An extension of the Goldberg-Sachs theorem in five and higher dimensions [1]

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[1] Joint work with V. Pravda, A. Pravdová and H. Reall, Class.Quant.Grav. 29 (2012) 205002, and Class.Quant.Grav. 30 (2013) 075016

Contents

Null frames

"Null" frame $(\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{m}^{(i)})$ in n-dimensions ($n \geq 4$):

$$\boldsymbol{l} \cdot \boldsymbol{n} = 1, \qquad \boldsymbol{m}^{(i)} \cdot \boldsymbol{m}^{(j)} = \delta_{ij} \quad (i = 2, \ldots, n-1)$$

Metric $g_{ab} = 2l_{(a}n_{b)} + \delta_{ij} m_a^{(i)} m_b^{(j)}$ invariant under:

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Null rotations

$$\boldsymbol{l}' = \boldsymbol{l}, \quad \boldsymbol{n}' = \boldsymbol{n} + z_i \boldsymbol{m}^{(i)} - \tfrac{1}{2} z^i z_i \boldsymbol{l}, \quad \boldsymbol{m}'^{(i)} = \boldsymbol{m}^{(i)} - z_i \boldsymbol{l}$$

Spatial rotations

$$\boldsymbol{l}' = \boldsymbol{l}, \qquad \boldsymbol{n}' = \boldsymbol{n}, \qquad \boldsymbol{m}'^{(i)} = X^i{}_j \boldsymbol{m}^{(j)}$$

Boosts

$$\boldsymbol{l}' = \lambda \boldsymbol{l}, \qquad \boldsymbol{n}' = \lambda^{-1} \boldsymbol{n}, \qquad \boldsymbol{m}'^{(i)} = \boldsymbol{m}^{(i)}$$

Classification of the Weyl tensor

Frame decomposition of Weyl:

$$\begin{aligned}
C_{abcd} = {} & \overbrace{4C_{0i0j}\, n_{\{a}m^{(i)}_{b}n_{c}m^{(j)}_{d\}}}^{b=+2} + \overbrace{8C_{010i}\, n_{\{a}l_{b}n_{c}m^{(i)}_{d\}} + 4C_{0ijk}\, n_{\{a}m^{(i)}_{b}m^{(j)}_{c}m^{(k)}_{d\}}}^{b=+1} \\
& \left.\begin{aligned} &+ 4C_{0101}\, n_{\{a}l_{b}n_{c}l_{d\}} + C_{01ij}\, n_{\{a}l_{b}m^{(i)}_{c}m^{(j)}_{d\}} \\ &+ 8C_{0i1j}\, n_{\{a}m^{(i)}_{b}l_{c}m^{(j)}_{d\}} + C_{ijkl}\, m^{(i)}_{\{a}m^{(j)}_{b}m^{(k)}_{c}m^{(l)}_{d\}} \end{aligned}\right\} b=0 \\
& + \underbrace{8C_{101i}\, l_{\{a}n_{b}l_{c}m^{(i)}_{d\}} + 4C_{1ijk}\, l_{\{a}m^{(i)}_{b}m^{(j)}_{c}m^{(k)}_{d\}}}_{b=-1} + \underbrace{4C_{1i1j}\, l_{\{a}m^{(i)}_{b}l_{c}m^{(j)}_{d\}}}_{b=-2},
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\end{aligned}
$$

Classification:

- find a frame "as aligned as possible"
- characterize C_{abcd} by the maximal *boost weight* $b \in \{+2, +1, 0, -1, -2\}$ of non-zero components.

[Coley-Milson-Pravda-Pravdová'04, Milson-Coley-Pravda-Pravdová'05]

Weyl types

- **G**: $b = +2$ (general case)
- **I**: $b = +1$ ($\boldsymbol{l}$ is a WAND≡Weyl Aligned Null Direction)
- **II**: $b = 0$ ($\boldsymbol{l}$ is a *double* WAND)
- **III**: $b = -1$ ($\boldsymbol{l}$ is a *triple* WAND)
- **N**: $b = -2$ ($\boldsymbol{l}$ is a *quadruple* WAND)

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The resulting classification scheme is:

- for $n = 4$: *equivalent* to the Petrov classification
- based on a preferred null vector field $\boldsymbol{l}$
- pointwise
- invariant (frame independent) (cf. also [M.O.'09, Senovilla'11])
- useful for studying spacetimes with vanishing invariants [Coley-Milson-Pravda-Pravdová'04, Hervik'11]

Any tensor can be classified similarly (e.g., F_{ab}, R_{abcd}, R_{ab}, ...).

