Boundary Feedback Stabilization of Hyperbolic Systems on Networks

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Quasilinear Hyperbolic System

We consider the following quasilinear hyperbolic system for $r_\pm^{(i)}(t,x)$ on a star-shaped network of N edges ($i \in \{1, ..., N\}$):

$$\begin{aligned} \frac{\partial}{\partial t} r_+^{(i)} &+ \Lambda_+^{(i)}(x, r_+^{(i)}, r_-^{(i)}) \frac{\partial}{\partial x} r_+^{(i)} &= \Psi_+^{(i)}(x, r_+^{(i)}, r_-^{(i)}), \\ \frac{\partial}{\partial t} r_-^{(i)} &+ \Lambda_-^{(i)}(x, r_+^{(i)}, r_-^{(i)}) \frac{\partial}{\partial x} r_-^{(i)} &= \Psi_-^{(i)}(x, r_+^{(i)}, r_-^{(i)}) \end{aligned}$$

with $t \in [0, T]$, $x \in [0, L^{(i)}]$ and C^1-functions $\Lambda_\pm^{(i)}$, $\Psi_\pm^{(i)}$ of the form

$$\begin{aligned} \Lambda_\pm^{(i)}(x, r_+^{(i)}, r_-^{(i)}) &= \lambda_\pm^{(i)}(x) + f_\pm^{(i)}(x, r_+^{(i)}, r_-^{(i)}), \\ \Psi_\pm^{(i)}(x, r_+^{(i)}, r_-^{(i)}) &= -(r_+^{(i)} + r_-^{(i)})\, \psi_\pm^{(i)}(x) + g_\pm^{(i)}(x, r_+^{(i)}, r_-^{(i)}) \end{aligned}$$

where

$$\lambda_+^{(i)} > 0, \quad \lambda_-^{(i)} < 0, \quad \psi_\pm^{(i)} > 0, \quad f_\pm^{(i)}(x,0,0) = 0,$$

$$g_\pm^{(i)}(x,0,0) = \frac{\partial}{\partial r_+^{(i)}} g_\pm^{(i)}(x,0,0) = \frac{\partial}{\partial r_-^{(i)}} g_\pm^{(i)}(x,0,0) = 0.$$

Coupling Condition and Feedback Controls

- Coupling conditions at the central node ω ($x = L^{(i)}$):

$$r_-^{(i)}(t, L^{(i)}) = \Xi^{(i)}(r_+^{(1)}(t, L^{(1)}), ..., r_+^{(N)}(t, L^{(N)}))$$

for $i \in \{1, ..., N\}$ with C^1-functions $\Xi^{(i)}$ with $\Xi^{(i)}(0, ..., 0) = 0$.

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- Feedback controls with time-varying delay at the free nodes ($x = 0$):

$$r_+^{(i)}(t, 0) = \begin{cases} \vartheta^{(i)}(t) & \text{for} \quad t \in [0, 2\overline{\tau}^{(i)}] \\ k^{(i)}\, r_-^{(i)}(t - \tau^{(i)}(t), 0) & \text{for} \quad t \in (2\overline{\tau}^{(i)}, T] \end{cases}$$

  with appropriate C^1-functions $\vartheta^{(i)}$, feedback constants $k^{(i)} \in (-1, 1)$ and time delay C^1-functions $\tau^{(i)}$ that satisfy

$$0 < \tau^{(i)}(t) \leq \overline{\tau}^{(i)} < \frac{T}{2}, \quad |\frac{d}{dt}\tau^{(i)}(t)| < 1.$$

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- Result by T. Li and Z. Wang:
  Existence of a C^1-solution $r_\pm^{(i)}(t, x)$ on a finite time interval $[0, T]$ for initial and boundary conditions with sufficiently small C^1-norms.

Network Lyapunov Function with Delay Terms

Network Lyapunov function $\mathcal{E}_\omega(t)$ for $r_\pm^{(i)}(t,x)$:

$$\mathcal{E}_\omega(t) \, = \, \sum_{i=1}^{N} \mathcal{E}^{(i)}(t) \, + \, \mathcal{D}^{(i)}(t)$$

with

$$\begin{aligned} \mathcal{E}^{(i)}(t) \quad &= \quad \int_0^{L^{(i)}} \frac{A_+^{(i)}}{\lambda_+^{(i)}(x)} h_+^{(i)}(x)(r_+^{(i)}(t,x))^2 \, + \, \frac{A_-^{(i)}}{|\lambda_-^{(i)}(x)|} h_-^{(i)}(x)(r_-^{(i)}(t,x))^2 \, dx, \\ \mathcal{D}^{(i)}(t) \quad &= \quad \int_0^{\tau^{(i)}(t)} A_-^{(i)} \, \exp(-\mu^{(i)} s) \, (r_-^{(i)}(t-s,0))^2 \, ds \end{aligned}$$

with appropriate constants $\mu^{(i)} > 0$, $A_\pm^{(i)} > 0$ and exponential weights

$$h_\pm^{(i)}(x) = \exp\left(-\mu^{(i)} \int_0^x \frac{1}{\lambda_\pm^{(i)}(\xi)} d\xi \right) .$$

