proper time (satellite world line). Uncertainties in this line –for one or various satellites– are responsible for the most important positioning errors
- In relativistic positioning, the four satellites must send codified signal which reach the user (receptor) at the same time. These signals contain the proper times $(\tau^1, \tau^2, \tau^3, \tau^4)$ of the four satellites when they were emitted. These proper times are called *EMISSION COORDINATES*

SATELLITE WORLD LINES IN THE ALMOST INERTIAL REFERENCE $\oplus$ EMISSION COORDINATES =====> USER INERTIAL COORDINATES (POSITIONING)

FROM INERTIAL TO EMISSION COORDINATES

FROM $x^\alpha \equiv (x, y, z, t)$ TO $(\tau^A \equiv \tau^1, \tau^2, \tau^3, \tau^4)$

It is assumed that photons move in the Minkowski space-time, whose metric has the covariant components $\eta_{\alpha\beta}$. This approach is good enough for us

Since photons follow null geodesics from emission to reception, the following algebraic equations must be satisfied:

$$\eta_{\alpha\beta}[x^\alpha - x^\alpha_A(\tau^A)][x^\beta - x^\beta_A(\tau^A)] = 0 \ . \qquad (1)$$

These four equations must be NUMERICALLY solved to get the four emission coordinates τ^A, where index A numerates the satellites.

The four proper times are the unknowns in the system (1), which may be easily solved by using the well known Newton-Raphson method. A code has been designed to implement this method. It uses multiple precision. Appropriate tests have been performed

Since the satellite world lines are known, functions $x^\alpha_A(\tau^A)$ may be calculated for any set of proper times $\tau^1, \tau^2, \tau^3, \tau^4$, thus, the left hand side of Eqs. (1) can be computed and, consequently, the Newton-Raphson method may be applied

FROM EMISSION TO INERTIAL COORDINATES

FROM $(\tau^A \equiv \tau^1, \tau^2, \tau^3, \tau^4)$ TO $x^\alpha \equiv (x, y, z, t)$

Given four emission coordinates τ^A, Eqs. (1) could be numerically solved to get the unknowns x^α, that is to say, the inertial coordinates; however, this numerical method is not used. It is better the use of an analytical formula giving x^α in terms of τ^A, which is due to B. Coll, J.J. Ferrando, & J.A. Morales-Lladosa (Class. Quantum Grav., 27, 2010, 065013)

The analytical formula is preferable because of the following reasons:

- The numerical method based on Eqs. (1) is more time consuming
- The analytical formulation of the problem allows us a systematic and clear discussion of the bifurcation problem, and also a study of the positioning errors close to situations of vanishing Jacobian

The analytical formula involves the function χ^2 and the discriminant Δ. These quantities may be calculated by using the satellite world lines and the emission coordinates. They are defined in the above paper by Coll, et al.

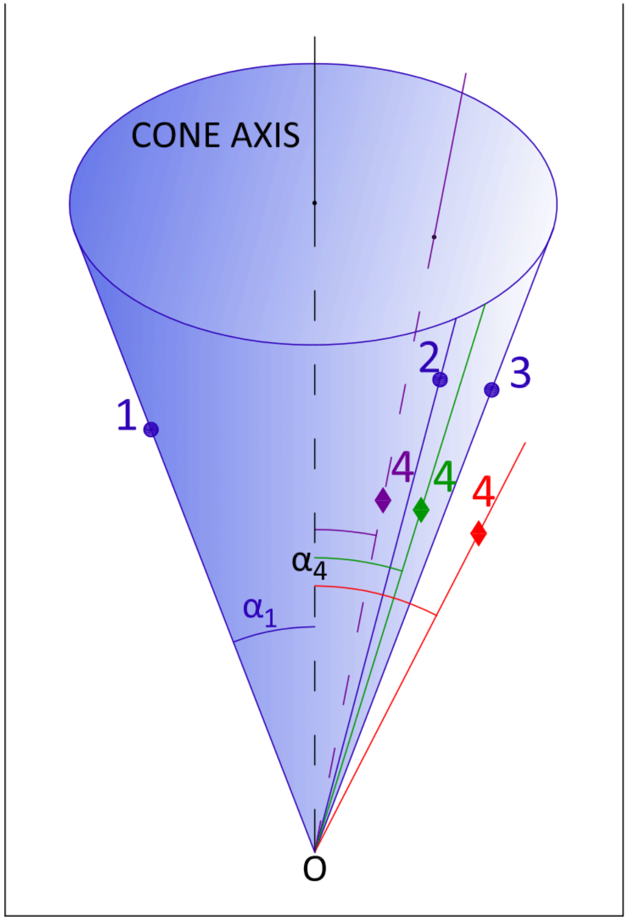
SOME PREVIOUS THEORETICAL RESULTS

By using the analytical formula giving the inertial coordinates from the emission ones, and some basic relations of Minkowski space-time, the following conclusions have been previously obtained

