Complete Classification of five-dimensional type D Einstein spacetimes [1]

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[1] Partially joint work with A. García-Parrado Gómez-Lobo

Contents

Definition in general dimensions

- Einsteinian Ricci tensor: $R_{ab} = 4\Lambda g_{ab}$

- Type D Weyl tensor [Coley-Milson-Pravda-Pravdová (2004)]: $\exists$ at least two double WANDs $\boldsymbol{l}$ and $\boldsymbol{n}$

  $\leftrightarrow$ $\exists$ a real null frame $\{\boldsymbol{e}_{(0)}, \boldsymbol{e}_{(1)}, \boldsymbol{e}_{(i)}\} = \{\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{m}_{(i)}\}$ s.t. only boost weight $b = 0$ Weyl components survive:

$$C_{abcd} = 4C_{0101}\, n_{\{a} l_b n_c l_{d\,\}} + C_{01ij}\, n_{\{a} l_b m^{(i)}_c m^{(j)}_{d\,\}} + 8C_{0i1j}\, n_{\{a} m^{(i)}_b l_c m^{(j)}_{d\,\}} + C_{ijkl}\, m^{(i)}_{\{a} m^{(j)}_b m^{(k)}_c m^{(l)}_{d\,\}}$$

  $\rightarrow$ in 5d: $C_{0101}, C_{01ij}, C_{ijkl}$ combinations of C_{0i1j}

  $\rightarrow$ Weyl tensor isotropic under boosts in $\Sigma = \langle \boldsymbol{l}, \boldsymbol{n} \rangle$

Pioneering work

Goldberg-Sachs (1962)

Kerr, Newman-Unti-Tamburino [NUT] (1963), Brill (1964), Newman (1965), Carter (1968), many others

Kinnersley (1969): full classification and integration of all type D vacua ($\tilde{\Lambda} = 0$)

Plebanski-Demianski (1976): generic type D electrovacuum (any Φ and $\tilde{\Lambda}$)

Completed work

Debever-Kamran-McLenaghan, Garcia (1984): full classification and integration of all type D electrovacua (any Φ and $\tilde{\Lambda}$)

$\rightarrow$ putting $\Phi = 0$ gives all 4d type D Einstein spacetimes

- metric components are rational functions of coordinates
- admit an abelian (sub)group G_2 of isometries
- only classifying constants of integration!

Griffiths-Podolsky (book 2009):
coordinate adaptations, physical meaning

The 4d metrics $(R_{ab} = 3\tilde{\Lambda} g_{ab})$

$$d\tilde{s}^2_{(1)} = \frac{1}{(1-axy)^2}\left\{\frac{K_1}{P_1}dx^2 + \frac{P_1}{K_1}(du + y^2 dv)^2 - 2(du - x^2 dv)\left[dy + \frac{Q_1}{2K_1}(du - x^2 dv)\right]\right.$$

$$P_1 = f_0 + 2lx - cx^2 + 2amx^3 - (a^2 f_0 + \tilde{\Lambda})x^4, \qquad K_1 = x^2 + y^2,$$

$$Q_1 = f_0 - 2my + cy^2 - 2aly^3 - (a^2 f_0 + \tilde{\Lambda})y^4, \qquad l + im \neq 0$$

$$d\tilde{s}^2_{(2\pm)} = \frac{K_2}{P_2}dx^2 \mp \frac{P_2}{K_2}(du + 2lydv)^2 + K_2\left(\frac{dy^2}{Q_2} \pm Q_2 dv^2\right)$$

$$P_2 = 3\tilde{\Lambda}l^4 - cl^2 + 2mx + (c - 6\tilde{\Lambda}l^2)x^2 - \tilde{\Lambda}x^4, \qquad Q_2 = 1 - cy^2, \qquad K_2 = x^2 + l^2,$$

