A hierarchical multi-levels energy method for under-controlled coupled systems of PDE's and applications

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Benasque, Spain, August 26 2013

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Outline

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Insensitizing controls

J. -L. Lions has introduced the notion of *insensitizing control* for heat type equations in 1989. Several results for parabolic type equations: heat (Bodart et al. 1995, de Teresa 2000, Bodart et al. 2004, de Teresa and Zuazua 2009, Kavian and de Teresa 2010, ...), Stokes: Guerrero 2007, Gueye 2012, Navier-Stokes Gueye 2013...
Wave equation: Dáger 2006, Tebou 2008, Tebou 2011.

The aim is to build controls that satisfy two requirements:

- to drive the initial state to a desired final state at time T
- and to be robust to small unknown perturbations of the initial data with respect to a given measurement of the solutions.

Let us describe formally this notion for the scalar wave equation in a bounded open set $\Omega \subset \mathbb{R}^n$ with a smooth boundary Γ

$$\begin{cases} y_{tt} - \Delta y = bv \text{ in } (0,T)\times\Omega\,, \\ y = 0 \text{ in } (0,T)\times\Gamma\,, \\ (y, y_t)(0,.) = (y^0 + \tau_0 z^0, y^1 + \tau_1 z^1) \text{ in } \Omega\,, \end{cases}$$

Here

- (y^0, y^1) are given known initial data
- (z^0, z^1) are unknown perturbations of the initial data and of norm 1 in appropriate functional spaces
- τ_0 and τ_1 are real (small) numbers measuring the amplitude of the perturbations (z^0, z^1)
- v is the control
- b is the control coefficient, which may vanish in some region of Ω.

We associate to the solution y the following measurement

$$\phi(y; \tau_0, \tau_1) = \frac{1}{2} \int_0^T \int_\Omega c(x) y^2 \, dx \, dt \,,$$

where c is the observation coefficient, which may also vanish in some regions of Ω and may have a disjoint support from that of the control coefficient b.

One says that the control v insensitizes ϕ if the following property holds

$$\frac{\partial \phi}{\partial \tau_0}(y;0,0) = \frac{\partial \phi}{\partial \tau_1}(y;0,0) = 0\,,$$

for all (z^0, z^1) of norm 1 in the appropriate spaces. We have formally

$$\frac{\partial \phi}{\partial \tau_0}(y;0,0) = \int_0^T \int_\Omega c(x) y_2 w \, dx \, dt\,,$$

$$\frac{\partial \phi}{\partial \tau_1}(y;0,0) = \int_0^T \int_\Omega c(x) y_2 z \, dx \, dt\,,$$

where y_2, w and z respectively solve

$$\begin{cases} y_{2,tt} - \Delta y_2 = bv \text{ in } (0,T)\times\Omega\,, \\ y_2 = 0 \text{ in } (0,T)\times\Gamma\,, \\ (y_2, y_{2,t})(0,.) = (y^0, y^1) \text{ in } \Omega\,, \end{cases} \qquad \begin{cases} w_{tt} - \Delta w = 0 \text{ in } (0,T)\times\Omega\,, \\ w = 0 \text{ in } (0,T)\times\Gamma\,, \\ (w, w_t)(0,.) = (z^0, 0) \text{ in } \Omega\,, \end{cases}$$

$$\begin{cases} z_{tt} - \Delta z = 0 \text{ in } (0,T)\times\Omega\,, \\ z = 0 \text{ in } (0,T)\times\Gamma\,, \\ (z, z_t)(0,.) = (0, z^1) \text{ in } \Omega\,. \end{cases}$$

Introducing the auxiliary equation

$$\begin{cases} y_{1,tt} - \Delta y_1 + c(x)y_2 = 0 \text{ in } (0,T)\times\Omega\,, \\ y_1 = 0 \text{ in } (0,T)\times\Gamma\,, \\ (y_1, y_{1,t})(0,.) = (y_1^0, y_1^1) \text{ in } \Omega\,. \end{cases}$$

Then, multiplying the above equation by w, integrating over $(0, T) \times \Omega$ and using the equation in w, we have formally

$$\frac{\partial \phi}{\partial \tau_0}(y;0,0) = \int_0^T \int_\Omega c(x) y_2 w \, dx \, dt = -\int_0^T \int_\Omega (y_{1,tt} - \Delta y_1) w = - \Big[\int_\Omega y_{1,t} w - y_1 w_t \Big]_0^T .$$