- for $\chi^2 \leq 0$, there is only a positioning (past-like) solution
- for $\chi^2 > 0$ there are two positioning solutions (bifurcation); namely, there are two sets of inertial coordinates (two physical real receivers) associated to the same emission coordinates $(\tau^1, \tau^2, \tau^3, \tau^4)$
- the Jacobian J of the transformation giving the emission coordinates in terms of the inertial ones vanishes if and only if the discriminant Δ vanishes
- the Jacobian J may only vanish if $\chi^2 > 0$; namely, in the region of double positioning
- The Jacobian J may only vanish if the lines of sight –at emission times– of the four satellites belong to the same cone)

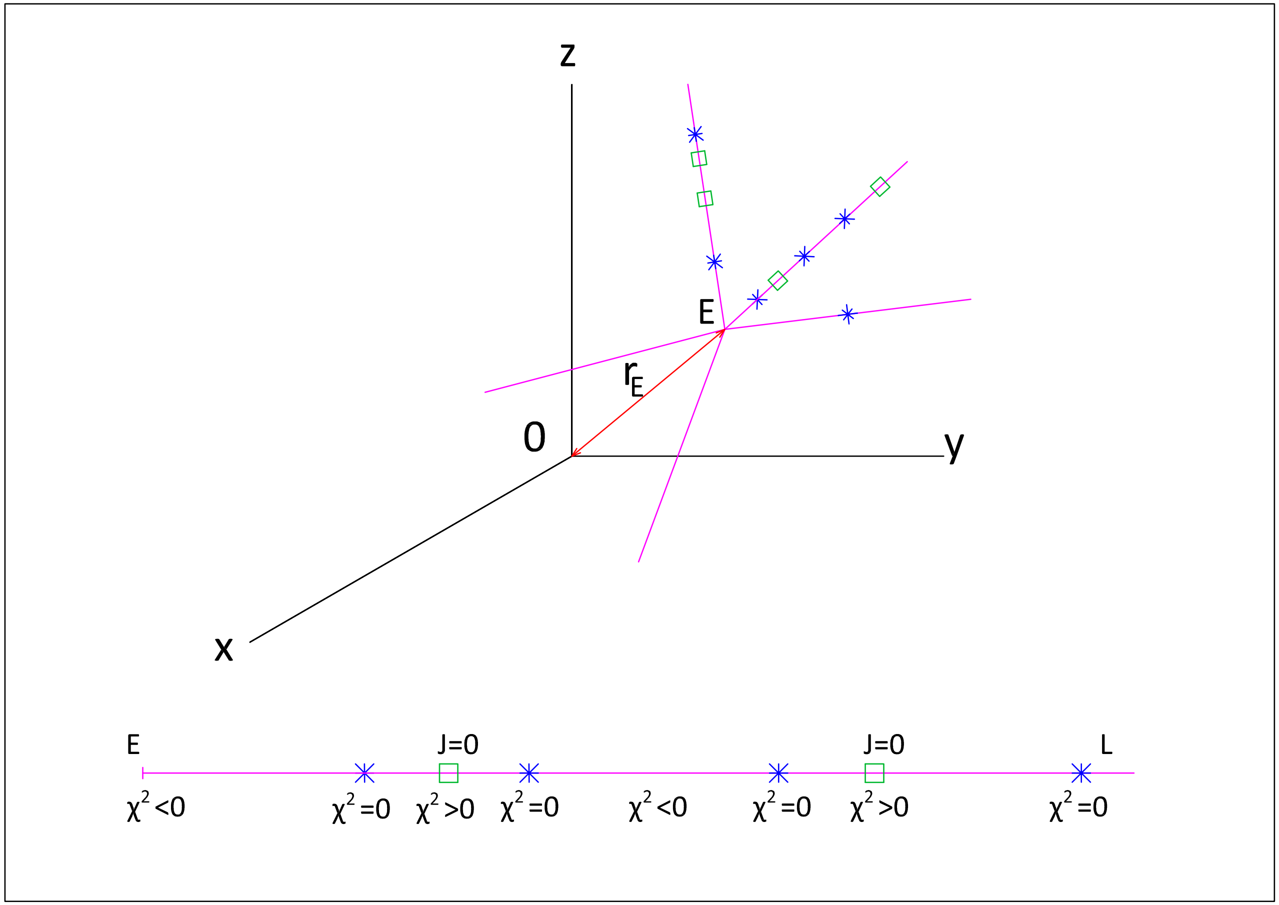
These conclusions are basic for the numerical estimates and discussions presented in next slides. In particular, the fifth item may be used to reject any set of four satellites leading to some point of vanishing Jacobian close to a certain user. Around these points with $J = 0$, positioning errors are very large