Exponential Stability

Network Lyapunov function $\mathcal{E}_\omega(t)$ for $r_\pm^{(i)}(t,x)$:

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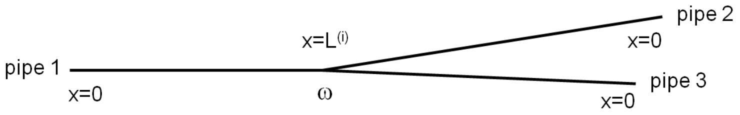
Exponential decay of the Lyapunov function with time:

If $|k^{(i)}|$ and the C^1-norms of the functions $\vartheta^{(i)}$ and of the initial data are small enough, we have ($\eta > 0$, $\tau_{\max} = \max\{\overline{\tau}^{(i)}\}$):

$$\mathcal{E}_\omega(t) \le \mathcal{E}_\omega(2\tau_{\max}) \exp(-\eta(t - 2\tau_{\max})) \quad \text{for} \quad t \in [2\tau_{\max}, T].$$

Isothermal Euler Equations with Friction



Conservation of mass: $$\frac{\partial}{\partial t}\rho^{(i)} + \frac{\partial}{\partial x}q^{(i)} = 0$$

Momentum equation: $$\frac{\partial}{\partial t}q^{(i)} + \frac{\partial}{\partial x}\left(\frac{(q^{(i)})^2}{\rho^{(i)}} + a^2\rho^{(i)}\right) = -\frac{\nu}{\delta}\frac{q^{(i)}|q|^{(i)}}{2\rho^{(i)}}$$

mass flux: $q^{(i)}(t,x) \neq 0$ *density:* $\rho^{(i)}(t,x) > 0$ *pipe diameter:* $\delta > 0$

friction factor: $\nu > 0$ *sonic speed:* $a > 0$ *subsonic states:* $\frac{|q^{(i)}|}{\rho^{(i)}} < a$

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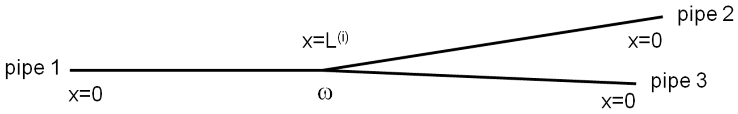

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System equation:

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho^{(i)} \\ q^{(i)}\end{pmatrix} + \begin{pmatrix} 0 & 1 \\ a^2 - \frac{(q^{(i)})^2}{(\rho^{(i)})^2} & 2\frac{q^{(i)}}{\rho^{(i)}}\end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix}\rho^{(i)} \\ q^{(i)}\end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{\nu}{\delta}\frac{q^{(i)}|q^{(i)}|}{2\rho^{(i)}}\end{pmatrix}$$

Euler Equations in Characteristic Variables

- Eigenvalues of the system matrix: $\mathcal{L}_{\pm}^{(i)} = \frac{q^{(i)}}{\rho^{(i)}} \pm a$

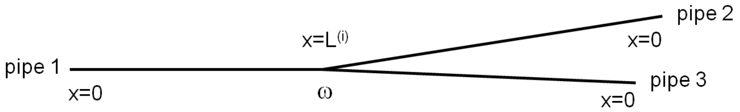
- Characteristic variables / Riemann invariants: $R_{\pm}^{(i)} = -\frac{q^{(i)}}{\rho^{(i)}} \mp a \ln(\rho^{(i)})$

- System equation in Riemann invariants:

$$\frac{\partial}{\partial t} \begin{pmatrix} R_{+}^{(i)} \\ R_{-}^{(i)} \end{pmatrix} + \begin{pmatrix} \mathcal{L}_{+}^{(i)} & 0 \\ 0 & \mathcal{L}_{-}^{(i)} \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} R_{+}^{(i)} \\ R_{-}^{(i)} \end{pmatrix} = -\frac{\nu}{8\delta}(R_{+}^{(i)}+R_{-}^{(i)})|R_{+}^{(i)}+R_{-}^{(i)}| \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

with $\mathcal{L}_{\pm}^{(i)} = -\frac{1}{2}(R_{+}^{(i)} + R_{-}^{(i)}) \pm a$

Coupling Conditions for a Star-Shaped Network



Continuity of the density: $\rho^{(1)}(t, L^{(1)}) = \rho^{(i)}(t, L^{(i)}) \ (i = 2, ..., N)$

