Universitat de les Illes Balears

3.5 post-Newtonian spin-orbit effects in the phasing of inspiralling compact binaries

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Class. Quantum Grav. 30 (2013) 055007
Class. Quantum Grav. 30 (2013) 075017
Class. Quantum Grav. 30 (2013) 135009

and A. Buonanno - Maryland

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Encuentros Relativistas Españoles
Benasque - 10/09/13

Outline

- Motivation and brief introduction to PN
- Ingredients of the computation of the 3.5 PN spin-orbit effects
    - The pole-dipole effective formalism
    - Reduction of the result (to an useful form)
    - Tests of the result
- Impact of these new results for data analysis

Motivation: building accurate templates for GW detection

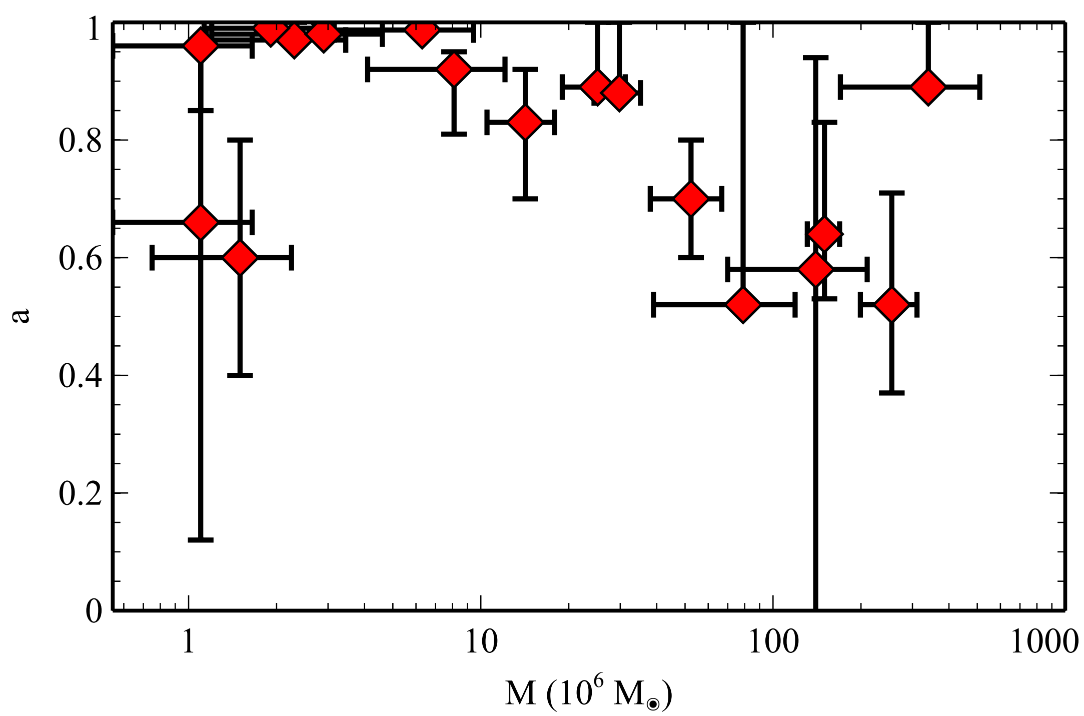
- Binaries of compact objects (black holes and/or neutron stars) are one of the most promising sources of GW that we hope to detect with the advanced versions of LIGO/Virgo and with a future space-based detector.

- Successfully extracting the very weak signal from the noise and estimating the parameters of the source with good precision can be achieved using matched filtering techniques provided that we have a very accurate modelling of the waveform.

- The post-Netwtonian approximation scheme enables to compute such accurate waveforms as an expansion in v/c for the inspiral phase. For non-spinning compact binaries, such templates are known to 3.5 PN order for the phase (3PN for the amplitude). The contributions from these high orders have a significant effect on parameter estimation (see Arun et al. 2005)

- Recent observational evidence indicates that black holes generically have large spins (close to maximal)

Reynolds 1302.3260 (2013)



→ include spin effects to the same level of accuracy

PN approximation scheme (1/3)

rewrite Einstein eqs

$$h^{\mu\nu} = \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu} \longrightarrow$$

$$\boxed{\begin{aligned} &\partial_\mu h^{\alpha\mu} = 0 \quad \text{harmonic gauge} \\ &\Box h^{\mu\nu} = \frac{16\pi G}{c^4}\tau^{\mu\nu} \end{aligned}}$$

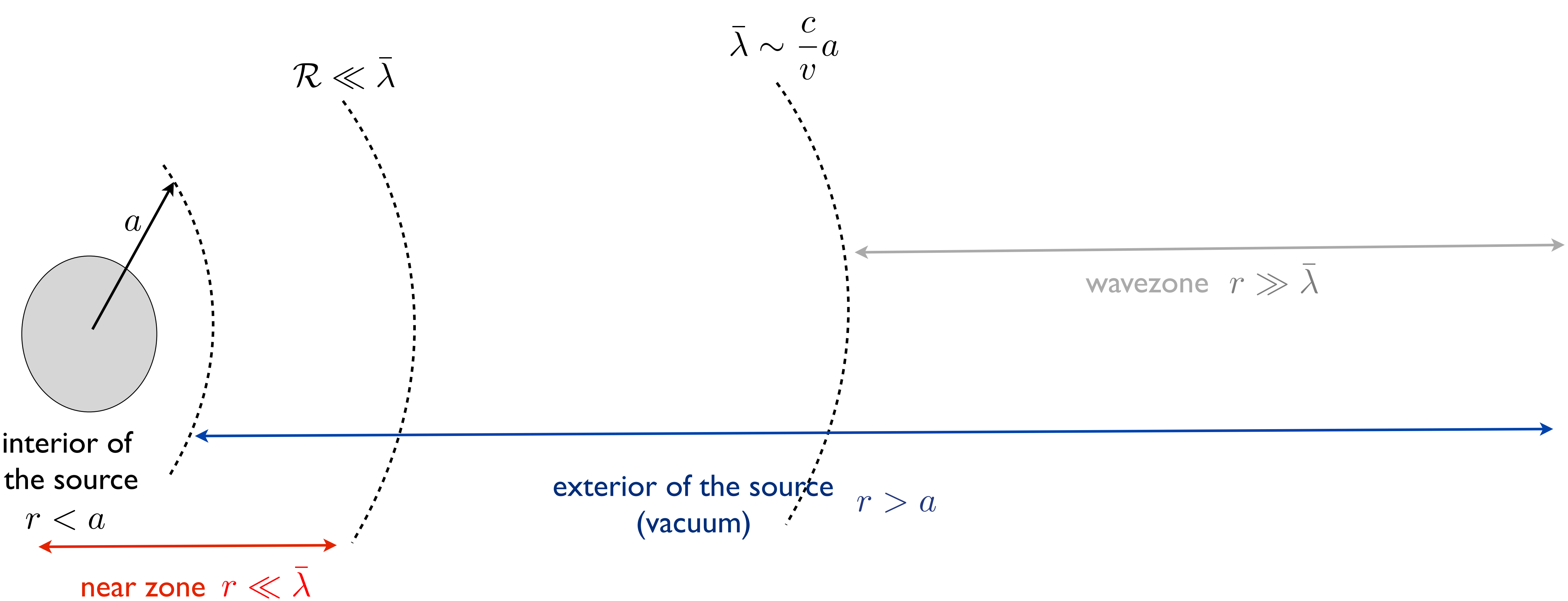
$\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ “flat” d’Alembertian

$$\Box^{-1}f = -\frac{1}{4\pi}\int \frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} f\left(\mathbf{x}', t - \frac{|\mathbf{x}'-\mathbf{x}|}{c}\right)$$