CONFIGURATIONS WITH $J = 0$



The user (in O) and the satellites 1, 2, and 3 generate a cone, α_1 is the cone angle, and α_4 is the angle between the line of sight of satellite 4 and the cone axis. This satellite is on the cone if and only if $\alpha_1 = \alpha_4$. In this case the Jacobian vanishes at O

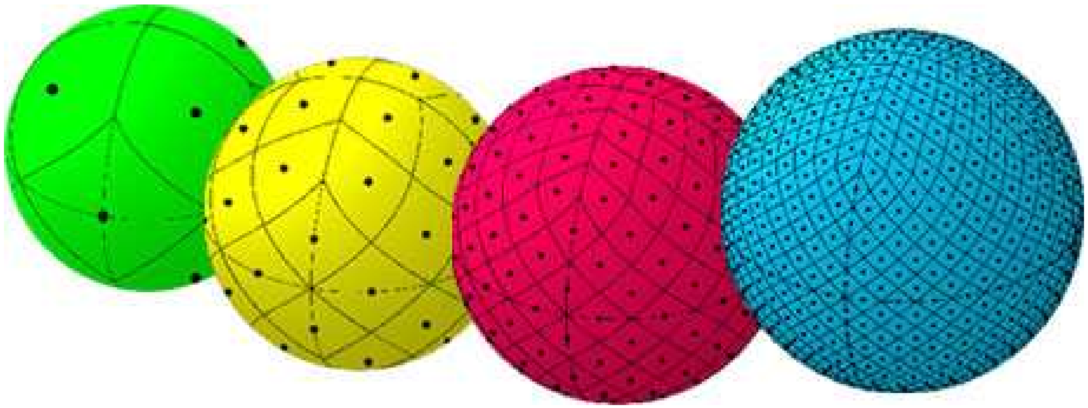
REGION STRUCTURE ANALYSIS



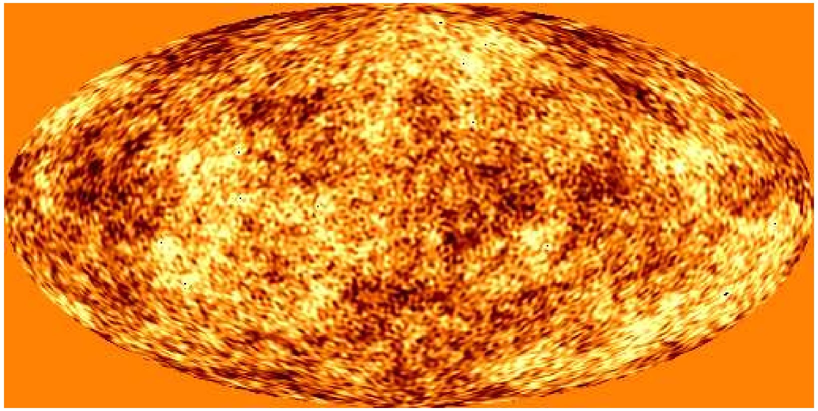
3D sections with $t = constant$ are considered. Point E is an arbitrary center. Its distance to the origin O is the Earth radius. 3072 directions starting from E cover the 3D sections. Along each direction our study is restricted to $0 < L < L_{max} = 10^5\ Km$. We look for the zeros of χ^2 and J. Quantity J only may vanish in the segments where $\chi 2 > 0$, which are limited by the first and second or by the third and fourth χ^2-zeros

REPRESENTATION TECHNIQUES

HEALPIX PIXELISATION(Hierarchical, Equal Area, and iso-Latitude PIXelisation)