In a similar way, we have

$$\frac{\partial \phi}{\partial \tau_1}(y;0,0) = \int_0^T \int_\Omega c(x) y_2 z \, dx \, dt = -\int_0^T \int_\Omega (y_{1,tt} - \Delta y_1) z = - \Big[\int_\Omega y_{1,t} z - y_1 z_t \Big]_0^T .$$

Hence, the insensitizing property will hold as soon as: for any (y^0, y^1) and any final state $(y_T, y_{2,T})$, given in appropriate energy space, there exists a control v (in a suitable space), such that the solution of the controlled 2 order cascade system

$$\begin{cases} y_{1,tt} - \Delta y_1 + c(x) y_2 = 0 \text{ in } (0,T) \times \Omega\,, \\ y_{2,tt} - \Delta y_2 = bv \text{ in } (0,T) \times \Omega\,, \\ y_1 = y_2 = 0 \text{ in } (0,T) \times \Gamma\,, \\ (y_1, y_{1,t})(0,.) = (y_1^0, y_1^1) \text{ in } \Omega\,, \\ (y_2, y_{2,t})(0,.) = (y^0, y^1) \text{ in } \Omega\,, \end{cases} \tag{1}$$

satisfies the following property:

$$(y_1, y_{1,t})(0,.) = (y_1, y_{1,t})(T,.) = 0\,, (y_2, y_{2,t})(T,.) = (y_T, y_{2,T}) \text{ in } \Omega\,.$$

Important: normally, one equation is forward in time, the other is backward in time. Here due to the reversibility property for the wave equation, it is not a problem.

A similar formal analysis can be performed for the building of insensitizing controls on a subset Γ_1 of the boundary Γ.

Hence the existence of insensitizing controls for the scalar wave equation is directly linked to an exact controllability result for a coupled system of two equations in cascade with a **single** control.

This problem can be reformulated as

$$Y'' + \mathcal{M}_2 Y = \mathcal{B}_2 v \, , (Y, Y')(0) = (y_1^0, y^0, y_1^1, y^1) \, ,$$

where $Y = (y_1, y_2)^t$, $\mathcal{B}_2 v = (0, bv)^t$ and where the involved matrix operator $\mathcal{M}_2$ has the following upper triangular form

$$\mathcal{M}_2 = \begin{pmatrix} A & cI \\ 0 & A \end{pmatrix}$$

where I stands for the identity operator in $L^2(\Omega)$ and $A = -\Delta$ stands for the homogeneous Dirichlet Laplacian.

• Dáger 2006 proves the insensitizing boundary controllability, and the ε-insensitizing locally distributed controllability for the one-dimensional wave equation.
In both situations, the control regions need not to meet the coupling region. More precisely, ω and O can be any arbitrary non-empty subsets of Ω.

• Tebou 2008 considers the same questions in the multi-dimensional framework for locally distributed control. He considers only situations for which the control region ω and coupling region O meet.

More precisely, he proves the ε-insensitizing locally distributed controllability for arbitrary open subsets ω and O such that $\omega \cap O \neq \emptyset$.

He also proves the insensitizing locally distributed controllability under a strong geometric assumption, namely that both the control and coupling regions contain the same neighbourhood of a part Γ_1 of the boundary, that satisfies the usual multiplier condition.

The results are based in both cases on a controllability result for a coupled system in cascade.

The difficulty is to compensate the lack of control on the component y_1. By duality, the difficulty is to recover the energy of the initial data of the first component in the dual cascade system.

We shall give in the sequel, a general result for the existence of insensitizing controls in the multi-dimensional case for both locally distributed and boundary controls, and in situations for which one can have $\omega \cap O = \emptyset$.

Several results are available for the insensitizing control of heat or parabolic systems: Bodart and Fabre 1995, Bodart and González-Burgos and Pérez-García 2004,... de Teresa 2000, Guerrero (Stokes 2007), de Teresa and Zuazua 2009, de Teresa and Kavian 2010, Gueye (Stokes 2012), Gueye (Navier-Stokes 2013)...

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Another motivation is the simultaneous control of systems coupled in parallel. Let us give an example.

Denote by $-\mathcal{L}$ a uniformly elliptic operator on $\Omega \subset \mathbb{R}^n$ with smooth coefficients, subjected to homogeneous Dirichlet boundary conditions.

