Swimming at low Reynolds number

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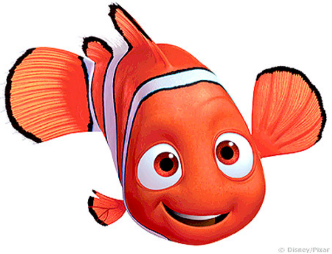
IECL–BCAM

Benasque 2013

Motivations

Swimming is seen as a control problem.

Given two points, does a fish can swim from one point to the other?



The motion of the fish is due to fluid-structure interactions.

The fluid

$$\text{Reynolds number: } Re = \frac{\rho U L}{\mu}.$$

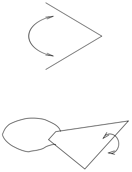
	$L\,(cm)$	$U\,(cm.s^{-1})$	$T\,(s)$	Re
Bacteria	10^{-5}	10^{-3}	10^{-4}	10^{-5}
Spermatozoon	10^{-3}	10^{-2}	10^{-2}	10^{-3}
Fish	50	100	0.5	5.10^{4}
Pigeon	25	10^{3}	5.10^{-1}	10^{5}

The Deformations I

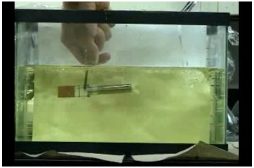
All deformations are not interesting from the point of view of the motion.

Theorem (Scallop, Purcell, 1977)

For a periodic motion described by one parameter, the displacement on one period is null.

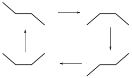


No motion
$\Rightarrow$
in Stokes fluid

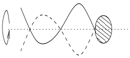


Taylor's experience

The Deformations II



Purcell's swimmer



Helical Deformation

Motion
$\Rightarrow$
in Stokes fluid

Taylor's experience

State of the art I

- Low Reynolds number:
  - Explicit solutions have been computed by J. Blake, 1973 and J. Happel and H. Brenner, 1983.
- Swimming model:
  - Experiences realised by G. Taylor, 1951.
  - Model and specificities of low Reynolds swimmers given by E. M. Purcell, 1977 and S. Childress, 1981.
  - First vision of the swimming problem as a control problem: A. Shapere and F. Wilczek, 1989.

State of the art II

- Controllability results:
  - Perfect fluid: T. Chambrion and A. Munnier, 2010.
  - Stokes fluid, with a swimmer formed by n spheres: F. Alouges, A. DeSimone and A. Lefebvre, 2009.
  - Stokes fluid, with a ciliated swimmer: J. San Martin, T. Takahashi and M. Tucsnak, 2007.

Domain

Let $B^\dagger(t)$ be the domain filled by the swimmer, $\Sigma^\dagger(t)$ it's boundary and $F^\dagger(t) = \mathbb{R}^3 \setminus \overline{B^\dagger(t)}$ the domain filled by the fluid.

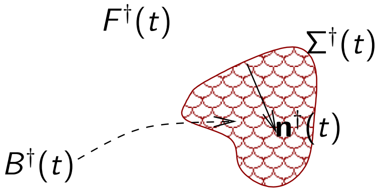


Figure: Domain

The fluid

Navier-Stokes Equations:

$$\begin{aligned} \rho\left(\frac{\partial \mathbf{u}^\dagger}{\partial t} + (\mathbf{u}^\dagger\cdot\nabla)\mathbf{u}^\dagger\right) + \nabla p^\dagger - \nu\Delta\mathbf{u}^\dagger &= 0 \quad \text{in } F^\dagger(t) \\ \operatorname{div}\mathbf{u}^\dagger &= 0 \quad \text{in } F^\dagger(t) \end{aligned} \tag{NS}$$

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The fluid is assumed to be at rest at infinity and to glue the swimmer,

$$\mathbf{u}^\dagger = \mathbf{v}_s \qquad \text{on } \Sigma(t)\,,$$

where $\mathbf{v}_s$ is the velocity of the swimmer.

