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Numeriwaves

Global Attractors for Semi-Linear PDEs Involving Degenerate Elliptic Operators

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Problem

$$\begin{aligned} \frac{\partial}{\partial t}u &= \Delta_\lambda u + f(u) && \Omega \times (0,\infty), \\ u|_{\partial\Omega} &= 0 && \partial\Omega \times [0,\infty), \\ u|_{t=0} &= u_0 && \Omega \times \{0\}, \end{aligned} \tag{1}$$

in a bounded, smooth domain $\Omega \subset \mathbb{R}^N$, where

$$\Delta_\lambda := \sum_{i=1}^{N} \partial_{x_i}(\lambda_i^2 \partial_{x_i}), \quad \lambda = (\lambda_1, \ldots, \lambda_N) \quad \text{is } \textit{sub-elliptic}.$$

- Local and global well-posedness
- Longtime behavior: Existence and finite fractal dimension of the global attractor, convergence to equilibria

Δ_λ-Laplacians

- Include, as a particular case, Grushin-type operators [1]
- First introduced and studied in 1983 [2], [3], [4]
- Existence and regularity of weak solutions of the semilinear sub-elliptic problem [5]

$$\Delta_\lambda u = f(u)$$

[1] **V.V. Grushin**, *On a Class of Hypoelliptic Operators*, Math. USSR Sbornik 12(3), 458–476, 1970.

[2] **B. Franchi, E. Lanconelli**, *Une métrique associée à une classe d'opérateurs elliptiques dégénérés*, Conference on linear partial and pseudodifferential operators (Torino, 1982), Rend. Sem. Mat. Univ. Politec. Torino 1983, Special Issue, 105–114, 1984.

[3] **B. Franchi, E. Lanconelli**, *An embedding theorem for Sobolev spaces related to non-smooth vectors fields and Harnack inequality*, Comm. Partial Differential Equations 9(13), 1237–1264, 1984.

[4] **B. Franchi, E. Lanconelli**, *Hölder regularity theorem for a class of nonuniformly elliptic operators with measurable coefficients*, Ann. Sc. Norm. Sup. Pisa Cl. Sci. 10(4), 523–541, 1983.

[5] **A.E. Kogoj, E. Lanconelli**, *On semilinear Δ_λ-Laplace equation*, Nonlinear Analysis **75**, 4637–4649, 2012.

Δ_λ-Laplacians: $\Delta_\lambda := \sum_{i=1}^N \partial_{x_i}(\lambda_i^2 \partial_{x_i})$

$\lambda_1, \ldots, \lambda_N$ continuous, strictly positive and C^1 outside the coordinate hyperplanes,

- $\lambda_1(x) \equiv 1,\ \lambda_i(x) = \lambda_i(x_1, \ldots, x_{i-1}),\ i = 2, \ldots, N$.
- $\lambda_i(x) = \lambda_i(x^*)$, where $x^* = (|x_1|, \ldots, |x_N|)$.
- There exists a group of dilations $(\delta_r)_{r>0}$
  $\delta_r(x_1, \ldots, x_N) = (r^{\varepsilon_1} x_1, \ldots, r^{\varepsilon_N} x_N),\ 1 \leq \varepsilon_1 \leq \varepsilon_2 \leq \cdots \leq \varepsilon_N$,
  such that λ_i is δ_r-*homogeneous of degree* $\varepsilon_i - 1$,
  $$\lambda_i(\delta_r(x)) = r^{\varepsilon_i - 1} \lambda_i(x) \quad \forall x \in \mathbb{R}^N,\ r > 0.$$

$Q := \varepsilon_1 + \cdots + \varepsilon_N$, is the *homogeneous dimension* of $\mathbb{R}^N$ with respect to $(\delta_r)_{r>0}$.

Examples

1. Grushin-type operators:

$$\Delta_\lambda = \Delta_x + |x|^{2\alpha}\Delta_y,$$

where $(x, y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ and $\alpha \geq 0$. We find

$$\delta_r(x, y) = \left(rx, r^{\alpha+1}y\right),$$

and $Q = N_1 + N_2(\alpha + 1)$.

Examples

2. Let $\alpha, \beta, \gamma \geq 0$. For the operator

$$\Delta_\lambda = \Delta_x + |x|^{2\alpha}\Delta_y + |x|^{2\beta}|y|^{2\gamma}\Delta_z,$$

where $(x, y, z) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3}$, we find

$$\delta_r(x, y, z) = \left(rx, r^{\alpha+1}y, r^{\beta+(\alpha+1)\gamma+1}z\right),$$

and $Q = N_1 + (\alpha+1)N_2 + (\beta + (\alpha+1)\gamma + 1)N_3$.

Abstract Semilinear Parabolic Problems

We consider

$$\begin{aligned} u_t &= Au + f(u) && t > 0, \\ u|_{t=0} &= u_0 && u_0 \in X^\gamma, \gamma \in [0,1[, \end{aligned}$$

where A is positive, sectorial in the Banach space X.