Set $p = (p_1, p_2, p_3)^t$, and use the notation $p_{tt} = (p_{1,tt}, \ldots, p_{3,tt})^t$, $\mathcal{L}p = (\mathcal{L}p_1, \ldots, \mathcal{L}p_3)^t$.

Let α and β be given functions on the set Ω. We consider the following controlled problem

$$\begin{cases} p_{1,tt} - \mathcal{L}p_1 - (3\alpha + \beta)p_1 + (\alpha + \beta)p_2 + (2\alpha + \beta)p_3 = v_1 \,, \\ p_{2,tt} - \mathcal{L}p_2 - (3\alpha - \beta)p_1 + (-\alpha + \beta)p_2 + (-2\alpha + \beta)p_3 = v_2 \,, \\ p_{3,tt} - \mathcal{L}p_3 - 6\alpha p_1 + 2\alpha p_2 + 4\alpha p_3 = v_3 \,, \end{cases}$$

where the initial conditions for p are known and where $v_1, v_2, v_3 \in L^2((0, T) \times \Omega)$ are the controls. Given any initial data, we want to find controls which drive back the solution to equilibrium at time T.

We want to control simultaneously these equations coupled in parallel.

This means that we look for controls which depend only on a scalar control.

More precisely we look for controls (v_1, v_2, v_3) such that

$$(v_1, v_2, v_3) = (\eta_1 h, \eta_2 h, h)\,,$$

where h is a scalar control in $L^2((0,T)\times\Omega)$, and where η_1, η_2 are fixed real coefficients.

Hence, for each given initial data we look for a scalar control h which could simultaneously drive back to equilibrium at time $T > 0$ each component of the system, i.e. which is such that $p_i(T) = p_{i,t}(T) = 0$, $i = 1, 2, 3$ for a sufficiently large time T.

Making an appropriate change of unknowns, we can transform this simultaneous control problem into a control problem for a bi-diagonal cascade problem by a single control. More precisely, set
$\eta_1 = 2\,, \eta_2 = 4$ and

$$\begin{aligned} y_1 &= \frac{1}{4}(-p_1 + p_2 - 2p_3), \\ y_2 &= \frac{1}{4}(3p_1 - p_2 - 2p_3), \\ y_3 &= (-p_1 + p_2 + p_3), \end{aligned}$$

Then $y = (y_1, y_2, y_3)^t$ is the solution of the following bi-diagonal 3-cascade system

$$\begin{cases} y_{1,tt} - \mathcal{L}y_1 + 6\alpha(x)y_2 = 0\,, t \in (0,T)\,, x \in \Omega\,, \\ y_{2,tt} - \mathcal{L}y_2 + \frac{\beta(x)}{2}y_3 = 0\,, t \in (0,T)\,, x \in \Omega\,, \\ y_{3,tt} - \mathcal{L}y_3 = 3h\,, t \in (0,T)\,, x \in \Omega\,. \end{cases}$$

This problem can be reformulated as

$$Y'' + \mathcal{M}_3 Y = \mathcal{B}_3 h,$$

where $Y = (y_1, y_2, y_3)^t$, $\mathcal{B}_3 v = (0, 0, 3h)^t$ and

$$\mathcal{B}_3 v = (0, 0, 3h)^t, \quad \mathcal{M}_3 = \begin{pmatrix} -\mathcal{L} & 6\alpha I & 0 \\ 0 & -\mathcal{L} & \frac{\beta}{2} I \\ 0 & 0 & -\mathcal{L} \end{pmatrix}$$

where I stands for the identity operator in $L^2(\Omega)$.

Here again, only the last equation is directly controlled and the goal is to determine whether if it is possible to drive any initial state to equilibrium by acting only on the last equation.

We can consider more generally bi-diagonal n-coupled controlled cascade system:

$$Y'' + \mathcal{M}_n Y = \mathcal{B}_n \mathbf{v}\,,$$

with $Y = (y_1, y_2, \ldots y_n)^t$, $\mathcal{B}_n \mathbf{v} = (0, \ldots, B_n v)^t$, and

$$\mathcal{M}_n = \begin{pmatrix} A & c_{21} I & 0 & \ldots & \\ 0 & A & c_{32} I & 0 & \ldots \\ \vdots & & & & \\ 0 & 0 & \ldots & A & c_{nn-1} I \\ 0 & 0 & \ldots & 0 & A \end{pmatrix},$$

Here once again, only the last equation is controlled, and the purpose is to control the full system.

Of course, there exist other forms of systems, such as for instance symmetric systems for which the matrix operator is symmetric and also more general systems with "less structure".