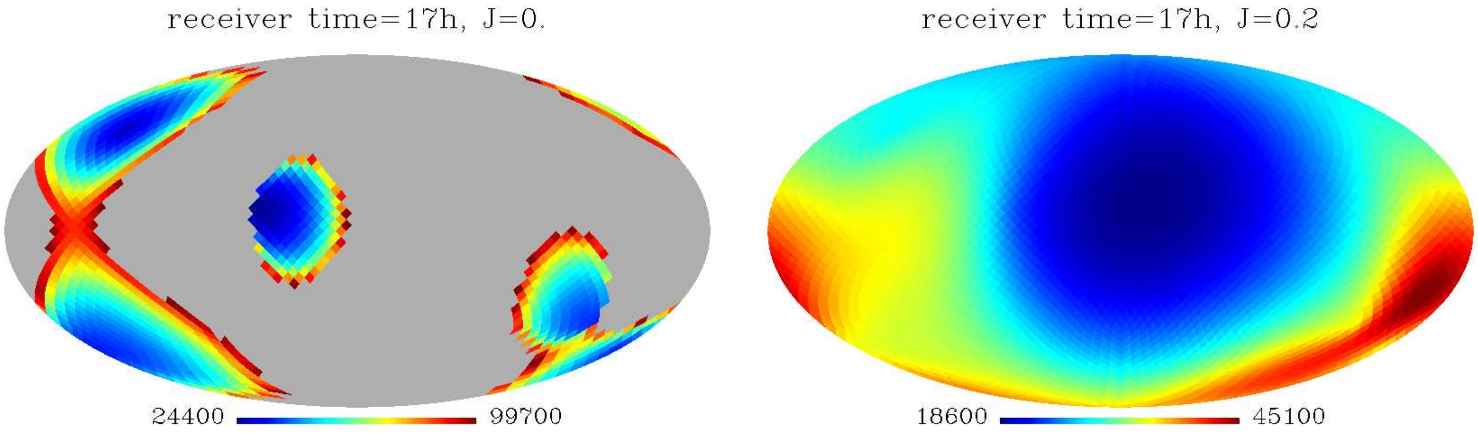


MOLLWEIDE PROJECTION



Our maps have 3072 equal area pixels. This area is 64 times the mean angular area of the full Moon

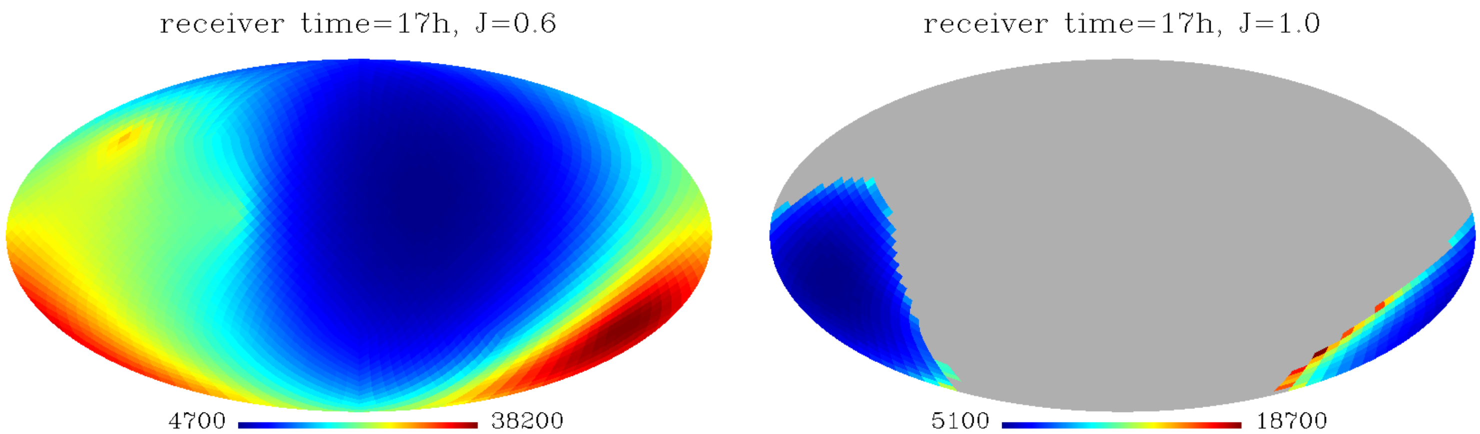
SPATIAL DISTRIBUTION OF JACOBIAN VALUES (I)



(a) (b)

- Galileo satellites 2, 5, 20 and 23, for $L < 10^5\ Km$.
- HEALPIX-MOLLWIDE representations with 3072 pixels. Distances are given in Km.
- In panels (a) and (b), the color bar measures the distance, L_J, from E to the closest point where the Jacobian takes on the value displayed in the top.
- In panel (a), we see that the Jacobian vanishes (big positioning errors) at distances L_J larger than $\sim 2 \times 10^4\ Km$. Gray pixels correspond to directions with $J \neq 0$.
- From panel (b), it follows that the condition $J = 0{,}2$ is satisfied for L_J values a little smaller than those of panel (a) ($J = 0$).