A generates an analytic semigroup $e^{-At}, t \geq 0$, in X,
$X^\gamma, \gamma \in [0,1[$, associated fractional power spaces,

$$\mathcal{D}(A) = X^1 \hookrightarrow X^\gamma \hookrightarrow X^0 = X.$$

Abstract Semilinear Parabolic Problems[1]

Theorem

If $f : X^\gamma \to X$ is Lipschitz on bounded subsets of X^γ, then $\forall u_0 \in X^\gamma$ there exists a unique solution, defined on the max. interval of existence $[0, T[$,

$$u \in C([0,T[; X^\gamma) \cap C^1((0,T); X^\beta) \qquad \forall \beta \in [0,1[,$$

either $T = \infty$ or, if $T < \infty$, then $\limsup_{t \to T} \|u(t)\|_{X^\gamma} = \infty$, and u satisfies the variation of constants formula

$$u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)} f(u(s))ds \qquad t \in [0, T[.$$

[1] **J.W. Cholewa, J. Dlotko,** *Global Attractors in Abstract Parabolic Problems*, Cambridge University Press, New York, 2000.

Functional Setting

Let $\mathring{W}^{1,2}_{\lambda}(\Omega)$ be the closure of $C^1_0(\Omega)$ with respect to

$$\|u\|_{\mathring{W}^{1,2}_{\lambda}(\Omega)} := \left(\int_{\Omega} |\nabla_{\lambda} u(x)|^2 dx \right)^{\frac{1}{2}},$$

where $\nabla_{\lambda} u = (\lambda_1 \partial_{x_1} u, \ldots, \lambda_1 \partial_{x_N} u)$, $|\nabla_{\lambda} u|^2 := \sum_{i=1}^{N} |\lambda_i \partial_{x_i} u|^2$.

Lemma (Poincaré-type inequality)

There exists $C > 0$ such that

$$\|u\|_{L^2(\Omega)} \leq C \|u\|_{\mathring{W}^{1,2}_{\lambda}(\Omega)} \qquad \forall u \in C^1_0(\Omega).$$

Furthermore, $-\Delta_\lambda$ is selfadjoint and densely defined in $L^2(\Omega)$
$\Longrightarrow A := -\Delta_\lambda$ positive, sectorial,

$$\mathcal{D}(A) = X^1 \hookrightarrow \mathring{W}_\lambda^{1,2}(\Omega) = X^{\frac{1}{2}} \hookrightarrow L^2(\Omega) = X^0.$$

A can be extended and restricted to a positive sectorial operator in X^α with domain $X^{\alpha+1},\ \alpha \geq -1$.

The analytic semigroups in X^α and X^β are obtained from each other by natural extension and restriction,

$$\|e^{-At}\|_{\mathcal{L}(X^\alpha;X^\beta)} \leq \frac{C_{\alpha,\beta}}{t^{\alpha-\beta}} \qquad -1 \leq \beta < \alpha < \infty,\ t > 0.$$

Embedding Properties[1]

$$\frac{\partial}{\partial t}u = \Delta_\lambda u + f(u),$$
$$u|_{\partial\Omega} = 0,$$
$$u|_{t=0} = u_0, \qquad\qquad u_0 \in \mathring{W}_\lambda^{1,2}(\Omega) = X^{\frac{1}{2}}.$$

Theorem (Sobolev-type embedding)

Let $2_\lambda^* := \frac{2Q}{Q-2}$. *Then, the embedding*

$$\mathring{W}_\lambda^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$$

is continuous for $p \in [1, 2_\lambda^*]$ *and compact for every* $p \in [1, 2_\lambda^*[$.

[1] **A.E. Kogoj, E. Lanconelli**, *On semilinear* Δ_λ *-Laplace equation*, Nonlinear Analysis **75**, 4637–4649, 2012.

Local Well-Posedness

We assume $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz and

$$|f(u) - f(v)| \leq c|u - v|(1 + |u|^\rho + |v|^\rho), \qquad 0 < \rho < \frac{4}{Q-2}.$$

The Sobolev-type embedding theorem implies

- Case 1 : $0 < \rho \leq \frac{2}{Q-2}$
  $f : X^{\frac{1}{2}} \to X^0$ is Lipschitz on bounded subsets of $X^{\frac{1}{2}}$.
- Case 2 : $\frac{2}{Q-2} < \rho < \frac{4}{Q-2}$
  There exists $\alpha \in (0, \frac{1}{2})$ such that $f : X^{\frac{1}{2}} \to X^{-\alpha}$ is Lipschitz on bounded subsets of $X^{\frac{1}{2}}$.