We see through the above examples that cascade systems arise naturally for several applications in control theory.

If we deal a single scalar controlled equation, then a wide literature has been developed with different methods to handle the controllability issues:

Multipliers method, micro-analysis approach, Carleman estimates, Fourier decomposition and Ingham types inequalities... by several authors: Russell, Fattorini, Lions, Ho, Zuazua, Komornik, Lasiecka, Triggiani, Bardos, Lebeau, Rauch, Burq, Gérard ...

Of course, there exist other forms of systems, such as for instance symmetric systems for which the matrix operator is symmetric and also more general systems with "less structure".

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Questions:

Here the main difficulty is due to the requirement to control a system of two (or more) coupled equations, by a single (or a reduced number of) control(s).

If the coupling vanishes then it is not possible to drive back to equilibrium the first uncontrolled equation, since it is then decoupled from the second one.

Hence if it is possible to control the coupled system, the coupling effects have to be active in some way.

This coupling should compensate the lack of controls in several equations: through which properties? And in particular, it is important to determine sufficient sharp geometric conditions on ω (resp. Γ_1) the active control region and O the active coupling region.

Is it possible to get necessary and sufficient (NS) conditions for such controllability properties to hold?

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We recall that by the Hilbert Uniqueness Method (HUM, Lions 1988), the exact controllability of the scalar wave equation is equivalent to an observability property for the homogeneous wave equation (dual problem). On the other hand, this observability property holds as soon as the Geometric Control Condition (GCC) of Bardos Lebeau Rauch (1992) is satisfied (it is a sufficient and almost necessary condition). We recall that (GCC) reads as follows

- **In the case of locally distributed observation:** an open subset ω of Ω satisfies (GCC) if there exists a time $T > 0$ such that every generalized bicharacteristic traveling at speed 1 in Ω meets ω at a time $t < T$.
- **Boundary observation:** a subset Γ_1 of the boundary Γ satisfies (GCC) if there exists a time $T > 0$ such that every generalized bicharacteristic traveling at speed 1 in Ω meets Γ_1 at a time $t < T$ in a non-diffractive point.

The above geometric conditions

as well as the fact that T has to be sufficiently large

are due to the finite speed of propagation for the wave equation.

Note that such conditions do not occur for the corresponding heat equation.

Other geometric conditions based on the multiplier method have also been derived, they hold for instance for star-shaped domains . . .

Once the observabilty inequalities are proved, the control v is build thanks to the HUM method.

Hence the exact controllability of the wave equation is well-known.

Cascade systems of two equations:

We first consider the following model example, namely the controlled cascade wave system in the case of a locally distributed control

$$\begin{cases} y_{1,tt} - \Delta y_1 + c y_2 = 0 \text{ in } (0,T) \times \Omega\,, \\ y_{2,tt} - \Delta y_2 = bv \text{ in } (0,T) \times \Omega \\ y_i = 0 \text{ in } (0,T) \times \Gamma \text{ for } i = 1, \ldots, 2\,, \\ (y_i, y_{i,t})(0) = (y_i^0, y_i^1) \text{ for } i = 1, \ldots, 2\,. \end{cases}$$

We set $H = L^2(\Omega)$, $A = -\Delta$ with domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ and the fractional powers of A as usual with the convention that $D(A^0) = H$. We recall in particular that $D(A^{1/2}) = H_0^1(\Omega)$.

We also set $Y_0 = (y_1^0, y_2^0, y_1^1, y_2^1)$. Then we have the following exact controllability result.

Theorem (A.-B. CRAS 2012, MCSS 2013)

We assume that :

- *the coefficient b satisfies*

$$\begin{cases} b \in \mathcal{C}(\overline{\Omega})\,, b \geq 0 \text{ on } \Omega\,, \\ \{b > 0\} \supset \overline{\omega} \text{ for some open subset } \omega \subset \Omega\,, \end{cases}$$

  where ω satisfies (GCC), the Geometric Control Condition (Bardos Lebeau and Rauch 1992)

- *the coefficient c satisfies*

$$\begin{cases} c \in W^{1,\infty}(\Omega) \\ c \geq 0 \text{ on } \Omega\,, \\ \{c > 0\} \supset \overline{O} \text{ for some open subsets } O \subset \Omega\,. \end{cases}$$

  where $O \subset \Omega$ satisfies (GCC)

Theorem (continued)