$$l(c - 4\tilde{\Lambda}l^2) + im \neq 0$$

$$d\tilde{s}^2_{(3)} = \frac{1}{(x+y)^2}\left[\frac{dx^2}{P_3} + P_3 du^2 + \frac{dy^2}{Q_3} - Q_3 dv^2\right]$$

$$P_3 = x^3 + Cx + D, \quad Q_3 = y^3 + Cy - D - \tilde{\Lambda}$$

$$d\tilde{s}^2_{(4)} = \frac{dx^2}{P_4} + P_4 du^2 + \frac{dy^2}{Q_4} - Q_4 dv^2, \qquad P_4 = 1 - 3\tilde{\Lambda}x^2, \quad Q_4 = 1 - 3\tilde{\Lambda}y^2, \quad \tilde{\Lambda} \neq 0$$

Definition

Property in 4d: a type D Weyl tensor is

- boost-isotropic in Σ = plane of double WANDs ($\equiv$ PNDs)
- spin-isotropic in $\Sigma' = \Sigma^\perp$

Remark: Σ and Σ' unique

Definition

$\mathcal{A}$ = subclass of 5d Einstein spaces for which the Weyl tensor is

- type D: boost-isotropic in *some* plane Σ of double WANDs
- spin-isotropic in *some* $\Sigma' \subset \Sigma^\perp$

Practical characterization of $\mathcal{A}$

- $\Sigma = \langle \boldsymbol{l}, \boldsymbol{n} \rangle$, $\Sigma' = \langle \boldsymbol{m}, \overline{\boldsymbol{m}} \rangle$

  complex 2+2+1 'NP' null frame:

$$\{\boldsymbol{e}_{(0)} = \boldsymbol{l}, \boldsymbol{e}_{(1)} = \boldsymbol{n} | \boldsymbol{m}, \overline{\boldsymbol{m}} | \boldsymbol{u}\}$$

- 9 Weyl components $C_{0\alpha 1\beta}$, $\alpha, \beta \in \{m, \bar{m}, u\}$

  $\rightarrow$ 6 with non-trivial boost or spin weight vanish:

$$C_{0m1m} = C_{0m1u} = C_{0u1m} = 0 \qquad \text{(and c.c.)}$$

  $\rightarrow$ 3 remaining:

$$\Psi_2 \equiv C_{0m1\bar{m}} \text{ (complex)}, \qquad \Psi_{11} \equiv -\tfrac{1}{2} C_{0u1u} \text{ (real)}$$

Classification result

- use 2+2+1 complex extension of GHP formalism (Σ-boost and Σ'-spin covariant)

- Bianchi and Ricci identities lead to

either $\Psi_2\bar{\Psi}_2 = 4\Psi_{11}^2$ or $\Psi_2 = \bar{\Psi}_2$ or $\Psi_{11} = 0$

$\rightarrow$ partition of $\mathcal{A}$ in 5 subclasses:

$$\begin{aligned}
\mathcal{A}\text{-I}: &\quad \Psi_{11} = 0 \neq \Psi_2 \\
\mathcal{A}\text{-II}: &\quad \Psi_2 = \bar{\Psi}_2,\ \Psi_2^2 \neq 4\Psi_{11}^2 \neq 0 \\
\mathcal{A}\text{-III}: &\quad \Psi_2 \neq \bar{\Psi}_2,\ |\Psi_2|^2 = 4\Psi_{11}^2 \\
\mathcal{A}^+: &\quad \Psi_2 = 2\Psi_{11} \neq 0 \\
\mathcal{A}^-: &\quad \Psi_2 = -2\Psi_{11} \neq 0
\end{aligned}$$

Uniqueness of Σ and Σ'

- $\mathcal{A}^+$: Σ' unique, Σ non-unique (isotropy in $\Sigma'^{\perp}$)
- $\mathcal{A}^-$: Σ unique, Σ' non-unique (isotropy in $\Sigma^{\perp}$)
- $\mathcal{A}$-I, $\mathcal{A}$-II, $\mathcal{A}$-III: Σ and Σ' unique