$\tau^{\mu\nu}$ stress-energy pseudo tensor of matter + gravitational fields

$$\tau^{\mu\nu} = |g|T^{\mu\nu} + \frac{c^4}{16\pi G}\Lambda^{\mu\nu}$$

$$\begin{aligned}\Lambda^{\alpha\beta} = &-h^{\mu\nu}\partial^2_{\mu\nu}h^{\alpha\beta} + \partial_\mu h^{\alpha\nu}\partial_\nu h^{\beta\mu} + \frac{1}{2}g^{\alpha\beta}g_{\mu\nu}\partial_\lambda h^{\mu\tau}\partial_\tau h^{\nu\lambda} \\ &- g^{\alpha\mu}g_{\nu\tau}\partial_\lambda h^{\beta\tau}\partial_\mu h^{\nu\lambda} - g^{\beta\mu}g_{\nu\tau}\partial_\lambda h^{\alpha\tau}\partial_\mu h^{\nu\lambda} + g_{\mu\nu}g^{\lambda\tau}\partial_\lambda h^{\alpha\mu}\partial_\tau h^{\beta\nu} \\ &+ \frac{1}{8}(2g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu})(2g_{\lambda\tau}g_{\epsilon\pi} - g_{\tau\epsilon}g_{\lambda\pi})\partial_\mu h^{\lambda\pi}\partial_\nu h^{\tau\epsilon}.\end{aligned}$$



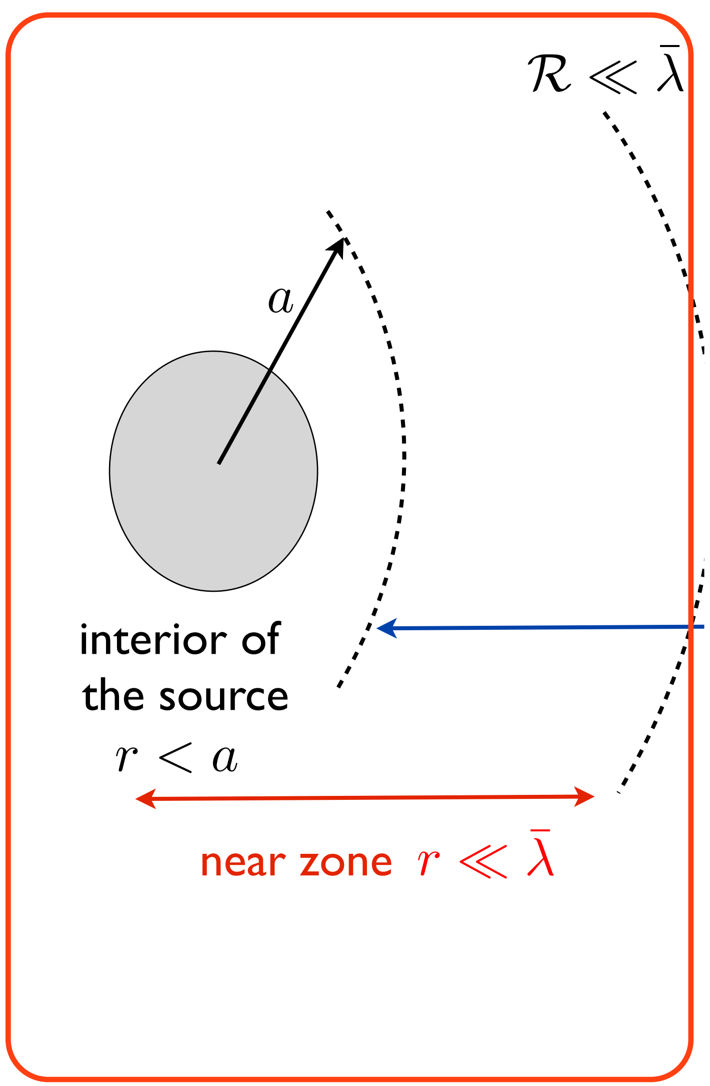
PN approximation scheme (2/3)

$$\partial_\mu h^{\alpha\mu} = 0$$
$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4}\tau^{\mu\nu}$$

Write the solution as formal PN series in powers of 1/c and solve iteratively order by order

$$\Box^{-1} f = -\frac{1}{4\pi}\int \frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} f\left(\mathbf{x}', t - \frac{|\mathbf{x}'-\mathbf{x}|}{c}\right)$$

retardation effects are small
we can (PN) expand inside the integrals



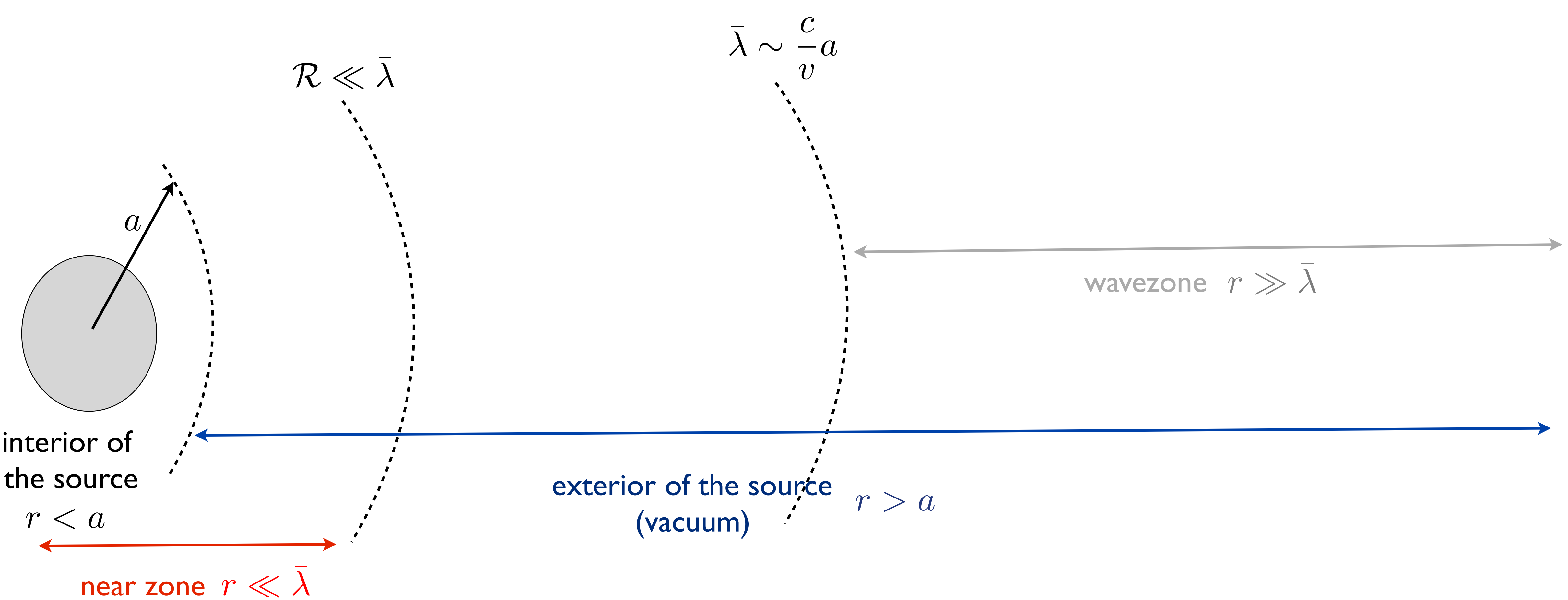
Beyond leading order, even if the source has compact support, the support of the integral diverges at spatial infinity... first need for a regularization