Local Well-Posedness

Theorem

For every $u_0 \in X^{\frac{1}{2}} = \mathring{W}^{1,2}_{\lambda}(\Omega)$ *there exists a unique solution, defined on the maximal interval of existence* $[0, T[$,

$$u \in C([0, T[; X^{\frac{1}{2}}) \cap C^1((0, T); X^{\frac{1}{2}}),$$

either $T = \infty$ *or, if* $T < \infty$, *then* $\limsup_{t \to T} \|u(t)\|_{X^{\frac{1}{2}}} = \infty$, *and u satisfies the variation of constants formula*

$$u(t) = e^{-At} u_0 + \int_0^t e^{-A(t-s)} f(u(s)) ds, \qquad t \in [0, T[.$$

Global Existence of Solutions

We additionally assume the dissipativity condition:

$$\limsup_{|u|\to\infty} \frac{f(u)}{u} < \mu,$$

where $\mu > 0$ denotes the first eigenvalue of $-\Delta_\lambda$ on Ω,

and consider the Lyapunov functional $\Phi : X^{\frac{1}{2}} \to \mathbb{R}$,

$$\Phi(u) := \int_\Omega \left(\frac{1}{2}|\nabla_\lambda u|^2 - F(u) \right), \qquad F(u) := \int_0^u f(s)ds.$$

Global Existence of Solutions

If u is a solution of (1), then

$$\frac{d}{dt}\Phi(u(t)) = -\|u_t(t)\|^2_{L^2(\Omega)} \qquad t > 0,$$

and moreover, for some constants $c_*, c^* \geq 0$,

$$c_*(1+\|u(t)\|^2_{X^{\frac{1}{2}}}) \leq \Phi(u(t)) \leq \Phi(u_0) \leq c^*(1+\|u_0\|^2_{X^{\frac{1}{2}}}+\|u_0\|^{\rho+2}_{L^{\rho+2}(\Omega)}).$$

Consequently, solutions exist globally.

Main Result [1]

Let $S(t), t \geq 0$, be the semigroup in $\mathring{W}^{1,2}_{\lambda}(\Omega)$ generated by Problem (1),

$$S(t)u_0 = u(t; u_0) \qquad t \geq 0.$$

Theorem

The semigroup $S(t), t \geq 0$, possesses a global attractor $\mathcal{A}$ in $\mathring{W}^{1,2}_{\lambda}(\Omega)$, which is connected and of finite fractal dimension. Furthermore,

$$\mathcal{A} = \mathcal{W}^u(\mathcal{E}),$$

the omega-limit set $\omega(u_0) \subset \mathcal{E} = \{u \,|\, \Delta_{\lambda} u + f(u) = 0\}$ and

$$\lim_{t\to\infty} \mathrm{dist}_H(S(t)u_0, \mathcal{E}) = 0, \qquad \forall u_0 \in \mathring{W}^{1,2}_{\lambda}(\Omega).$$

[1] **A.E. Kogoj, S. Sonner,** *Attractors for a class of semi-linear degenerate parabolic equations,* Journal of Evolution Equations 13, 675-691, 2013.

Generalizations

Equations involving *X-elliptic operators* [1]

$$\mathcal{L} = \sum_{i,j=1}^{N} \partial_i (a_{ij} \partial_j u), \quad a_{ij} = a_{ji},$$

$X := \{X_1, \ldots, X_m\}$ vector fields in $\mathbb{R}^N$, $X_j = \sum_{k=1}^{N} \alpha_{jk} \partial_{x_k}$.

$$\frac{1}{C} \sum_{j=1}^{m} \langle X_j(x), \xi \rangle^2 \leq \sum_{i,j=1}^{N} a_{ij}(x) \xi_i \xi_j \leq C \sum_{j=1}^{m} \langle X_j(x), \xi \rangle^2 \quad \forall x, \xi \in \mathbb{R}^N,$$

$$\langle X_j(x), \xi \rangle = \sum_{k=1}^{N} \alpha_{jk}(x) \xi_k, \qquad j = 1, \ldots, m.$$

[1] **E. Lanconelli, A.E. Kogoj,** *X-elliptic operators and X-control distances,* Contributions in honor of the memory of Ennio De Giorgi, Ric. Mat. 49 suppl., 223–243, 2000.

Examples of Admissible X-elliptic Operators

- Δ_λ-Laplacians,
  e.g., Grushin-type operators

$$\Delta_x + |x|^{2\alpha}\Delta_y$$

- Sub-Laplacians on Carnot groups,
  e.g., Kohn Laplacian on the Heisenberg group

$$\Delta_{\mathbb{H}^N} = \sum_{j=1}^{N}(X_j^2 + Y_j^2),$$

where the vector fields

$$X_j = \partial_{x_j} + 2y_j\partial_z, \quad Y_j = \partial_{y_j} - 2x_j\partial_z, \quad (x, y, z) \in \mathbb{R}^{2N+1}.$$

Generalizations [1]

Global well-posedness, existence and finite fractal dimension of global attractors for

- semi-linear degenerate parabolic problems

$$u_t = \mathcal{L}u + f(u),$$

- semi-linear degenerate damped hyperbolic problems

$$u_{tt} + \beta u_t = \mathcal{L}u + f(u),$$

where $\mathcal{L}$ is X-elliptic, f is dissipative and satisfies appropriate growth restrictions determined by the vector fields $\{X_1, \ldots, X_m\}$.

[1] **A.E. Kogoj, S. Sonner,** *Attractors met X-elliptic operators,* submitted.

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