Then there exists $T^* > 0$ such that

- *for all $T > T^*$,*
- *and for all $Y_0 \in D(A) \times D(A^{1/2}) \times D(A^{1/2}) \times H$,*

there exists a control function $v \in L^2((0, T); L^2(\Omega))$,

such that the solution $Y = (y_1, y_2, y_1', y_2')$ of the controlled cascade system with initial data Y_0 satisfies

$$Y(T) = 0\,.$$

We consider the following controlled cascade wave system in the case of a boundary control

$$\begin{cases} y_{1,tt} - \Delta y_1 + c y_2 = 0 \text{ in } (0,T) \times \Omega \,, \\ y_{2,tt} - \Delta y_2 = 0 \text{ in } (0,T) \times \Omega \\ y_1 = 0 \text{ in } (0,T) \times \Gamma \,, y_2 = bv \text{ in } (0,T) \times \Gamma \,, \\ (y_i, y_{i,t})(0) = (y_i^0, y_i^1) \text{ for } i = 1, \ldots, 2 \,. \end{cases}$$

We denote by $D(A^{-1/2})$ the dual space of $D(A^{1/2})$ with respect to the pivot space H. Here $D(A^{-1/2}) = H^{-1}(\Omega)$.

Theorem (A.-B. CRAS 2012, MCSS 2013)

Assume that :

- *the coefficient b satisfies*

$$\begin{cases} b \in \mathcal{C}(\overline{\Gamma})\,, b \geq 0 \text{ on } \Gamma\,, \\ \{b > 0\} \supset \overline{\Gamma_1} \text{ for some subset } \Gamma_1 \subset \Gamma\,, \end{cases}$$

  where Γ_1 satisfies (GCC)
- *the coefficient c satisfies $(A2)$ where $O \subset \Omega$ satisfies (GCC)*

Then there exists $T^* > 0$ such that

- *for all $T > T^*$,*
- *and for all $Y_0 \in D(A^{1/2}) \times H \times H \times D(A^{-1/2})$,*

there exists a control function $v \in L^2((0,T); L^2(\Omega))$ such that

$$Y(T) = 0\,.$$

We can make several remarks:

- One has to define rigorously the notion of transposition solutions for boundary controlled cascade systems. This relies on the dual homogeneous system which is no longer conservative as for the scalar case. However, one can naturally extend this notion.
- One important point in the above results is that, the geometric conditions on the supports of the control and of the coupling coefficient allow disjoint intersections between the two supports, since the condition is that they both satisfy (GCC).
- The proof is constructive so that the time $T^\star$ can be characterized, but is not optimal.
- We prove the above results for general self-adjoint coercive operators A and general bounded coupling operators.
- The control v can be chosen in weaker or stronger spaces.

The proof of the above theorems relies on appropriate observability estimates for the dual homogeneous system:

$$\begin{cases} u_{1,tt} - \Delta u_1 = 0 \quad \text{in } (0,T) \times \Omega\,, \\ u_{2,tt} - \Delta u_2 + c(x) u_1 = 0 \quad \text{in } (0,T) \times \Omega\,, \\ u_i = 0 \text{ for } i = 1, \ldots 2 \quad \text{in } (0,T) \times \partial\Omega\,, \\ (u_i, u_{i,t})(0) = (u_i^0, u_i^1) \text{ for } i = 1, \ldots 2 \quad \text{in } \Omega\,, \end{cases}$$

Here the coupling operator is the multiplication operator by the coefficient c. Moreover, in view of applications to insensitzing controls, one can note that $c \geq 0$ in Ω, but may vanish over some parts of the domain.

Theorem (A.-B. 2012)

We assume that c satisfies the above smoothness property and $\{c > 0\} \supset \overline{O}$ where $O \subset \Omega$ satisfies (GCC). Then we have the following result in the case of boundary observation.
Let b be a given function defined on Γ such that $\{b \in \Gamma, b > 0\} \supset \overline{\Gamma_1}$ holds where Γ_1 satisfies (GCC). Then there exists $T^* > 0$ such that for all $T > T^*$, there exist constants $c_{i,2}(T) > 0$, $i = 1, 2$ such that for all $U^0 \in L^2(\Omega) \times H_0^1(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega)$ the following observability inequalities hold

$$c_{1,2}(T) ||(u_1^0, u_1^1)||^2_{L^2(\Omega)\times H^{-1}(\Omega))} \leq \int_0^T \int_\Gamma b \Big| \frac{\partial u_2}{\partial \nu} \Big|^2 d\sigma \, dt \,,$$

$$c_{2,2}(T) ||(u_2^0, u_2^1)||^2_{H_0^1(\Omega)\times L^2(\Omega)} \leq \int_0^T \int_\Gamma b \Big| \frac{\partial u_2}{\partial \nu} \Big|^2 d\sigma \, dt \,,$$

Assume that $\partial\Omega$ has no contact of infinite order with its tangent. Then the above condition on O and ω (resp. on O and Γ_1) for the case of locally (resp. boundary) distributed control are necessary.