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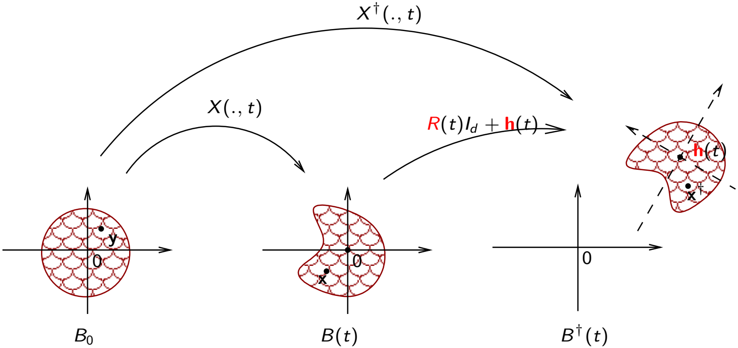
Let $\sigma = \nu\left(\nabla\mathbf{u}^\dagger + (\nabla\mathbf{u}^\dagger)^T\right) - p^\dagger I_3 \in \mathbb{R}^{3\times 3}$ be the Cauchy stress tensor. The force exerted by the fluid on a part $\mathrm{d}\Gamma$ of $\Sigma(t)$ is $\sigma\mathbf{n}\,\mathrm{d}\Gamma$.

The swimmer
Deformation

The swimmer is located by the position of it's center of mass $\mathbf{h} \in \mathbb{R}^3$ and an angular position $R \in O^+(3)$.



The swimmer
Deformation speed

The velocity of a point $x^\dagger = X^\dagger(y,t)$ of $B^\dagger(t)$ is:

$$\mathbf{v}_S = \dot{\mathbf{h}} + R\boldsymbol{\omega} \times (x^\dagger - \mathbf{h}) + R\,\mathbf{w}(x^\dagger, t)\,,$$

with:

- $\mathbf{w}$ the non-rigid velocity of the swimmer:

$$\mathbf{w}(x^\dagger, t) = \dot{X}\left(X(.,t)^{-1}(R^T(x^\dagger - \mathbf{h}(t))),\ t\right)\,.$$

- $\boldsymbol{\omega}$ the angular velocity:

$$\dot{R} = RA(\boldsymbol{\omega})\,,$$

where, $A(\boldsymbol{\omega}) = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}\,.$

The deformation $X(t)$ must be:

- a C^1-diffeomorphism of $\mathbb{R}^3$;

and must preserve:

- the mass:

$$\longrightarrow \qquad \rho(\cdot, t) = \frac{1}{\left|\det\left(\mathrm{Jac} X(\cdot, t)\right)\right|}$$

- the position of the mass center:

$$0 = \int_{B(t)} \rho(x, t) x \, \mathrm{d}x \, ;$$

- the angular momentum:

$$0 = \int_{B(t)} \rho(x, t) x \times \dot{X} \left(X(., t)^{-1}(x), \, t \right) \, \mathrm{d}x \, .$$

Newton's principle gives:

$$\begin{aligned} m\ddot{\mathbf{h}} &= \int_{\Sigma^\dagger(t)} \sigma \mathbf{n}^\dagger \,\mathrm{d}\Gamma\,, \\ \frac{\mathrm{d}\, J\boldsymbol{\omega}}{\mathrm{d}t} &= \int_{\Sigma^\dagger(t)} (x - \mathbf{h}) \times \sigma \mathbf{n}^\dagger \,\mathrm{d}x\,, \end{aligned} \qquad \text{(PFD)}$$

with $J(t)$ the inertial matrix at time t.

The coupled problem

In dimensionless variables, taking the formal limit $L \to 0$, we obtain the *quasi-static* problem:

$$\left\{\begin{array}{rcll} 0 & = & \nabla p^\dagger - \Delta \mathbf{u}^\dagger\,, & \text{in } F^\dagger(t) \\ 0 & = & \operatorname{div} \mathbf{u}^\dagger\,, & \text{in } F^\dagger(t) \\ & & \displaystyle\lim_{|\mathbf{x}|\to\infty} \mathbf{u}^\dagger(x) = 0 & \end{array}\right. \qquad (\mathrm{S}^\dagger)$$

$$\mathbf{u}^\dagger = \dot{\mathbf{h}} + R\boldsymbol{\omega} \times (x - \mathbf{h}) + R\mathbf{w}\,, \text{on } \Sigma^\dagger(t) \qquad (\mathrm{BC}^\dagger)$$