How to impose the no incoming radiation condition?
... the definition of the appropriate inverse operator requires knowledge from the far-zone

$$\Box^{-1} f = -\frac{1}{4\pi}\sum_n \frac{(-1)^n}{n!}\left(\frac{\partial}{c\,\partial t}\right)^n \mathrm{FP}_{B=0}\int d^3\mathbf{x}'|\mathbf{x}-\mathbf{x}'|^{n-1} f(\mathbf{x}', t)$$

see e.g. Blanchet's Living Review for a detailed construction of the solution

PN approximation scheme (2/3)



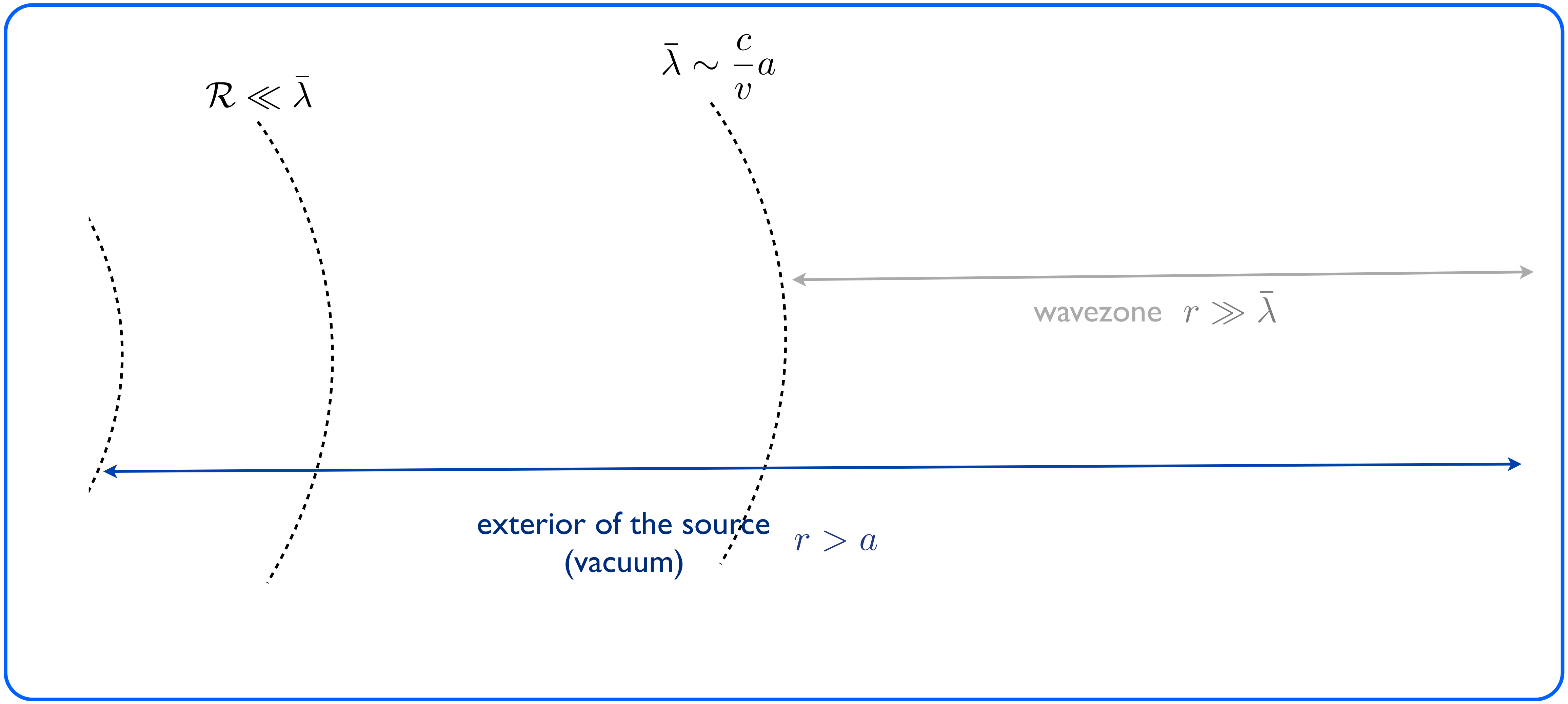
PN approximation scheme (2/3)

$$\partial_\mu h^{\alpha\mu} = 0 \quad \text{harmonic gauge}$$

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4}\tau^{\mu\nu}$$

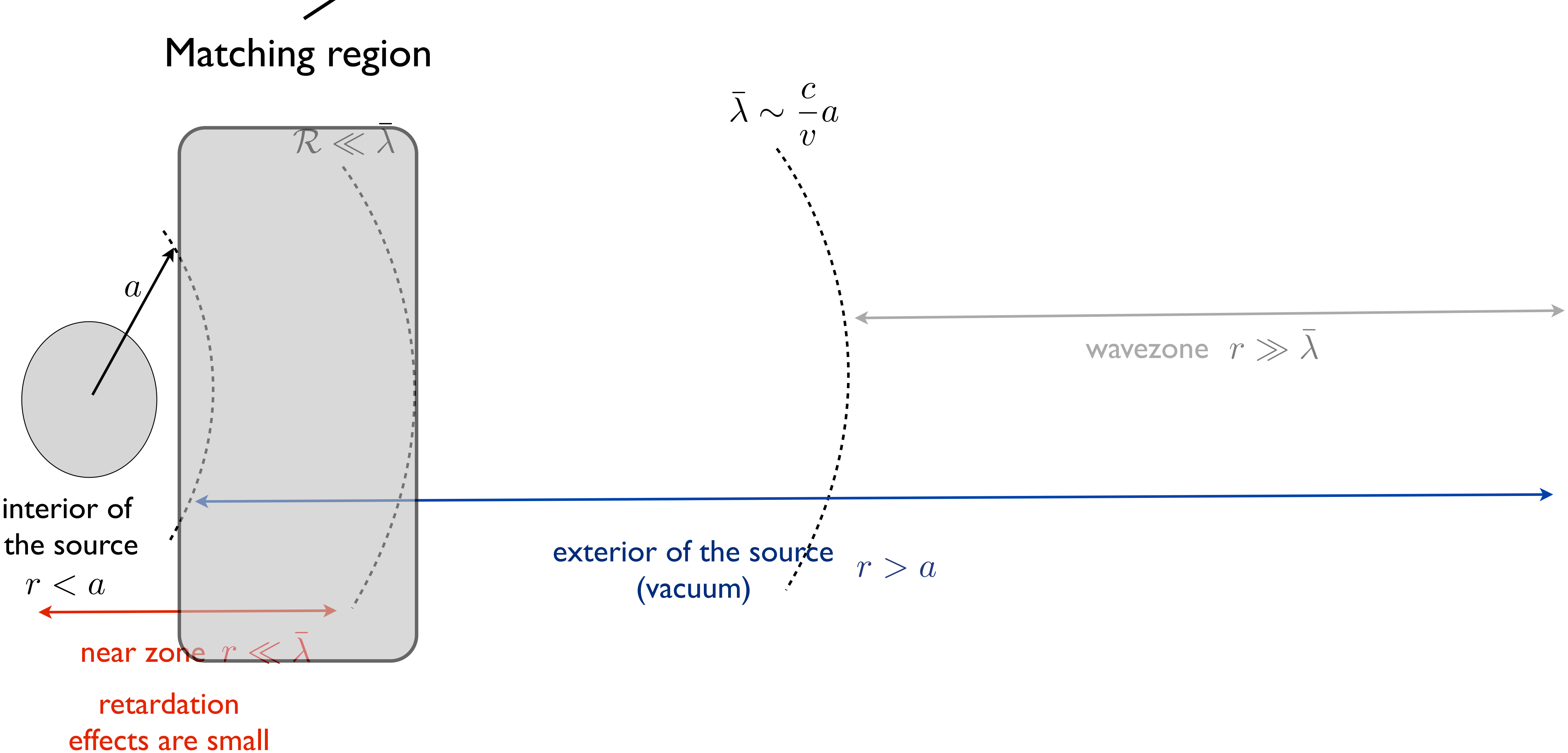
The most general solution in vacuum can be obtained by combining post-Minkowskian expansion + multipole expansion