Geometric optics

Frame components of the covariant derivative of $\boldsymbol{l}$:

$$l_{a;b} = \gamma l_a l_b + \varepsilon l_a n_b + \beta_i l_a m_b^{(i)} + \tau_i m_a^{(i)} l_b + \kappa_i m_a^{(i)} n_b + \rho_{ij} m_a^{(i)} m_b^{(j)}$$

Geodesic vector field $\boldsymbol{l}$

$$0 = l_{a;b} l^b = \varepsilon l_a + \kappa_i m_a^{(i)} \Leftrightarrow \varepsilon = 0 = \kappa_i$$

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Optics of $\boldsymbol{l}$ → matrix $\rho_{ij} \equiv l_{a;b} m^{(i)a} m^{(j)b} = (\sigma_{ij} + \theta \delta_{ij}) + \rho_{[ij]}$.

Optical scalars

expansion	$\theta = \frac{1}{n-2} \rho_{ii} = \frac{1}{n-2} l^a{}_{;a}$
twist	$\omega^2 \equiv \rho_{[ij]} \rho_{[ij]} = -l_{[a;b]} l^{a;b}$
shear	$\sigma^2 \equiv \sigma_{ij} \sigma_{ij} = l_{(a;b)} l^{a;b} - \frac{1}{n-2} (l^a{}_{;b})^2$

[Frolov-Stojković'03, Pravda-Pravdová-Coley-Milson'04, Lewandowski-Pawlowski'05]

Interpretation for $n = 4$:

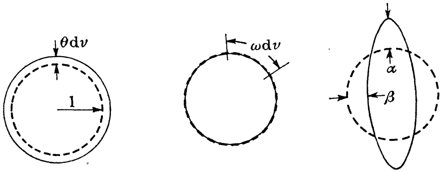

FIGURE 3. The shadow (solid lines) cast by an object will differ from the parallel displace of the object (dotted lines) by an expansion $\theta\,\mathrm{d}v$, a rotation $\omega\,\mathrm{d}v$, and shear $|\sigma|\,\mathrm{d}v = \frac{1}{2}(\alpha+\beta)$.

From [Sachs'61]

Goldberg-Sachs in 4D

Goldberg-Sachs theorem $(n = 4)$ [Goldberg-Sachs'62, Newman-Penrose'62]

In *vacuum*, $\boldsymbol{l}$ is a multiple WAND (i.e., type II or more special) $\Leftrightarrow$ $\boldsymbol{l}$ is geodesic $(\kappa_i = 0)$ + shearfree $(\sigma = 0)$

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- analogy with "null" Maxwell fields (Mariot-Robinson theorem)
- $\kappa_i = 0 = \sigma$ drastically simplifies NP equations, e.g.:
  - Kerr metric [Kerr'63]
  - all type D vacua [Kinnersley'69]

Goldberg-Sachs in higher dimensions?

Goldberg-Sachs extends to $n > 4$? How?

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Early results:

	Weyl type	Geodesic	Shearfree
Schwarzschild	D	$\surd$	$\surd$
Myers-Perry	D	$\surd$	X
Black string	D	$\surd$	X
Expanding III/N	III, N	$\surd$	X
VSI	III, N, O	$\surd$	$\surd$
$\subset$ Kerr-Schild	II, D	$\surd$	X
$\subset$ Robinson-Trautman	D	X	

[Myers-Perry'86, Horowitz-Ross'98, Frolov-Stojković'03, Pravda-Pravdová-Coley-Milson'04, Coley-Milson-Pravda-Pravdová'04, Podolský-M.O.'06, Pravda-Pravdová-M.O.'07, M.O.-Pravda-Pravdová'09]

Review papers: [Coley'08, M.O.-Pravda-Pravdová'13]

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Natural to split the HD study into two parts:

1. "geodesic part"
2. "shearfree part".