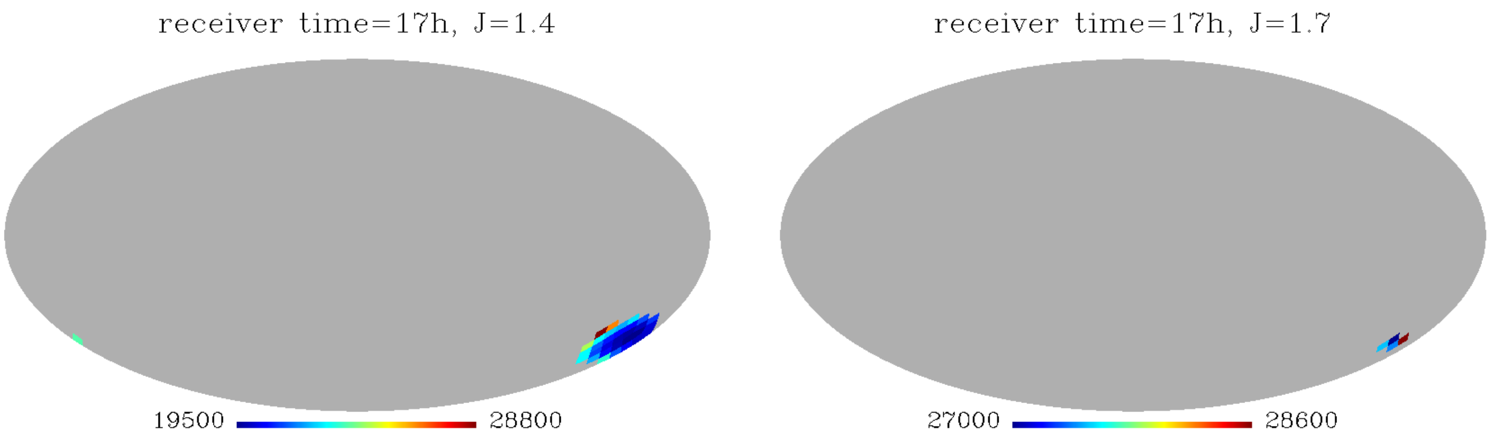
SPATIAL DISTRIBUTION OF JACOBIAN VALUES (II)



(c) (d)

- Same as in previous slide for $J = 0{,}6$ [panel (c)] and $J = 1{,}0$ [panel (d)].
- By comparing panels (b) and (c), it follows that the condition $J = 0{,}6$ [panel (c)] is satisfied for L_J values smaller than those of panel (b) ($J = 0{,}2$). In the region where $J \sim 0{,}6$, positioning errors are expected to be small enough.
- In the region where $J \sim 1{,}0$, positioning errors are expected to be better; nevertheless, panel (d) shows an extended gray region corresponding to directions whose J values are always smaller than $1{,}0$

SPATIAL DISTRIBUTION OF JACOBIAN VALUES (III)



(e) (f)

- Same as in the two last slides for $J = 1{,}4$ [panel (e)] and $J = 1{,}7$ [panel (f)]
- Panels (d), (e) and (f) show that, as J increases, the area of the gray region grows, which means that the directions containing J values greater than 1.0 are restricted to cover small areas; in other words, these directions become more and more scarce
- From panels (e) and (f), it follows that J values in the interval $(1{,}4, 1{,}7)$ appear –in scarce directions– at L_J distances greater than $1{,}9 \times 10^4\ Km$. In most directions, the Jacobian values are well below $1{,}0$ for similar distances to point E.
- No directions with $|J|$ greater than about 2.0 have been found in this case.

POSITIONING ERRORS: THEORY (I)

The background world lines of the satellites are known circumferences $y^\alpha = x^\alpha_A(\tau^A)$

Given the inertial coordinate x^α of an user, the above circumferences, Eqs. (1), the Newton-Raphson method, and multiple precision may be used to find the emission coordinates $\tau^1, \tau^2, \tau^3, \tau^4$ with very high accuracy.

Finally, the inertial coordinates x^α may be recovered from the emission ones –with very high accuracy– by using the analytical solution found by Coll, Ferrando, & Morales-Lladosa. This is a severe test for our codes, which must recover the expected number of figures

Let us now suppose that there are uncertainties in the satellite world lines, whose equations are $y^\alpha = x^\alpha_A(\tau^A) + \xi^\alpha_A$, where ξ^α_A are deviations with respect to the background world lines.