$$h^{\mu\nu}_{\text{ext}} = \sum_{n=1}^{+\infty} G^n h^{\mu\nu}_{(n)}[I_L, J_L, W_L, X_L, Y_L, Z_L]$$



PN approximation scheme (2/3)

Both expansions are valid. A matching procedure provides an expression of the multipole moments as integrals over the matter and the gravitational fields in the source.



PN approximation scheme (3/3)

In practice, the calculation is divided into two (coupled) sub-problems

Computation of the dynamics up to n-th PN order (near-zone resolution of the Einstein eqs)

Newtonian-like equation of motion

$$\frac{\mathrm{d}v_1^i}{\mathrm{d}t} = A_{\mathrm{N}}^i + \frac{1}{c^2}A_{1\mathrm{PN}}^i + \frac{1}{c^4}A_{2\mathrm{PN}}^i + \frac{1}{c^5}A_{2.5\mathrm{PN}}^i + \frac{1}{c^6}A_{3\mathrm{PN}}^i + \frac{1}{c^7}A_{3.5\mathrm{PN}}^i + \mathcal{O}(8)$$

quasi-circular orbits in the CM frame $x = \left(\frac{Gm\omega}{c^3}\right)^{2/3} \sim \mathcal{O}(1/c^2)$

"conserved" Energy

$$E = -\frac{\mu c^2 x}{2}\left[1 + e_1 x + e_2 x^2 + e_3 x^3 + O(1/c^8)\right]$$

Computation of the radiation up to n-th PN order

flux $$\mathcal{F} = \frac{32c^5}{5G}x^5\nu^2\left[1 + f_1\,x + f_{1.5}\,x^{3/2} + f_2\,x^2 + f_{2.5}\,x^{5/2} + f_3\,x^3 + f_{3.5}\,x^{7/2} + \mathcal{O}(8)\right]$$

Finally, the balance equation $\frac{dE}{dt} = -\mathcal{F}$ provides the phase evolution

Progress of the spin PN computations: dynamics

We redefine our spin variable as $S \equiv c\, S_{\rm phys} = \chi G m^2$

so that S is of Newtonian order for maximally spinning compact objects.

$$\frac{\mathrm{d}v_1^i}{\mathrm{d}t} = A_{\rm N}^i + \frac{1}{c^2} A_{\rm 1PN}^i + \frac{1}{c^3} {A_S^i}_{\rm 1.5PN} + \frac{1}{c^4}\left[A_{\rm 2PN}^i + {A_{SS}^i}_{\rm 2PN}\right] + \frac{1}{c^5}\left[A_{\rm 2.5PN}^i + {A_S^i}_{\rm 2.5PN}\right]$$
$$+ \frac{1}{c^6}\left[A_{\rm 3PN}^i + {A_{SS}^i}_{\rm 3PN}\right] + \frac{1}{c^7}\left[A_{\rm 3.5PN}^i + {A_S^i}_{\rm 3.5PN}\right] + \mathcal{O}(8)$$

LO Spin-Orbit (1/c^3):

Barker and O'Connell (75, 79)
Goldberger, Rothstein (06) (EFT approach)

NLO Spin-Orbit (1/c^5):

Tagoshi, Ohashi, Owen (98, 01)
Blanchet, Buonanno, Faye (06)
Damour, Jaranowski, Schäfer, (08) (ADM formalism)
Levi (10), Porto (10) (EFT)

Spin-Spin effects:

LO (1/c^4): Kidder, Will, Wiseman, (93)
Porto (05) (EFT)
Buonanno, Faye, Hinderer (13)
NLO (1/c^6): Steinhoff, Hergt, Schäfer (08,10) (ADM)
Porto, Rothstein (10), Levi (11) (EFT)
NNLO (1/c^8) spin1-spin2:
Hartung, Steinhoff (11) (ADM)
Levi (12) (EFT)

Here we compute the 3.5PN spin-orbit (linear in spin) correction together with the evolution equations for the spins

NNLO (1/c^7):

Hartung Steinhoff (11) (ADM)
Marsat, Bohe, Faye, Blanchet, (12)

Progress of the spin PN computations: Radiation

So far, a wave generation formalism has only been derived in the harmonic gauge formulation (although EFT on the way (cf Porto (06))

$$\mathcal{F} = \frac{32c^5}{5G} x^5 \nu^2 \left[1 + f_1\, x + f_{1.5}\, x^{3/2} + f_2\, x^2 + f_{2.5}\, x^{5/2} + f_3\, x^3 + f_{3.5}\, x^{7/2} + \mathcal{O}(4) \right]$$

For the flux

Spin-Orbit effects

LO ($1/c^3$): Kidder, Will, Wiseman (93, 95)

NLO ($1/c^5$): Blanchet, Buonanno, Faye (06)

NNLO ($1/c^7$): Bohe, Marsat, Blanchet, (13)

Tail SO effects

LO ($1/c^6$): Blanchet, Buonanno, Faye (06)

NLO ($1/c^8$): Marsat, Bohe, Blanchet, Buonanno

Spin-Spin effects

LO ($1/c^4$): Mikoczi, Vasuth, Gergely (05)

For the polarizations

SO LO ($1/c^3$): Kidder, Will, Wiseman (93, 95)
Arun, Buonanno, Faye, Ochsner (09)

SS LO ($1/c^4$): Kidder, Will, Wiseman (95, 96) Spin1-Spin2
Buonanno, Faye, Hinderer Spin1-Spin1

tail LO ($1/c^6$): Blanchet, Buonanno, Faye (06)

Progress

- Motivation and brief introduction to PN
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  - The pole-dipole effective formalism
  - Reduction of the result (to an useful form)
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Description of the system: pole-dipole formalism