Goldberg-Sachs in HD: geodesic part

We have:

- type III, N $\Rightarrow$ $\boldsymbol{l}$ geodesic [Pravda-Pravdová-Coley-Milson'04]
- type II $\Rightarrow$ $\boldsymbol{l}$ geodesic [Durkee-Reall'09]
- type D $\not\Rightarrow$ $\boldsymbol{l}$ geodesic [Pravda-Pravdová-M.O.'07]

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However,

Proposition (Geodesic part [Durkee-Reall'09])

An $n > 4$ Einstein spacetime admits a multiple WAND $\Leftrightarrow$ it admits a geodesic multiple WAND.

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However,

Proposition (Geodesic part [Durkee-Reall'09])

An $n > 4$ Einstein spacetime admits a multiple WAND $\Leftrightarrow$ it admits a geodesic multiple WAND.

All such spacetimes explicitly known for $n = 5$ [Durkee-Reall'09] (e.g., $dS_3 \times S^2$).

Goldberg-Sachs in HD: "shearfree" part

Shear is generically non-zero. How to generalize "$\sigma = 0$" to $n > 4$?

Goldberg-Sachs in HD: "shearfree" part

Shear is generically non-zero. How to generalize "$\sigma = 0$" to $n > 4$? Note:

- $n = 4$:

$$\sigma = 0 \Leftrightarrow \boldsymbol{\rho} = b \begin{pmatrix} 1 & a \\ -a & 1 \end{pmatrix} .$$

- type III/N [Pravda-Pravdová-Coley-Milson'04]:

$$\boldsymbol{\rho} = \mathrm{diag}\left(b \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix}, 0, \ldots, 0 \right) .$$

- Kerr-Schild [M.O.-Pravda-Pravdová'09]:

$$\boldsymbol{\rho} = \alpha \mathrm{diag}\left(b_1 \begin{bmatrix} 1 & a_1 \\ -a_1 & 1 \end{bmatrix}, \ldots, b_\nu \begin{bmatrix} 1 & a_\nu \\ -a_\nu & 1 \end{bmatrix}, 1, \ldots, 1, 0, \ldots, 0 \right) .$$

Goldberg-Sachs in HD: "shearfree" part

Idea: look for canonical forms of $\rho_{ij} \equiv l_{a;b} m^a_{(i)} m^b_{(j)}$.

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Full result known for $n = 5$:

Proposition ("Shearfree" part $n = 5$ [M.O.-Pravda-Pravdová-Reall'12])

Five-dimensional Einstein spacetimes of type II (or more special) admit a geodesic multiple WAND $\boldsymbol{l}$ for which, in a suitable frame, $\boldsymbol{\rho}$ takes one of the forms (rank($\boldsymbol{\rho}$)=3,2,1,0)

$$i) \quad \boldsymbol{\rho} = b \begin{pmatrix} 1 & a & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1+a^2 \end{pmatrix}, \qquad ii) \quad \boldsymbol{\rho} = b \begin{pmatrix} 1 & a & 0 \\ -a & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$iii) \quad \boldsymbol{\rho} = b \begin{pmatrix} 1 & a & 0 \\ -a & -a^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad iv) \quad \boldsymbol{\rho} = 0.$$

- obtained from Bianchi/Ricci identities (plus derivatives)
- i)–iii) are generically shearing
- invariant meaning (integrability of null distributions, "optical structures", . . .)
- examples are known for each case:
  - i) Myers-Perry
  - ii) Kerr black string
  - iii) $dS_3 \times S^2$
  - iv) Kundt
- conditions on ρ are necessary but not sufficient, in general (counterexample)

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- conditions on $\boldsymbol{\rho}$ are necessary but not sufficient, in general (counterexample)

Used to find all 5D non-twisting vacuum solutions of type II, III, N [Reall-Graham-Turner'13].

For $n > 5$, form of $\boldsymbol{\rho}$ determined in special cases:

- type III/N [Pravda-Pravdová-Coley-Milson'04]
- Kerr-Schild [M.O.-Pravda-Pravdová'09a, Málek-Pravda'11]
- asymptotically flat [M.O.-Pravda-Pravdová'09b]

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Systematic study in the **non-twisting** case [M.O.-Pravda-Pravdová'13]:

- constraints on $\boldsymbol{\rho}$ (at least one multiple eigenvalue)
- many subcases depending on rank$(\boldsymbol{\rho})$, form of C_{abcd}, ...
- geometric meaning
- again, many of the obtained conditions are only necessary
- m distinct non-zero eigenvalues are possible (contrary to $n = 4, 5$), e.g. $\mathrm{dS}_{m+2} \times \mathrm{S}_{n-2-m}$ $(1 \leq m \leq n-4)$