The above results can be proved in a more general context, that is for abstract controlled cascade systems, for which A is a uniformly elliptic operator (for instance with variable coefficients).

It can also be associated to different types of boundary conditions.

Hence these results apply to much more general PDE's than the wave equation: elasticity, plates, beams . . .

The controllability results are deduced from observability results for the corresponding dual homogeneous abstract cascade systems via duality methods (here HUM).

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We consider the abstract dual cascade system

$$\begin{cases} u_1'' + Au_1 = 0\,, \\ u_2'' + Au_2 + C_{21}\,u_1 = 0\,, \\ (u_i, u_i')(0) = (u_i^0, u_i^1) \text{ for } i = 1, 2\,, \end{cases}$$

where

- H is an Hilbert space with norm $|\cdot|$ and scalar product $\langle\, ,\, \rangle$
- C_{21} is a bounded operator in H
- A satisfies:

$$\begin{cases} A : D(A) \subset H \mapsto H\,, A^* = A\,, \\ \exists\, \omega > 0\,, |Au| \geq \omega |u| \quad \forall\, u \in D(A)\,, \\ A \text{ has a compact resolvent}\,. \end{cases}$$

We want to study the observability properties for this cascade systems by a single observation on the second component in suitable functional spaces.

For this we need some notation:

- We set $H_k = D(A^{k/2})$ for $k \in \mathbb{N}$ and by convention $H_0 = H$ equipped with the corresponding norm and scalar product.
- H_{-k} denotes the dual space of H_k with the pivot space H
- For $V = (v_1, v_2) \in H_k \times H_{k-1}$, we define the energies of level k as

$$e_k(V)(t) = \frac{1}{2}\Big(|A^{k/2}v_1|^2 + |A^{(k-1)/2}v_2|^2\Big)\,,\ k \in \mathbb{Z}\,, i = 1, 2\,.$$

- $\mathcal{H} = H_1^2 \times H^2\,.$

Let G be a given Hilbert space, identified with its dual and let $\mathbf{B}^* \in \mathcal{L}(H_2 \times H; G)$ be a given operator. It is the observation operator. We assume that this operator is admissible. This means that the solution of the free wave equation over any given time interval $(0, T)$:

$$w'' + Aw = 0\,, (w, w')(0) = (w^0, w^1) \in H_1 \times H\,,$$

satisfies the so-called "direct inequality"

$$\int_0^T \|\mathbf{B}^*(w, w')\|_G^2 dt \leq Ce_1(W)(0)$$

where $W = (w, w')$ and where $C = C(T)$ does not depend on the initial data for W. As a consequence, it allows to define rigorously the observation for solutions of finite energy (in the case of boundary control).

In view of systems, we shall need a refined admissibility property, for solutions of the "forced" wave equation

$$w'' + Aw = f\,, (w, w')(0) = (w^0, w^1) \in H_1 \times H\,,$$

where $f \in L^2((0, T); H)$ with the suitable changes (the energy $e_1(W)$ of the solution is no longer conserved through time). This will be precised later on.

We recall that the operator $\mathbf{B}^*$ is said to satisfy an observability inequality for $T > T_0 > 0$ if

$$\begin{cases} \forall\ T > T_0\,, \exists\ C_1(T) > 0 \text{ such that} \\ \forall\ (w^0, w^1) \in H_1 \times H\,, \text{ the solution } w \text{ of} \\ w'' + Aw = 0\,, (w, w')(0) = (w^0, w^1) \text{ satisfies} \\ \int_0^T \|\mathbf{B}^*(w, w')\|_G^2 dt \geq C_1(T) e_1(W)(0)\,. \end{cases}$$

Let us go back to the dual cascade system

$$\begin{cases} u_1'' + Au_1 = 0\,, \\ u_2'' + Au_2 + C_{21}\,u_1 = 0\,, \\ (u_i, u_i')(0) = (u_i^0, u_i^1) \text{ for } i = 1, 2\,, \end{cases}$$

We want to give necessary and sufficient conditions on the coupling operator C_{21} and on the operator $\mathbf{B}^*$ for the following observability property

$$\int_0^T ||\mathbf{B}^* U_2||_G^2 \geq C(T)\Big(e_0(U_1)(0) + e_1(U_2)(0)\Big)\,,$$

to hold.