Effective description in terms of spinning point particles: pole-dipole approximation

$$T^{\mu\nu} = c^2 \int_{-\infty}^{+\infty} \mathrm{d}\tau \; p^{(\mu} u^{\nu)} \frac{\delta^{(4)}(x - y(\tau))}{\sqrt{-g(x)}} - c \int_{-\infty}^{+\infty} \mathrm{d}\tau \; \nabla_\rho \left[S^{\rho(\mu} u^{\nu)} \frac{\delta^{(4)}(x - y(\tau))}{\sqrt{-g(x)}} \right] .$$

Formlism developped by
Mathisson, Papapetrou, Tulczyjew
generalized by Dixon, Bailey & Israel

$$\frac{\mathrm{D}S^{\mu\nu}}{\mathrm{d}\tau} = c^2 \left(p^\mu u^\nu - p^\nu u^\mu \right) ,$$
$$\frac{\mathrm{D}p^\mu}{\mathrm{d}\tau} = -\frac{1}{2} R^\mu{}_{\nu\rho\sigma} u^\nu S^{\rho\sigma}$$

Mathisson-Papapetrou
equations of motion

We work with the covariant Tulczyjew supplementary spin condition $S^{\mu\nu} p_\nu = 0$ and we restrict to effects linear in the spins. The equations of motion reduce to

$$\boxed{\begin{aligned} \frac{\mathrm{D}S^{\mu\nu}}{\mathrm{d}\tau} &= \mathcal{O}(S^2) , \\ mc \frac{\mathrm{D}u^\mu}{\mathrm{d}\tau} &= -\frac{1}{2} R^\mu{}_{\nu\rho\sigma} u^\nu S^{\rho\sigma} + \mathcal{O}(S^2) \end{aligned}}$$

Such a point particle description has to be supplemented with some UV regularization procedure.
(Hadamard regularization, dimensional regularization)

Tests of the result

- Existence of 10 conserved integrals of the motion
 (when neglecting radiation reaction terms)

 Energy, Linear Momentum, Angular Momentum, Center of Mass Position

 Determined using the method of undetermined coefficients

- Lorentz invariance

 The harmonic gauge condition is manifestly Lorentz invariant so our equation of motion must take the same form in two frames related to one another by a boost

- Test-mass limit

 Recover the motion of a test mass around Kerr and of a spinning test mass around Schwarzschild (linear effects in spin)

- Equivalence with the ADM result

 Extended the "contact" transformation

$$\mathbf{Y}_1 = \overline{\mathbf{x}}_1 + \frac{1}{c^3}\underset{S}{\mathbf{Y}}{}_1^{1.5\mathrm{PN}} + \frac{1}{c^4}\mathbf{Y}_1^{2\mathrm{PN}} + \frac{1}{c^5}\underset{S}{\mathbf{Y}}{}_1^{2.5\mathrm{PN}} + \frac{1}{c^6}\mathbf{Y}_1^{3\mathrm{PN}} + \frac{1}{c^7}\underset{S}{\mathbf{Y}}{}_1^{3.5\mathrm{PN}} + \mathcal{O}\left(\frac{1}{c^8}\right)$$

 together with the relation between both spin variables

Reduction of the result

We first rewrite our result in term of spin variables S^i of conserved Euclidian norm $\delta_{ij}S^iS^j = s^2$

The spin evolution equations reduce to simple precession equations $\frac{\mathrm{d}\mathbf{S}_1}{\mathrm{d}t} = \mathbf{\Omega}_1 \times \mathbf{S}_1$ $\mathbf{\Omega}_1$ up to 3PN

Conserved spins are secularly constant at spin orbit level (required for Taylor approximants)

We then reduce to the center of mass frame defined by $P^i = 0, G^i = 0$

Everything is expressed in terms of
$$\mathbf{x} = \mathbf{y}_1 - \mathbf{y}_2$$
$$r = |\mathbf{x}|$$
$$\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2,$$
and
$$\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$$
$$\mathbf{\Sigma} = m\left(\frac{\mathbf{S}_2}{m_2} + \frac{\mathbf{S}_1}{m_1}\right)$$

Finally, we are mostly interested in quasi-circular orbits

The emission of GW circularizes the orbit. We look for solutions for which the separation r only varies due to radiation reaction $\dot{r} = \mathcal{O}(1/c^5)$ and change variable to $x = \left(\frac{Gm\omega}{c^3}\right)^{2/3}$

$$\begin{aligned}
E = -\frac{\mu c^2 x}{2}\Bigg\{ & 1 + x\left(-\frac{3}{4} - \frac{1}{12}\nu\right) + x^2\left(-\frac{27}{8} + \frac{19}{8}\nu - \frac{1}{24}\nu^2\right) \\
& + x^3\left(-\frac{675}{64} + \left[\frac{34445}{576} - \frac{205}{96}\pi^2\right]\nu - \frac{155}{96}\nu^2 - \frac{35}{5184}\nu^3\right) \\
& + \frac{x^{3/2}}{Gm^2}\left[\frac{14}{3}S_\ell + 2\frac{\delta m}{m}\Sigma_\ell\right] + \frac{x^{5/2}}{Gm^2}\left[\left(11 - \frac{61}{9}\nu\right)S_\ell + \frac{\delta m}{m}\left(3 - \frac{10}{3}\nu\right)\Sigma_\ell\right] \\
& + \frac{x^{7/2}}{Gm^2}\left[\left(\frac{135}{4} - \frac{367}{4}\nu + \frac{29}{12}\nu^2\right)S_\ell + \frac{\delta m}{m}\left(\frac{27}{4} - 39\nu + \frac{5}{4}\nu^2\right)\Sigma_\ell\right] \\
& + \mathcal{O}\left(\frac{1}{c^8}\right)
\end{aligned}$$

Flux calculation

The flux can be expressed in terms of the (derivatives of) multipole moments

$$\mathcal{F} = \frac{G}{c^5}\left\{\frac{1}{5}I^{(3)}_{ij}I^{(3)}_{ij} + \frac{1}{c^2}\left[\frac{1}{189}I^{(4)}_{ijk}I^{(4)}_{ijk} + \frac{16}{45}J^{(3)}_{ij}J^{(3)}_{ij}\right]\right.$$
$$\left. + \frac{1}{c^4}\left[\frac{1}{9072}I^{(5)}_{ijkl}I^{(5)}_{ijkl} + \frac{1}{84}J^{(4)}_{ijk}J^{(4)}_{ijk}\right] + \frac{1}{c^6}\left[\frac{4}{14175}J^{(5)}_{ijkl}J^{(5)}_{ijkl}\right] + (\text{tails}) + \mathcal{O}\left(\frac{1}{c^8}\right)\right\}$$

... which can be expressed as integrals over the matter and the gravitational fields in the source

$$I_L(t) = \mathrm{FP}_{B=0}\int d^3\mathbf{x}\int_{-1}^{1} dz\left\{\frac{1}{c^2}\delta_l \hat{x}_L(\tau^{00}+\tau^{ii}) - \frac{4(2l+1)}{c^3(l+1)(2l+3)}\delta_{l+1}\hat{x}_{iL}\tau^{0i(1)}\right.$$
$$\left. + \frac{2(2l+1)}{c^4(l+1)(l+2)(2l+5)}\delta_{l+2}\hat{x}_{ijL}\tau^{ij(2)}\right\}(\mathbf{x}, t+z|\mathbf{x}|/c)$$