$$\left\{\begin{array}{rcl} 0 & = & \displaystyle\int_{\Sigma^\dagger(t)} \sigma(\mathbf{u}^\dagger, p^\dagger)\mathbf{n}^\dagger \,\mathrm{d}\Gamma \\ 0 & = & \displaystyle\int_{\Sigma^\dagger(t)} (\mathbf{x} - \mathbf{h}) \times \sigma(\mathbf{u}^\dagger, p^\dagger)\mathbf{n}^\dagger \,\mathrm{d}\Gamma \end{array}\right. \qquad (\mathrm{CM}^\dagger)$$

Examples

- **Ciliated organism**:
  X is constant but $\mathbf{w} \neq 0$.
  - 2007: J. San Martin, T. Takahashi and M. Tucsnak proved that with six independent controls on **w**, the swimmer is exactly controllable.
  - 2008: M. Sigalotti and J.-C. Vivalda proved that generically with respect to the shape of the swimmer only three control are need.
- **Golestanian's swimmer**:
  $B(t)$ is the union of three aligned spheres.
  - 2009: F. Alouges, A. DeSimone and A. Lefebvre proved the controllability of this swimmer and studied optimal controls.

Equations in the axi-symmetric case

The system (S†)-(BC†)-(CM†) in the axi-symmetric case writes:

$$\left\{\begin{array}{rcll} 0 & = & \nabla p - \Delta \mathbf{u}\,, & \text{in } F(t)\,,\\ 0 & = & \operatorname{div} \mathbf{u}\,, & \text{in } F(t)\,,\\ & & & \\ & & \lim_{|\mathbf{x}|\to\infty} \mathbf{u}(x) = 0 & \end{array}\right. \tag{S}$$

$$\mathbf{u} = \dot{h}\mathbf{e}_z + \mathbf{w}\,, \qquad \text{on } \Sigma(t)\,, \tag{BC}$$

$$0 = \left(\int_{\Sigma(t)} \sigma(\mathbf{u}, p)\mathbf{n}\,\mathrm{d}\Gamma\right) \cdot \mathbf{e}_z\,, \tag{CM}$$

with $\mathbf{w}(x,t) = \dot{X}\left(X(\cdot,t)^{-1}(x), t\right)$.

ODE I

Let assume that the diffeomorphism X is given by:

$$X(t,x) = x + \sum_{i=1}^{n} \mathbf{s}_i(t) D_i(x)\,.$$

One can redefine X by $X(\mathbf{s}) = I_D + \sum_{i=1}^{n} \mathbf{s}_i D_i$. $\mathbf{s}$ is the deformation parameter.
For every $i \in \{1,\dots,n\}$, we define $\big(\mathbf{v}_i(\mathbf{s}), q_i(\mathbf{s})\big)$ the Stokes solution with boundary condition $\mathbf{v}_i(\mathbf{s}) = D_i \circ X(\mathbf{s})^{-1}$ on $\Sigma(\mathbf{s}) = X(\mathbf{s})(\Sigma)$.
We also define $\big(\mathbf{v}_0(\mathbf{s}), q_0(\mathbf{s})\big)$ the Stokes solution with boundary condition $\mathbf{v}_i(\mathbf{s}) = \mathbf{e}_z$ on $\Sigma(\mathbf{s})$.

ODE II

Expanding (CM), we obtain:

$$\int_{\Sigma(\mathbf{s})} \sigma\big(\mathbf{v}_0(\mathbf{s}), q_0(\mathbf{s})\big)\,\mathrm{d}\Gamma \cdot \mathbf{e}_z\ \dot{h} = -\sum_{i=1}^{n} \int_{\Sigma(\mathbf{s})} \sigma\big(\mathbf{v}_i(\mathbf{s}), q_i(\mathbf{s})\big)\,\mathrm{d}\Gamma \cdot \mathbf{e}_z\ \dot{\mathbf{s}}_i\,.$$