5d type D in general:

$$\begin{aligned}\mathcal{A}^+ &= \Sigma \text{ non-unique} \\ &= \exists \text{ more than 2 double WANDs} \\ &\quad (\text{circle of double WANDs}) \\ &= \exists \text{ non-geodesic double WANDs}\end{aligned}$$

Class $\mathcal{A}^+$ $(\Psi_2 = 2\Psi_{11})$

Theorem [Durkee-Reall (2009)

$\mathcal{A}^+$ is the union of spacetimes with metrics

(1) $\mathrm{d}\Omega^3_-[k] + \mathrm{d}\Omega^2_+[2k]$, $k \neq 0$ $(\mathrm{dS}_3 \times \mathrm{S}^2$ or $\mathrm{AdS}_3 \times \mathrm{H}^2)$

(2) $$\mathrm{d}s^2 = r^2 \mathrm{d}\Omega^3_-[k] + \frac{\mathrm{d}r^2}{U(r)} + U(r)\mathrm{d}z^2$$

$$U(r) = k - \frac{m}{r^2} - \Lambda r^2, \qquad k \in -1, 0, 1$$

(anal. cont. 'Schwarzschild')

- circle of double WANDs (tangent to $r =$const., $z =$ constant)
- $\exists$ non-geodesic double WAND fields
  $\exists$ geodesic double WAND fields $\boldsymbol{l}$ with zero optical matrix $(\boldsymbol{\rho} \equiv [l_{i;j}] = 0)$: double Kundt solutions

Class $\mathcal{A}^-$ $(\Psi_2 = -2\Psi_{11})$

Theorem

$\mathcal{A}^-$ is the union of spacetimes with metrics

$$(1) \quad d\Omega^3_+[k] + d\Omega^2_-[2k],\ k \neq 0 \quad (dS_2 \times S^3 \text{ or } AdS_2 \times H^3)$$

$$(2) \quad ds^2 = r^2 d\Omega^3_+[k] + \frac{dr^2}{U(r)} - U(r)dt^2$$

$$U(r) = k - \frac{m}{r^2} - \Lambda r^2, \qquad k \in -1, 0, 1$$

Robinson-Trautman sols [Podolský-Ortaggio (2006)]

Sachs dual of $\mathcal{A}^+$

Class $\mathcal{A}$-I $\quad (\Psi_{11} = 0 \neq \Psi_2)$

5d type D 'black brane' class = Brinkmann (1925) warps

$$(1) \quad \mathrm{d}s^2 = \mathrm{d}z^2 + e^{-2\sqrt{-\Lambda}z}\mathrm{d}\tilde{s}^2[\tilde{\Lambda} = 0]$$

$$(2) \quad \mathrm{d}s^2 = \frac{\mathrm{d}x^2}{(x^2+\Lambda)^2} + \frac{1}{|x^2+\Lambda|}\mathrm{d}\tilde{s}^2[\tilde{\Lambda} = \delta], \qquad \delta = \pm 1.$$

$\mathrm{d}\tilde{s}^2[\tilde{\Lambda}]$ = 4d type D Einstein spacetime with cosm. const. $3\tilde{\Lambda}$

isometry group G_r compared to $\tilde{G}_{\tilde{r}}$:

- $\tilde{\Lambda} \neq 0$: $r = \tilde{r}$
- $\tilde{\Lambda} = 0 \neq \Lambda$: $r = \tilde{r}$ except for special values of the constants
- $\tilde{\Lambda} = \Lambda = 0$: $r = \tilde{r} + 1$ (direct product case)