New regularized integrals to compute + use equations of motion to 3.5PN to compute time derivatives

$$\mathcal{F}_S = \frac{32c^5}{5G}x^5\nu^2\left(\frac{x^{3/2}}{G\,m^2}\right)\left\{-4S_\ell - \frac{5}{4}\frac{\delta m}{m}\Sigma_\ell + x\left[\left(-\frac{9}{2}+\frac{272}{9}\nu\right)S_\ell + \left(-\frac{13}{16}+\frac{43}{4}\nu\right)\frac{\delta m}{m}\Sigma_\ell\right]\right.$$
$$\left. + x^2\left[\left(\frac{476645}{6804}+\frac{6172}{189}\nu-\frac{2810}{27}\nu^2\right)S_\ell + \left(\frac{9535}{336}+\frac{1849}{126}\nu-\frac{1501}{36}\nu^2\right)\frac{\delta m}{m}\Sigma_\ell\right]\right\}$$
$$+ (\text{NS}) + (\text{tails}) + \mathcal{O}\left(\frac{1}{c^8}\right).$$

Tests of the result:

- test mass limit (see Tagoshi et al 1996)
- source moments for a single boosted Kerr black hole

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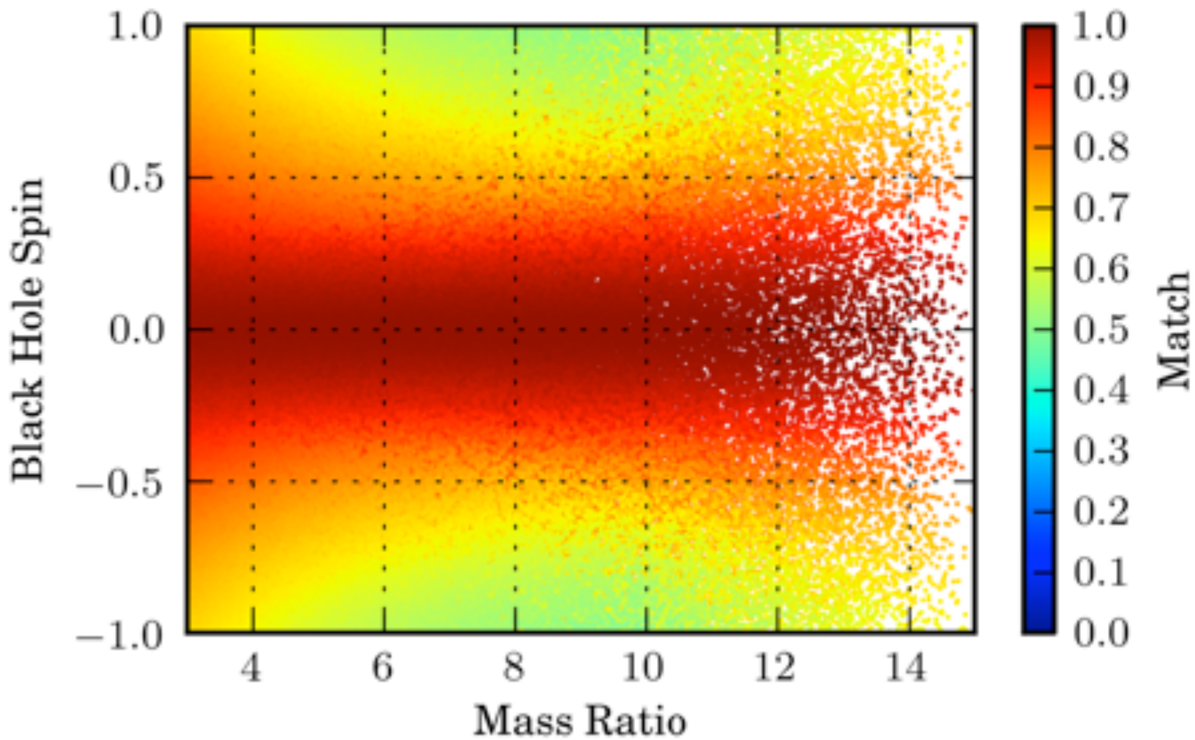
Contribution to the matches



Nitz, Lundgren, Brown, Ochsner, Keppel, Harry (July 2013)

Match between waveforms with and without 3.5PN SO corrections (+3PN SO tail) computed with Advanced LIGO noise curve (TaylorF2, typically 1.4+10 M systems so no need for NR)

Comparable picture for other approximants. Need to push the series further!

Computation of the 4PN SO tail term in the flux   Marsat, Bohe, Blanchet, Buonanno arXiv:1307.6793

$$\mathcal{F} = \frac{32c^5}{5G} x^5 \nu^2 \left[1 + f_1\, x + f_{1.5}\, x^{3/2} + f_2\, x^2 + f_{2.5}\, x^{5/2} + f_3\, x^3 + f_{3.5}\, x^{7/2} + f_4\, x^4 + \mathcal{O}_{\mathrm{NS}}(x^4) \right]$$

SO tail contributions (non-linearities in the propagation of the waves)

Phase estimates

LIGO/Virgo	$1.4M_\odot + 1.4M_\odot$	$10M_\odot + 1.4M_\odot$	$10M_\odot + 10M_\odot$
Newtonian	15952.6	3558.9	598.8
1PN	439.5	212.4	59.1
1.5PN	$-210.3 + 65.6\kappa_1\chi_1 + 65.6\kappa_2\chi_2$	$-180.9 + 114.0\kappa_1\chi_1 + 11.7\kappa_2\chi_2$	$-51.2 + 16.0\kappa_1\chi_1 + 16.0\kappa_2\chi_2$
2PN	9.9	9.8	4.0
2.5PN	$-11.7 + 9.3\kappa_1\chi_1 + 9.3\kappa_2\chi_2$	$-20.0 + 33.8\kappa_1\chi_1 + 2.9\kappa_2\chi_2$	$-7.1 + 5.7\kappa_1\chi_1 + 5.7\kappa_2\chi_2$
3PN	$2.6 - 3.2\kappa_1\chi_1 - 3.2\kappa_2\chi_2$	$2.3 - 13.2\kappa_1\chi_1 - 1.3\kappa_2\chi_2$	$2.2 - 2.6\kappa_1\chi_1 - 2.6\kappa_2\chi_2$
3.5PN	$-0.9 + 1.9\kappa_1\chi_1 + 1.9\kappa_2\chi_2$	$-1.8 + 11.1\kappa_1\chi_1 + 0.8\kappa_2\chi_2$	$-0.8 + 1.7\kappa_1\chi_1 + 1.7\kappa_2\chi_2$
4PN	$(\mathrm{NS}) - 1.5\kappa_1\chi_1 - 1.5\kappa_2\chi_2$	$(\mathrm{NS}) - 8.0\kappa_1\chi_1 - 0.7\kappa_2\chi_2$	$(\mathrm{NS}) - 1.5\kappa_1\chi_1 - 1.5\kappa_2\chi_2$

TABLE I. Spin-orbit contributions to the number of gravitational-wave cycles $\mathcal{N}_{\rm GW} = (\phi_{\rm max} - \phi_{\rm min})/\pi$. For binaries detectable by ground-based detectors LIGO/Virgo, we show the number of cycles accumulated from $\omega_{\rm min} = \pi \times 10\,\mathrm{Hz}$ to $\omega_{\rm max} = \omega_{\rm ISCO} = c^3/(6^{3/2}Gm)$. For each compact object we define the magnitude χ_A and the orientation κ_A of the spin by $\mathbf{S}_A \equiv G\,m_A^2\,\chi_A\,\hat{\mathbf{S}}_A$ and $\kappa_A \equiv \hat{\mathbf{S}}_A \cdot \boldsymbol{\ell}$. For comparison, we give all the non-spin contributions up to 3.5PN order, but the non-spin 4PN terms (NS) are yet unknown. We neglect all the spin-spin terms.

crude phase estimate, no match, T2
... but comparable magnitude

Conclusions

We have computed the NNLO spin-orbit effects (3.5PN for maximally spinning bodies) in the dynamics of the binary and in the emitted flux .