Then, new inertial coordinates $x^\alpha + \Delta(x^\alpha)$ may be obtained, from the emission ones τ^A (whose observational values cannot be changed), the perturbed world lines $y^\alpha = x^\alpha_A(\tau^A) + \xi^\alpha_A$, and the Coll et al. analytical solution.

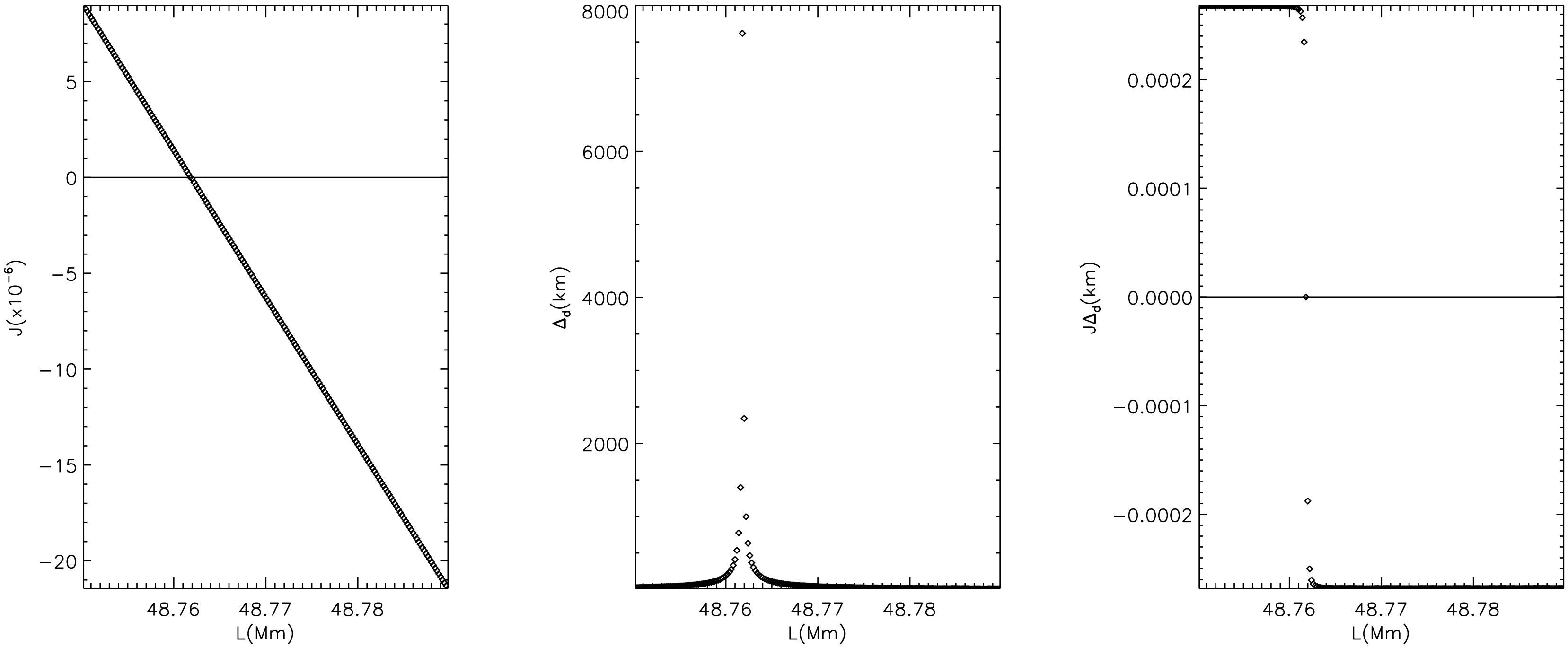
Quantity $\Delta_d = [\Delta^2(x^1) + \Delta^2(x^2) + \Delta^2(x^3)]^{1/2}$ is a good estimator of the positioning errors produced by the ξ^α_A uncertainties in the satellite motions.

POSITIONING ERRORS: THEORY (II)

We have taken intervals of $200\ Km$ having zeros of J. Each interval corresponds to a certain direction. The estimator Δ_d has been calculated in many points around $J = 0$.

In order to do that, random deviations ξ_A^{α} have been generated around each satellite (with a maximum amplitude of $1\ m$). These random deviations have been used in each point of the $200\ Km$ interval where quantity Δ_d has been calculated.