Class $\mathcal{A}$-II $\quad (\Psi_2 = \bar{\Psi}_2,\ \Psi_2^2 \neq 4\Psi_{11}^2 \neq 0)$

cohomogeneity-1 metrics

$$\mathrm{d}s^2 = (\mathrm{d}x)^2 - \frac{2\mathrm{d}u\mathrm{d}v}{f(x)^2\left(1 - \frac{k}{2}uv\right)^2} + \frac{2\mathrm{d}\zeta\mathrm{d}\bar{\zeta}}{g(x)^2\left(1 + \frac{k'}{2}\zeta\bar{\zeta}\right)^2}.$$

dynamical system

$$\begin{aligned}
\mathrm{d}\Psi_{11} &= [\upsilon(\Lambda + \mathfrak{e}^2 - 2\Psi_{11}) + \mathfrak{e}(\Lambda + \upsilon^2 + 2\Psi_{11})]\mathrm{d}x \\
\mathrm{d}\upsilon &= (2\Psi_{11} + \Lambda - \upsilon^2)\mathrm{d}x \\
\mathrm{d}\mathfrak{e} &= (2\Psi_{11} - \Lambda + \mathfrak{e}^2)\mathrm{d}x
\end{aligned}$$

2 x 2 possibilities

- $k = 0, \quad \mathrm{d}f = \mathfrak{e}f\mathrm{d}x$
  $k = \pm 1, \quad f(x)^2 = |\mathfrak{e}^2 - 2\upsilon\mathfrak{e} - 3\Lambda - 2\Psi_{11}|$
- $k' = 0, \quad \mathrm{d}g = -\upsilon g\mathrm{d}x$
  $k' = \pm 1, \quad f(x)^2 = |\upsilon^2 - 2\upsilon\mathfrak{e} - 3\Lambda + 2\Psi_{11}|$

Class $\mathcal{A}$-II $\quad (\Psi_2 = \bar{\Psi}_2,\ \Psi_2^2 \neq 4\Psi_{11}^2 \neq 0)$

- general solution depends on two 'initial values' of reduced dynamical system
- group $G_6 = G_3 \times G_3$ of isometries
- known solutions in closed form in the cases [Gregory (1996)]:
  - $k = k' = 0$
  - $\Lambda = 0$ and $[k = 0,\ k' = \pm 1$ or $k = \pm 1,\ k' = 0]$

Class $\mathcal{A}$-III $\quad (\Psi_2 \neq \bar{\Psi}_2,\ |\Psi_2|^2 = 4\Psi_{11}^2)$

$$\Psi_{11} = \frac{p}{z^2 \bar{z}^2}, \qquad \Psi_2 = 2\Psi_{11}\frac{z}{\bar{z}} = \frac{2p}{z\bar{z}^3}$$

$$z = y + ir, \qquad p = \mathrm{sgn}(\Psi_{11})$$

r constant $\Leftrightarrow$ metric Σ-boost isotropic
y constant $\Leftrightarrow$ metric Σ'-spin isotropic

- y- and r-integration directly from GHP system
- use presence of abelian G_3 in all cases to define good Killing coordinates

Class $\mathcal{A}$-III $\quad (\Psi_2 \neq \bar{\Psi}_2, \; |\Psi_2|^2 = 4\Psi_{11}^2)$

(1) y and r non-constant $\to$ maximal abelian G_3

(1a) non-null orbits: Kerr-(A)dS solutions, including Myers-Perry (1986)

$$ds^2 = \frac{dr^2}{P} - P(dv + y^2 du)^2 + \frac{dy^2}{Q} + Q(dv - r^2 du)^2$$
$$+ \left[yr\, dt + \frac{J}{yr} \left(dv + (y^2 - r^2) du \right) \right]^2$$
$$P = \frac{P^*(r)}{y^2 + r^2}, \quad P^*(r) = -\Lambda r^4 + C_1 r^2 + (C_2 + p) + J^2/r^2$$
$$Q = \frac{Q^*(y)}{y^2 + r^2}, \quad Q^*(y) = -\Lambda y^4 - C_1 y^2 + (C_2 - p) - J^2/y^2$$

(1b) null orbits $\to$ new, single Kundt solution

$$ds^2 = -2dr(dv + y^2 du) + (y^2 + r^2) dy^2 + \frac{(dv - r^2 du)^2}{y^2 + r^2} + y^2 r^2 dt^2$$