Setting $f_i(\mathbf{s}) = -\dfrac{\int_{\Sigma(s)} \sigma(\mathbf{v}_i, q_i)\,\mathrm{d}\Gamma \cdot \mathbf{e}_z}{\int_{\Sigma(s)} \sigma(\mathbf{v}_0, q_0)\,\mathrm{d}\Gamma \cdot \mathbf{e}_z}$, we have:

$$\dot{h} = \sum_{i=1}^{n} f_i(s)\lambda_i\,,$$
$$\dot{s} = \lambda\,.$$

Chow's theorem I

Let us consider a dynamical system under the form:

$$\dot{z} = \sum_{i=1}^{n} f_i(z) u_i \,, \qquad (*)$$

set on $\mathbb{R}^n$.
We associated to this system the Lie algebra $\mathrm{Lie}\{f_1, \dots, f_n\}$ which is the smallest Lie algebra containing $\{f_1, \dots, f_n\}$ stable for the Lie bracket:

$$\begin{array}{rccl} [f,g] \,:\, \mathbb{R}^n & \to & \mathbb{R}^n \\ z & \mapsto & \mathrm{D}_z g \cdot f(z) - \mathrm{D}_z f \cdot g(z) \,. \end{array}$$

Chow's theorem II

Theorem (Chow)

Let assume that $f_i \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ and $u_i(t) \in B_{\mathbb{R}^m}(0, r)$ $(r > 0)$.
If for every $z \in \mathbb{R}^n$, $\mathrm{Lie}_z\{f_1, \ldots, f_m\} = \mathbb{R}^n$, then the system (*) is controllable.

Control result

Let consider the control problem:

Problem

Given $h^f \in \mathbb{R}^*$ does-it exists $T > 0$ and $\boldsymbol{\lambda} \in C^1([0,T], \mathbb{R}^n)$ such that:

$$h(T) = h^f \quad \text{and} \quad \mathbf{s}(T) = 0\,,$$

with the initial conditions:

$$h(0) = 0 \quad \text{and} \quad \mathbf{s}(0) = 0\,?$$

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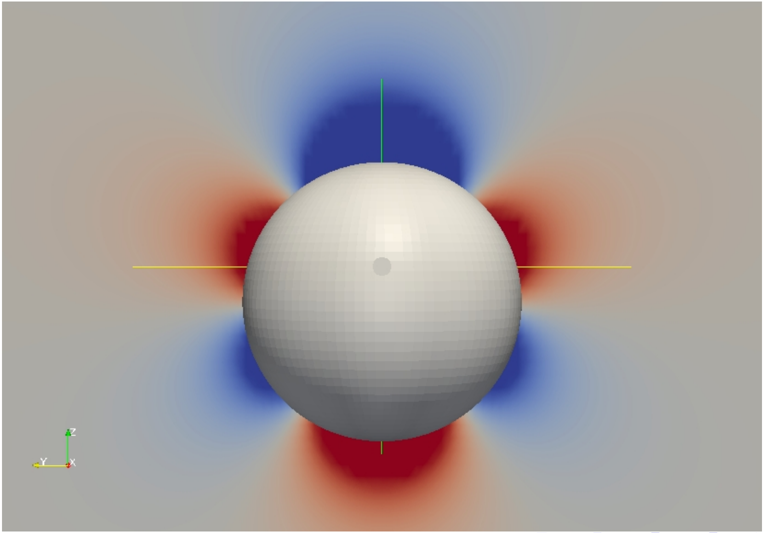
Using shape differentiation, explicit solution (given by Lamb) and Chow's theorem, we prove that the answer is positive for the elementary deformations given by :

$$D_1(r,\theta,\phi) = P_2(\cos\theta)\chi(r)\mathbf{e}_r(\theta,\phi)\,,$$
$$D_2(r,\theta,\phi) = P_3(\cos\theta)\chi(r)\mathbf{e}_r(\theta,\phi)\,.$$

Example of control



Conclusion

Other results

- Controllability holds if we had rotations, we then need four elementary deformations.
- Generically with respect to the shape, we can do motion planning (both for the rigid and the non-rigid deformations).
- There exists optimal controls. In the axi-symmetric case, we looked at the time optimal control problem.

Conclusion

Open problems

- Minimal number of independent controls?
- Swimming in a bounded domain?
- Swimming with a flagella?
- Collective swimming?
- If we had inertia to the system?
- Does microorganisms try to minimize a cost function? Which one?
- Numerical simulations?