These new corrections produce significant mismatches with previous lower order waveforms at least in certain regions of parameter space. They have to be incorporated into the data-analysis pipelines.

We have also computed the NLO spin-orbit contribution to the tail effect which seems to be (crude estimate!) of comparable magnitude...

PN iteration of the Einstein's equations in harm gauge

rewrite Einstein eqs
$h^{\mu\nu} = \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu}$ $\longrightarrow$

$\partial_\mu h^{\alpha\mu} = 0$ harmonic gauge

$\Box h^{\mu\nu} = \frac{16\pi G}{c^4}\tau^{\mu\nu}$

$\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ "flat" d'Alembertian

$\tau^{\mu\nu}$ stress-energy pseudo tensor of matter + gravitational fields

$$\tau^{\mu\nu} = |g|T^{\mu\nu} + \frac{c^4}{16\pi G}\Lambda^{\mu\nu}$$

$$\begin{aligned}\Lambda^{\alpha\beta} = &-h^{\mu\nu}\partial^2_{\mu\nu}h^{\alpha\beta} + \partial_\mu h^{\alpha\nu}\partial_\nu h^{\beta\mu} + \frac{1}{2}g^{\alpha\beta}g_{\mu\nu}\partial_\lambda h^{\mu\tau}\partial_\tau h^{\nu\lambda} \\ &-g^{\alpha\mu}g_{\nu\tau}\partial_\lambda h^{\beta\tau}\partial_\mu h^{\nu\lambda} - g^{\beta\mu}g_{\nu\tau}\partial_\lambda h^{\alpha\tau}\partial_\mu h^{\nu\lambda} + g_{\mu\nu}g^{\lambda\tau}\partial_\lambda h^{\alpha\mu}\partial_\tau h^{\beta\nu} \\ &+\frac{1}{8}(2g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu})(2g_{\lambda\tau}g_{\epsilon\pi} - g_{\tau\epsilon}g_{\lambda\pi})\partial_\mu h^{\lambda\pi}\partial_\nu h^{\tau\epsilon}.\end{aligned}$$

The metric is parametrized via a set of «potentials»

$$\begin{aligned} g_{00} &= -1 + \frac{2}{c^2}V - \frac{2}{c^4}V^2 + \frac{8}{c^6}\left(\hat{X} + V_iV_i + \frac{V^3}{6}\right) + \frac{32}{c^8}\left(\hat{T} - \frac{1}{2}V\hat{X} + \hat{R}_iV_i - \frac{1}{2}VV_iV_i - \frac{V^4}{48}\right) + \mathcal{O}(10)\,, \\ g_{0i} &= -\frac{4}{c^3}V_i - \frac{8}{c^5}\hat{R}_i - \frac{16}{c^7}\left(\hat{Y}_i + \frac{1}{2}\hat{W}_{ij}V_j + \frac{1}{2}V^2V_i\right) + \mathcal{O}(9)\,, \\ g_{ij} &= \delta_{ij}\left[1 + \frac{2}{c^2}V + \frac{2}{c^4}V^2 + \frac{8}{c^6}\left(\hat{X} + V_kV_k + \frac{V^3}{6}\right)\right] + \frac{4}{c^4}\hat{W}_{ij} + \frac{16}{c^6}\left(\hat{Z}_{ij} + \frac{1}{2}V\hat{W}_{ij} - V_iV_j\right) + \mathcal{O}(8) \end{aligned}$$

which obey inhomogeneous flat d'Alembertian equations sourced by $T^{\mu\nu}$ and by the lower order potentials

At 1PN, $\Box V = -4\pi G\,\sigma$, $\Box V_i = -4\pi G\,\sigma_i$ with $\sigma = (T^{00} + T^{ii})/c^2$, $\sigma_i = T^{0i}/c$

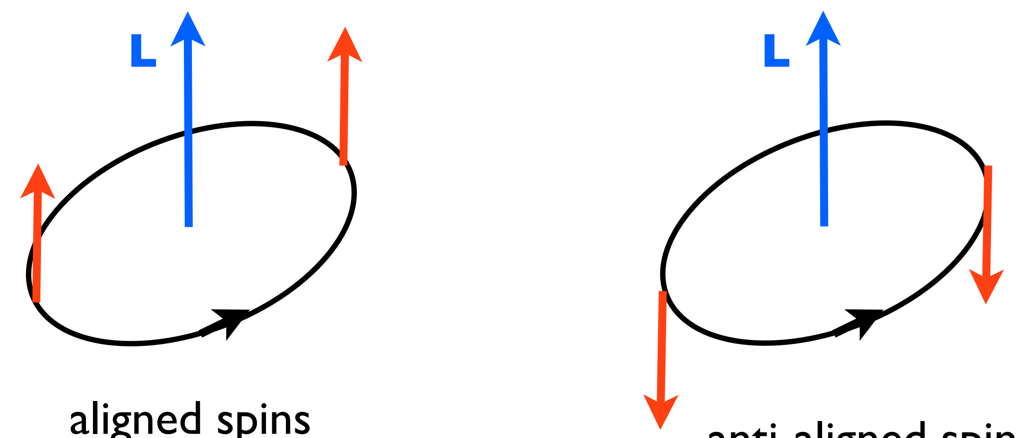
At 2PN, $\Box\hat{W}_{ij} = -4\pi G\,(\sigma_{ij} - \delta_{ij}\sigma_{kk}) - \partial_i V\partial_j V$ with $\sigma_{ij} = T^{ij}$ ...