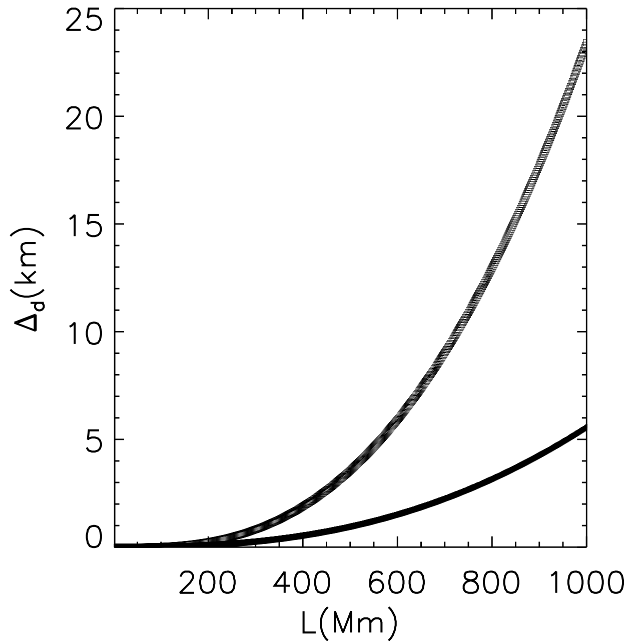
POSITIONING ERRORS: NUMERICAL CALCULATIONS



(g) (h) (i)

- A segment of $200\ Km$ where J vanishes
- panel (g) displays the point where J vanishes
- Panel (h) shows that positioning errors strongly grow close to the $J = 0$ point
- From panel (i), it follows that $J\Delta_d$ vanishes at the same point as J.

POSITIONING ERRORS FOR $L > 10^5\ Km$



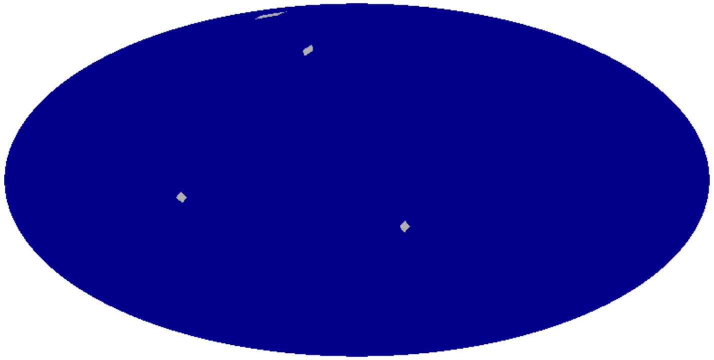
- It is assumed that the errors in the position of the GALILEO satellites have an amplitude of $1\ m$.
- The two lines correspond to different observation directions. The errors quickly grow as the distance L to point E increases
- For $L = 4 \times 10^5\ Km$, the positioning errors become as large as $\sim 2\ Km$.
- For a more realistic amplitude of $\sim 10\ m$ in the GALILEO satellite position uncertainties, the positioning errors – for $L = 4 \times 10^5\ Km$ – become as large as $\sim 20\ Km$.

NAVIGATION IN THE SOLAR SYSTEM: PULSARS (1)

- Errors close to $20\ km$ at a distance from Earth of $\sim 4 \times 10^5\ Km$ (GALILEO expectations) are similar to the positioning errors expected for pulsar navigation (in the solar system). Hence, we may conclude that, for distances greater than the mean distance Earth-Moon, we should not use satellite constellations. Pulsars might be an advantageous alternative.
- WE FIRST REPLACE FOUR PULSARS BY FOUR IDEAL SATELLITES FORECASTING THEIR PROPER TIMES. THUS, THE SAME METHODS AS IN GNNSs (GALILEO, GPS, AND SO ON) MAY BE USED
- It does not matter that we are dealing with an ideal satellite configuration, which may not be realized in practice. Anyway, some conclusions based on this unrealistic configuration strongly suggest some conditions, which should be satisfied by realistic designs of pulsar based positioning
- In the case of pulsars, it is frequent to work in an almost inertial reference centered in the solar system barycenter. We also use this reference, in which, the satellite (pulsar) world lines must be known. Uncertainties in these lines would lead to positioning errors as it has been emphasized above

NAVIGATION IN THE SOLAR SYSTEM: PULSARS (2)

- In a first step, pulsars (satellites) are considered at rest in our system of reference and, consequently, the world line of each satellite is fully defined by its two angular coordinates and its distance to the barycenter. In the absence of uncertainties in these three coordinates, positioning would be exact.