For example, for $n = 6$:

Case	rank($\boldsymbol{\rho}$)	Possible form of $\boldsymbol{\rho}$	Examples
$\Phi_{ij} \neq 0$	4	$\mathrm{diag}(a,a,a,a)$	Rob.-Tra.
	3	$\mathrm{diag}(a,a,b,0)$	?(†)
	2	$\mathrm{diag}(a,b,0,0)$	$dS_4 \times S^2$
	1	$\mathrm{diag}(a,0,0,0)$	$dS_3 \times S^3$
	0	$\mathrm{diag}(0,0,0,0)$	Kundt
$\Phi_{ij} = 0$	4	$\mathrm{diag}(a,a,a,a)$[2]	Rob.-Tra. ("$\mu = 0$")
	3	X	X
	2	$\mathrm{diag}(a,a,0,0)$	?
	1	X	X
	0	$\mathrm{diag}(0,0,0,0)$	Kundt

[2] Cf. also [Reall-Graham-Turner'13]

Compare possible forms for $n = 4, 5, 6$ (**non-twisting**):

rank($\boldsymbol{\rho}$)	$n = 4$	$n = 5$	$n = 6,\ \Phi_{ij} \neq 0$	$n = 6,\ \Phi_{ij} = 0$
0	$\mathrm{diag}(00)$	$\mathrm{diag}(000)$	$\mathrm{diag}(0000)$	$\mathrm{diag}(0000)$
1	X	$\mathrm{diag}(a00)$	$\mathrm{diag}(a000)$	X
2	$\mathrm{diag}(aa)$	$\mathrm{diag}(aa0)$	$\mathrm{diag}(ab00)$	$\mathrm{diag}(aa00)$?
3	–	$\mathrm{diag}(aaa)$	$\mathrm{diag}(aab0)$?	X
4	–	–	$\mathrm{diag}(aaaa)$	$\mathrm{diag}(aaaa)$

A geometric perspective on Goldberg-Sachs

In 4D, if $\{\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{m}, \bar{\boldsymbol{m}}\}$ is a null frame:

$$[\boldsymbol{m}, \boldsymbol{l}] = (\bar{\alpha} + \beta - \pi)\boldsymbol{l} + \kappa \boldsymbol{n} - \sigma \bar{\boldsymbol{m}} - (\bar{\rho} + \epsilon - \bar{\epsilon})\boldsymbol{m}$$

i.e., $\boldsymbol{l}$ is geodesic and shearfree $\Leftrightarrow \mathcal{D} \equiv \mathrm{Span}\{\boldsymbol{m}, \boldsymbol{l}\}$ is integrable.

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- emphasis on the complex 2-dimensional totally null distribution $\mathcal{D}$ (rather than on $\boldsymbol{l}$)
- natural for extensions to arbitrary signature and *complex geometries* [Plebański-Hacyan'75] (cf. [Gover-Hill-Nurowski'10])

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- partial results also in HD (e.g., "optical structure" in 5D)
- or viceversa, *inequivalent formulation of HD Goldberg-Sachs*: conditions on the Weyl tensor that *ensure* certain integrability properties [Taghavi-Chabert'11,'12].

An example

Einstein spacetime with

- a twistfree multiple WAND $\boldsymbol{l}$
- $\det \boldsymbol{\rho} \neq 0$
- *distinct*, non-zero eigenvalues.

Metric ($R \sim \lambda$, $n \geq 7$):

$$\mathrm{d}s^2 = \lambda r^2 \mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}r + (\lambda u r - 1)^2 \tilde{g}_{AB} \mathrm{d}x^A \mathrm{d}x^B + r^2 \hat{g}_{IJ} \mathrm{d}x^I \mathrm{d}x^J .$$

with $\tilde{g}_{AB}$ Einstein and $\hat{g}_{IJ}$ Ricci-flat.
Multiple WAND:

$$\ell_a \mathrm{d}x^a = \mathrm{d}u, \qquad \ell^a \partial_a = \partial_r .$$

Optical matrix:

$$\rho_{AB} = \frac{\lambda u}{\lambda u r - 1} \delta_{AB}, \qquad \rho_{IJ} = \frac{1}{r} \delta_{IJ} .$$

For $n = 7$: $\boldsymbol{\rho} = \mathrm{diag}(a, a, a, a, b)$.