In the near zone, solution computed with the retarded inverse d'Alembertian (PN expansion of the retardations)

$$\Box P = S \longrightarrow P(\mathbf{x}) = -\frac{1}{4\pi}\sum_n \frac{(-1)^n}{n!}\left(\frac{\partial}{c\,\partial t}\right)^n \mathrm{FP}_{B=0}\int d^3\mathbf{x}'|\mathbf{x} - \mathbf{x}'|^{n-1}S(\mathbf{x}', t)$$

Effect of the spin on the inspiral

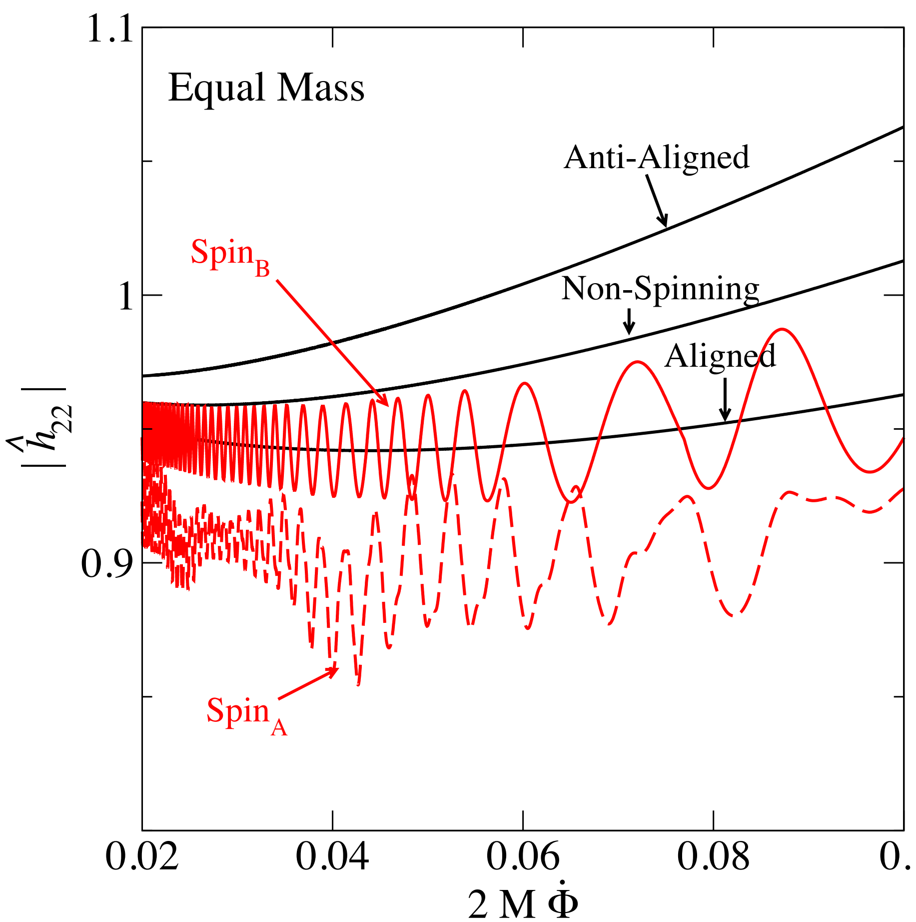
The components of the spins that are orthogonal to the orbital plane change the inspiral rate, i.e. in particular the phase



aligned spins
slower inspiral

anti-aligned spins
faster inspiral

The components of the spins in the orbital plane cause the orbital plane to precess: strong amplitude modulations



taken from Arun, Buonanno, Faye, Ochsner (09)

Spin ``power counting”

The spin of a rotating compact body is of the order of $S_{\rm true} \sim m\, l\, v_{\rm spin}$ with $l \sim \dfrac{Gm}{c^2}$

⟶ For maximally rotating bodies, $v_{\rm spin} \sim c$ so $S_{\rm true} \sim \chi \dfrac{Gm^2}{c}$ is formally 0.5 PN

⟶ For slowly rotating bodies, $v_{\rm spin} \ll c$ so $S_{\rm true}$ is formally 1 PN

We adopt the following spin (re-)definition $S \equiv c\, S_{\rm true} = \chi G m^2$

⟶ For maximally rotating objects, our spin variable is Newtonian

With this definition, the spin enters the Newtonian-like equation of motion at the following orders:

$$\frac{\mathrm{d}v_1^i}{\mathrm{d}t} = A^i_{\rm N} + \frac{1}{c^2}A^i_{\rm 1PN} + \frac{1}{c^3}{A_S^i}_{\rm 1.5PN} + \frac{1}{c^4}\left[A^i_{\rm 2PN} + {A_{SS}^i}_{\rm 2PN}\right] + \frac{1}{c^5}\left[A^i_{\rm 2.5PN} + {A_S^i}_{\rm 2.5PN}\right] + \frac{1}{c^6}\left[A^i_{\rm 3PN} + {A_{SS}^i}_{\rm 3PN}\right] + \frac{1}{c^7}\left[A^i_{\rm 3.5PN} + {A_S^i}_{\rm 3.5PN}\right] + \mathcal{O}(8)$$

Hadamard regularization of the potentials

The point particle description has to be supplemented with some UV regularization procedure to make sense of

$F(\mathbf{y}_1),\ \int F(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}_1)\,d^3\mathbf{x},\ \int F(\mathbf{x})\,d^3\mathbf{x}$ for functions F singular at the positions of the particles (typically the potentials or their derivatives).

$$F(\mathbf{x}) = \sum_{a_0\leq a\leq n} r_1^a f_a(\mathbf{n}_1) + o(r_1^n) \qquad r_1 = |\mathbf{x}-\mathbf{y}_1| \quad \mathbf{n}_1 = (\mathbf{x}-\mathbf{y}_1)/r_1$$

For most of the calculation, the pure Hadamard-Schwartz (pHS) prescription proved sufficient

- Hadamard partie finie $(F)_1 = \int \frac{d\Omega_1}{4\pi} f_0(\mathbf{n}_1)$
- Compact support integrals $\int F(\mathbf{x})\delta_1\,\mathrm{d}^3\mathbf{x} = (F)_1$
- For non-compact support integrals $\mathrm{Pf}_{s_1,s_2}\int d^3\mathbf{x}\ F = \lim_{s\to 0}\left\{\int_{\mathbb{R}^3\setminus\mathcal{B}_1(s)\cup\mathcal{B}_2(s)} d^3\mathbf{x}\ F + \sum_{a+3<0}\frac{s^{a+3}}{a+3}\int d\Omega_1 f_a + \ln\left(\frac{s}{s_1}\right)\int d\Omega_1 f_{-3} + 1\leftrightarrow 2\right\}$
- Gel'Fand-Shilov formula for homogeneous functions to compute the distributional parts of the derivatives

$$D_i\left(\frac{n^L}{r^m}\right) = 4\pi\frac{(-)^m 2^m(\ell+1)!(\frac{\ell+m-1}{2})!}{(\ell+m)!}\sum_{p=p_0}^{[m/2]}\frac{\Delta^{p-1}\partial_{(M-2P}\,\delta_{iL+2P-M)}}{2^{2p}(p-1)!(m-2p)!(\frac{\ell+1-m}{2}+p)!}$$

when l+m is even
0 otherwise

However, for one of the terms needed to compute the acceleration, namely $\partial_{jk}\hat{Y}_i^{\mathrm{NC}}$, the pHS regularization yields an ambiguous result.

$$(\partial_{jk}\hat{Y}_i^{\mathrm{NC}})_1 = -\frac{1}{4\pi}\mathrm{Pf}_{s_1,s_2}\int d^3\mathbf{x}\frac{3n_1^{jk}-\delta^{jk}}{r_1^3}S_{\hat{Y}_i}^{\mathrm{NC}} + \ln\left(\frac{r_1'}{s_1}\right)\left((3n_1^{jk}-\delta^{jk})S_{\hat{Y}_i}^{\mathrm{NC}}\right)_1 + \frac{\delta^{jk}}{3}\left(S_{\hat{Y}_i}^{\mathrm{NC}}\right)_1$$

the final pHS result depends on s_2 and r_1'