Note that the above inequality involves two *different* levels of energies: the natural energy for the directly observed component U_2 and a weakened energy for the unobserved component U_1.

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$$\begin{cases} u_1'' + Au_1 = 0\,, \\ u_2'' + Au_2 + C_{21}\,u_1 = 0\,, \\ (u_i, u_i')(0) = (u_i^0, u_i^1) \text{ for } i = 1, 2\,, \end{cases}$$

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to hold.

Note that the above inequality involves two *different* levels of energies: the natural energy for the directly observed component U_2 and a weakened energy for the unobserved component U_1.

For this, we shall assume the following hypotheses $(A2)-(A5)$.

We assume that the coupling operator C_{21} satisfies

$$(A2)\begin{cases} C_{21}^* \in \mathcal{L}(H_k) \text{ for } k\in\{0,1\}\,, \\ ||C_{21}|| = \beta\,, |C_{21}w|^2 \leq \beta\langle C_{21}w\,,w\rangle \quad \forall\, w\in H\,, \end{cases}$$

$$(A3)\begin{cases} \exists\; T_0>0, \forall\; T> T_0\,,\exists\; C_2(T)>0 \text{ such that} \\ \forall\,(w^0,w^1)\in H_1\times H \text{ the solution } w \text{ of} \\ w''+Aw=0\,,(w,w')(0)=(w^0,w^1) \text{ satisfies} \\ \int_0^T |C_{21}w'|^2 dt \geq C_2(T)e_1(W)(0)\,, \end{cases}$$

where $W=(w,w')$. We denote by G a given Hilbert space with norm $||\;||_G$ and scalar product $\langle\,,\rangle_G$. The space G will be identified to its dual space in all the sequel. We make the following assumptions on the observability operator $\mathbf{B}^*$ (the dual operator of the control operator $\mathbf{B}$).

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We assume that $\mathbf{B}^*$ is an admissible observation operator for one equation, that is

$$(A4)\begin{cases}\mathbf{B}^* \in \mathcal{L}(H_2 \times H; G)\,, \forall\ T > 0\ \exists\ C > 0, \\ \text{ such that for all } (w^0, w^1) \in H_1 \times H\,, \text{ and } f \in L^2([0,T]; H)\,, \\ \text{the solution } w \text{ of } w'' + Aw = f\,, (w, w')(0) = (w^0, w^1) \text{ satisfies} \\ \int_0^T \|\mathbf{B}^*(w, w')\|_G^2 dt \leq C(e_1(W)(0) + e_1(W)(T) + \\ \int_0^T e_1(W)(t)dt + \int_0^T |f|^2 dt)\,, \end{cases}$$

where $W = (w, w')$.
We assume the following observability inequality for a single equation

$$(A5)\begin{cases}\exists\ T_0 > 0, \forall\ T > T_0\,, \exists\ C_1(T) > 0 \text{ such that} \\ \forall\ (w^0, w^1) \in H_1 \times H\,, \text{ the solution } w \text{ of} \\ w'' + Aw = 0\,, (w, w')(0) = (w^0, w^1) \text{ satisfies} \\ \int_0^T \|\mathbf{B}^*(w, w')\|_G^2 dt \geq C_1(T) e_1(W)(0)\,. \end{cases}$$

Remark

The minimal times for which the two observability inequalities hold in $(A3)$ and $(A5)$ are not necessarily the same for the two observability operators.

Lemma (A.-B. MCSS 2013)

(Admissibility property) Assume $(A1)$, $(A4)$ and that $C_{21} \in \mathcal{L}(H)$, then for all $T > 0$, there exists a constant $C = C(T) > 0$ such that for all initial data $U^0 \in \mathcal{H}$, the solution of the homogeneous dual cascade system satisfies the following direct inequality

$$\int_0^T ||\mathbf{B}^* U_2||_G^2 \, dt \leq C\Big(e_0(U_1)(0) + e_1(U_2)(0)\Big) .$$

Remark

This Lemma establishes a hidden regularity property of the solutions: for all $U^0 \in \mathcal{H}$, $\mathbf{B}^* U_2 \in L^2([0,T]; G)$.