Class $\mathcal{A}$-III $\quad (\Psi_2 \neq \bar{\Psi}_2,\ |\Psi_2|^2 = 4\Psi_{11}^2)$

(2) y constant, r non-constant $\to G_5$ (Σ'-spin isotropy)

$$ds^2 = \frac{dr^2}{P} - P(dv + 2xy\,du)^2 + (y^2 + r^2)\left[\frac{dx^2}{1 - K'x^2} + (1 - K'x^2)du^2\right]$$
$$+ \left[yr\,dt + \frac{J}{yr}\left(dv + \frac{2x}{y}(y^2 + r^2)du\right)\right]^2,$$
$$P = P(r) = \frac{2p}{y^2 + r^2} - (y^2 + r^2)\left(\Lambda - \frac{J^2}{y^4 r^2}\right), \qquad K' = 4\Lambda y^2 + \frac{4J^2}{y^4}$$

Class $\mathcal{A}$-III $\quad (\Psi_2 \neq \bar{\Psi}_2,\ |\Psi_2|^2 = 4\Psi_{11}^2)$

(3) y non-constant, r constant $\to G_5$ (Σ-boost isotropy)

$$ds^2 = \frac{dy^2}{Q} - Q(dv + 2xr\,du)^2 + (y^2 + r^2)\left[\frac{dx^2}{1 - Kx^2} - (1 - Kx^2)du^2\right]$$
$$+ \left[yr\,dt + \frac{J}{yr}\left(dv + \frac{2x}{r}(y^2 + r^2)du\right)\right]^2$$
$$Q = Q(y) = \frac{-2p}{y^2 + r^2} - (y^2 + r^2)\left(\Lambda + \frac{J^2}{y^2 r^4}\right), \qquad K = 4\Lambda r^2 - \frac{4J^2}{r^4}$$

- new, double Kundt solutions
- Sachs dual of case (2)

$$y \leftrightarrow r, \qquad u \to iu, \qquad v \to iv, \qquad p \to -p, \qquad J \to -iJ$$

Class $\mathcal{A}$-III $\quad (\Psi_2 \neq \bar{\Psi}_2, \; |\Psi_2|^2 = 4\Psi_{11}^2)$

(4) y and r constant $\to G_7$ (homogenous, both isotropies)

$$c \equiv \frac{y}{r}$$

$$ds^2 = \frac{dq^2}{1 - Kq^2} - (1 - Kq^2)du^2 + \frac{dx^2}{1 - K'x^2} + (1 - K'x^2)dv^2 + [dt + \mathfrak{e}(qdu + cxdv)]^2$$

$$K = 2(2c^2 - 1)\mathfrak{e}^2 \neq 0, \qquad K' = 2(c^2 - 2)\mathfrak{e}^2 \neq 0$$

$$12\Lambda = K + K' = 6\mathfrak{e}^2(c^2 - 1)$$

Origin: the Hd "shearfree" GS result [Ortaggio+Pravda+Pravdová+Reall (2012)]

- genuine type II Einstein spacetimes
- "geodesic part" of Goldberg-Sachs in Hd
  $\rightarrow \exists$ geodesic double WAND
- take real null frame $\{\boldsymbol{e}_{(0)}, \boldsymbol{e}_{(1)}, \boldsymbol{e}_{(i)}\} = \{\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{m}_{(i)}\}$
  put $\rho_{ij} \equiv l_{i;j}$ and $\Phi_{ij} \equiv C_{0i1j}$
- take 2+3 covariant GHP formalism: Bianchi-Ricci system gives chain of integrability conditions

$$Þ^i \mathcal{F} = 0, \qquad \mathcal{F} = P(\rho_{ij}, \Phi_{ij})$$

$\rightarrow$ can be solved in 5d

Intermezzo: characterization of $\mathcal{A}$

in general:

$$\Phi_{ij} = \Phi^S_{ij} + \Phi^A_{ij} = \Phi^S_{ij} + \epsilon_{ijk} w^k$$

characterization of $\mathcal{A}$: $\exists$ real null frame $\{\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{m}_{(i)}\}$ s.t.