- The angular pulsar coordinates have been chosen in such a way that the lines of sight point towards the white pixels. Evidently, these pulsars are not in the same cone ($J \neq 0$); hence, the estimated positioning errors are not large due to a bad choice of pulsars with $J = 0$.
- Pulsar distances to the barycenter have been taken to be 0.98, 0.60, 0.89, and 0.50 Kpc.

NAVIGATION IN THE SOLAR SYSTEM: PULSARS (3)

- Users, Q, have been assumed to be inside a sphere, S, centered in the solar system barycenter with radius $50\ A.U.$
- As a test, we have taken the *inertial coordinates* of an user Q, to compute its emission coordinates by using the Newton-Raphson method and, then, the initial inertial coordinates have been recovered with the analytical formula. It has been verified that a large number of figures is recovered; namely, it has been proved that code precisions are high enough.

NAVIGATION WITH PULSARS: ERRORS

- Uncertainties in the pulsar (satellite) world lines have been now considered. It has been assumed that pulsar coordinates have errors with amplitudes of $10^7\ Km$. The corresponding positioning errors have been calculated as for GNSSs.
- Since the size of the sphere S ($100\ A.U. \simeq 1{,}5 \times 10^{10}\ Km$) is much smaller than the distance to the pulsars, which is of the order of $1\ Kpc \simeq 3 \times 10^{16}\ Km$, positioning errors are very similar for all the users inside S. They have many common figures.
- If the position of an user with respect to the barycentric reference is known, e.g., an user on Earth, we may find its position by using pulsars, and the difference between both positions is the main common part, Ξ, of the positioning errors inside S.
- If the positioning errors are corrected inside S by subtracting the main common part Ξ everywhere, we get residual errors of the order of $10\ Km$, which is the order of magnitude of the errors estimated –in the technical literature– for positioning based on pulses emitted from X-ray pulsars
- Recently, the distance to PSR J2222-0137 has been measured with the greatest accuracy reached until now (Deller et al., 2013). The relative error is $\sim 3 \times 10^{-3}$ and the absolute one is of the order of $10^{13}\ Km$. These large errors in the pulsar world lines lead to inadmissible positioning errors of the order of $10^7\ Km \sim 0{,}1\ A.U.$

GENERAL DISCUSSION I

- In our approach, satellites move in Schwarzschild space-time, so the effect of the Earth gravitational field on the clocks is taken into account, e.g., GPS clocks run more rapid than clocks at rest on Earth by about 38.4 microseconds per day. It has been verified and taken into account.
- However, it is assumed that photons move in Minkowski space-time. This procedure is good enough since the Earth gravitational field produces a very small effect on photons while they travel from the satellites to the receiver. The distance traveled is not large and the gravitational field is weak.
- The emission coordinates may be obtained from the inertial ones by using accurate numerical codes based on the Newton-Raphson method.
- The inertial coordinates can be found from the emission ones by using the analytic transformation law of Coll, Ferrando & Morales-Lladosa.
- Small uncertainties in the satellite world lines produce large positioning errors if $J \simeq 0$.
- If possible, the four emitters (satellites of the chosen GNSS) must be selected to avoid both bifurcation and situations with $J \simeq 0$.

GENERAL DISCUSSION II

- Positioning on Earth surface always correspond to $\chi^2 \leq 0$, and the Jacobian does not vanish in this case. Nevertheless, our study applies to the case of objects located away from Earth surface. For distances to E smaller than $\sim 2 \times 10^4\ Km$, positioning errors are admissible since J takes on appropriate non vanishing values.
- At distances close to $\sim 4 \times 10^5\ Km$, Galileo satellites lead to positioning errors as greater as $\sim 20\ Km$ in some directions. For greater distances, pulsar based positioning would be advantageous.
- Pulsar based relativistic positioning in the solar system must be strictly local, without considering the pulsar distances as basic elements for location. If these distances are only used to calculate corrections to a rather well estimated position, the errors in the corrections (due to uncertainties in pulsar distances) must be evaluated in each case.
- In Feng et al., J Zhejiang Univ-Sci C (Comput and Electron), 14, 2013, 133, A local non relativistic method for navigation with pulsars is described.
- In S.I. Sheikh et al., Journal of Guidance, Control, and Dynamics, 29, 2006, 49, pulsar distances are involved in some formulas. The procedure may be right, but it requires a detailed analysis according to previous comments

ACKNOWLEDGMENTS

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