Conservation of mass: $\sum_{i=1}^{N} q^{(i)}(t, L^{(i)}) = 0$

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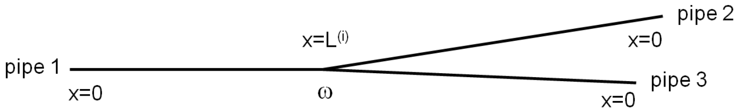

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Coupling conditions in Riemann invariants:

$$\begin{pmatrix} R_-^{(1)}(t, L^{(1)}) \\ \vdots \\ R_-^{(N)}(t, L^{(N)}) \end{pmatrix} = A_\omega \begin{pmatrix} R_+^{(1)}(t, L^{(1)}) \\ \vdots \\ R_+^{(N)}(t, L^{(N)}) \end{pmatrix}$$

with an orthogonal, symmetric matrix A_ω $(A_\omega^2 = I)$

Nonstationary States

For a given stationary state $\bar{R}_{\pm}^{(i)}(x)$ we consider a nonstationary state $\bar{R}_{\pm}^{(i)}(x) + r_{\pm}^{(i)}(t,x)$ in a local C^1-neighborhood of $\bar{R}_{\pm}^{(i)}(x)$:

Quasilinear system for $r_{\pm}^{(i)}(t,x)$:

$$\begin{aligned} \tfrac{\partial}{\partial t} r_{+}^{(i)} &+ \Lambda_{+}^{(i)}(x, r_{+}^{(i)}, r_{-}^{(i)}) \tfrac{\partial}{\partial x} r_{+}^{(i)} &= \Psi_{+}^{(i)}(x, r_{+}^{(i)}, r_{-}^{(i)}), \\ \tfrac{\partial}{\partial t} r_{-}^{(i)} &+ \Lambda_{-}^{(i)}(x, r_{+}^{(i)}, r_{-}^{(i)}) \tfrac{\partial}{\partial x} r_{-}^{(i)} &= \Psi_{-}^{(i)}(x, r_{+}^{(i)}, r_{-}^{(i)}) \end{aligned}$$

with

$$\begin{aligned} \Lambda_{\pm}^{(i)}(x, r_{+}^{(i)}, r_{-}^{(i)}) &= \lambda_{\pm}^{(i)}(x) - \tfrac{1}{2}(r_{+}^{(i)} + r_{-}^{(i)}), \\ \Psi_{\pm}^{(i)}(x, r_{+}^{(i)}, r_{-}^{(i)}) &= -(r_{+}^{(i)} + r_{-}^{(i)})\, \psi_{\pm}^{(i)}(x) - \mathrm{sign}(\bar{R}_{+}^{(i)} + \bar{R}_{-}^{(i)}) \frac{\nu}{8\delta} (r_{+}^{(i)} + r_{-}^{(i)})^2 \end{aligned}$$

where $\lambda_{\pm}^{(i)}$ and $\psi_{\pm}^{(i)}$ only depend on the given stationary state.

Feedback Stabilization and Exponential Stability

- Coupling conditions at the central node ω ($x = L^{(i)}$):

$$\begin{pmatrix} r_-^{(1)}(t, L^{(1)}) \\ \vdots \\ r_-^{(N)}(t, L^{(N)}) \end{pmatrix} = A_\omega \begin{pmatrix} r_+^{(1)}(t, L^{(1)}) \\ \vdots \\ r_+^{(N)}(t, L^{(N)}) \end{pmatrix}$$

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- Boundary controls at the free nodes $(x = 0)$:

$$r_+^{(i)}(t, 0) = \begin{cases} \vartheta^{(i)}(t) & \text{for} \quad t \in [0, 2\overline{\tau}^{(i)}] \\ k^{(i)}\, r_-^{(i)}(t - \tau^{(i)}(t), 0) & \text{for} \quad t \in (2\overline{\tau}^{(i)}, T] \end{cases}$$

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- Lyapunov function: $\mathcal{E}_\omega(t) = \sum_{i=1}^{N} \mathcal{E}^{(i)}(t) + \mathcal{D}^{(i)}(t)$

- For an appropriate choice of $\mu^{(i)}, A_\pm^{(i)}, k^{(i)}$ ($\eta > 0$, $\tau_{\max} = \max\{\overline{\tau}^{(i)}\}$):

$$\mathcal{E}_\omega(t) \le \mathcal{E}_\omega(2\tau_{\max}) \exp(-\eta(t - 2\tau_{\max})) \quad \text{for} \quad t \in [2\tau_{\max}, T]$$

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