$$[\Phi^S_{ij}] = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix}, \quad [\Phi^A_{ij}] = \begin{bmatrix} 0 & c & 0 \\ -c & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \leftrightarrow [w^i] = \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix}$$

$$[\Phi_{ij}] = \begin{bmatrix} a & c & 0 \\ -c & a & 0 \\ 0 & 0 & b \end{bmatrix}$$

$\leftrightarrow$ spin types $\{(11)1\}_{\|}$, $\{(11)1\}_0$, $\{(000)\}_{\|}$, $\{(000)\}_0$
[Coley-Hervik-Ortaggio-Wylleman (2012)]

The 5d "shearfree" GS result

The 5d "shearfree" GS theorem [Ortaggio+Pravda+Pravdová+Reall (2012)]

Any 5d genuine type II Einstein spacetime admits a geodesic double WAND $\boldsymbol{l}$ for which, in a suitable real null frame, the matrix $\boldsymbol{\rho} = [\rho_{ij}]$ takes one of the forms

$$i) \quad \boldsymbol{\rho} = b \begin{pmatrix} 1 & a & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1+a^2 \end{pmatrix}, \qquad ii) \quad \boldsymbol{\rho} = b \begin{pmatrix} 1 & a & 0 \\ -a & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$iii) \quad \boldsymbol{\rho} = b \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & a & 0 \end{pmatrix}, \qquad iv) \quad \boldsymbol{\rho} = 0.$$

The 5d "shearfree" GS result

The 5d "shearfree" GS theorem: continuation [Ortaggio+Pravda+Pravdová+Reall (2012)]

For cases (i)-(iii) the matrix $\boldsymbol{\Phi} = [\Phi_{ij}]$ takes one of the following forms:

$$i) \quad \boldsymbol{\Phi} = \Phi_{44} \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad ii) \quad \boldsymbol{\Phi} = \begin{pmatrix} \Phi_{22} & \Phi_{23} & 0 \\ \Phi_{32} & \Phi_{33} & 0 \\ \Phi_{42} & \Phi_{43} & 0 \end{pmatrix},$$

$$iii) \quad \boldsymbol{\Phi} = \mathrm{diag}(\Phi, \Phi, -\Phi) \quad = \mathcal{A}^+.$$

Corollary

A 5d type D Einstein spacetime which admits a geodesic double WAND with a case (i) or (iii) optical matrix belongs to $\mathcal{A}$.

Classification theorem

One can prove:

Proposition

A 5d type D Einstein spacetime which admits a geodesic double WAND with a case (ii) optical matrix belongs to $\mathcal{A}$-I.

[Coley-Hervik-Papadopoulos-Pelavas (2009)] implies:

Classification theorem

A 5d type D Einstein spacetime either belongs to $\mathcal{A}$ or is "double Kundt": it admits a distribution Σ of double WANDs with vanishing optical matrix, in which it is boost-isotropic.

This distribution is either

- integrable $\to$ free functions: 1 of 3 coords and 2 of 1 coord
- non-integrable $\to$ free functions: 2 of 1 coord

GHP description ok, coordinates under construction

Summary and outlook

- studied 5d type D Einstein spacetimes (arbitrary Λ)
- fully classified and integrated the subclass $\mathcal{A}$ of spacetimes for which the Weyl tensor admits spin isotropy
  $\rightarrow$ only constants, minimal G_2 (cf. 4d)
  $\rightarrow$ new examples in $\mathcal{A}$-III
- other spacetimes are necessarily “double Kundt”
  $\rightarrow$ free functions arise
- remark: type II (even non-twisting) is less restrictive [Reall-Graham-Turner (2013)]

- interpretation of solutions
- classifications in $D \geq 5$? [Ortaggio-Pravda-Pravdová (2013)]