Theorem (Sufficient conditions, A.-B. MCSS 2013)

Assume the hypotheses $(A1)-(A5)$. Then there exists $T_* > 0$ such that for all $T > T_*$, and all initial data $U^0 \in \mathcal{H}$, the solution of the dual homogeneous cascade system satisfies the observability estimates

$$\begin{cases} d_1(T)\int_0^T ||\mathbf{B}^* U_2||_G^2 \geq e_0(U_1)(0)\,, \\ d_2(T)\int_0^T ||\mathbf{B}^* U_2||_G^2 \geq e_1(U_2)(0)\,, \end{cases}$$

where the constants $d_i(T) > 0$ depend on T and satisfy for T sufficiently large, suitable asymptotic properties with respect to T.

Note that the above observability inequality is in a *decoupled* form, that is it does not involve the sum of the initial energies of the two components, but each of them separately.

We also prove that the above conditions are optimal in the following theorem.

Theorem (Necessary conditions, A.-B. MCSS 2013)

Assume the hypotheses $(A1)$ *and* $(A2)$. *Assume that* C_{21} *does not satisfy the observability property given in* $(A3)$ *or that* $\mathbf{B}$ *does not satisfy* $(A5)$. *Then there does not exist* $T_* > 0$ *such that for all* $T > T_*$, *the following property holds*

$$(OBS) \begin{cases} \exists\, C > 0 \text{ such that } \forall\, U^0 \in \mathcal{H} \text{ the solution satisfies} \\ C(e_0(U_1)(0) + e_1(U_2)(0)) \leq C \int_0^T ||\mathbf{B}^* U_2||_G^2 \, dt\,. \end{cases}$$

Corollary (A.-B. MCSS 2013)

Assume $(A1)$ *and* $(A2)$. *Then* (OBS) *holds if and only if* $(A3)$ *and* $(A5)$ *hold.*

Here we assumed that provided that the coupling operator C_{21} satisfies the *partial coercivity property given in* $(A2)$.

Then to summarize, we prove

Theorem (A.-B. MCSS 2013)

Assume that $\mathbf{B}^*$ *is admissible ("forced" wave equation) and that* C_{21} *satisfies the partial coercivity property and regularity properties given in* $(A2)$. *Then there exists* $T_* > 0$ *such that for all* $T > T_*$, *and all initial data* $U^0 = (u_1^0, u_2^0, u_1^1, u_2^1) \in H \times H_1 \times H_{-1} \times H$ *the solution of the dual cascade system satisfies the observability estimate*

$$\int_0^T ||\mathbf{B}^* U_2||_G^2 \geq C_1(T)\Big(e_0(U_1)(0) + e_1(U_2)(0)\Big),$$

if and only if

Theorem (continued)

The operator $\mathbf{B}^*$ *satisfies an observability inequality for all* $T \geq T_0$ *for a single scalar abstract equation (property* $(A5)$*)*

and C_{21} *satisfies the observability inequality* $(A3)$.

The proof is based on

- *the two-level energy method (A.-B. 2001, 2003) introduced for systems coupled symmetrically and coercive couplings*
- *its extension to handle partially coercive couplings (A.-B. and Léautaud 2011, 2012) for symmetrically coupled systems*

The spirit of the proof is to:

compensate the lack of observation of the second component by a balance effect between the natural energy of the observed component and the weakened energy of the unobserved one.

This means that we have to work with the $H_1 \times H$ norm of the observed component, whereas we have to consider the $H_{-1} \times H$ norm of the unobserved component. This is a key point.

Note that if C_{21} vanishes then one cannot get any information on the initial data for u_1. Hence the result cannot hold true without some assumptions on C_{21}.

One needs first to prove an admissibility result for the coupled system. This is also used to define rigorously the solutions by transposition of the abstract controlled problem in Y.

The ingredients for the two-level energy method are:

- a key estimate due to the coercivity properties of the coupling.
- observability assumption for a forcing source term, uniform with respect to sufficiently large times T. This is a key property introduced already in A.-B. 2003 but proved in applications to PDE's by multipliers methods, and generalized later on by A.-B. and Léautaud in JMPA 2012 for the abstract forced wave equation (**in a form invariant by time-translation**).
- energy type estimates (several ones are required).
- conservation of the total natural and weakened energies and suitable balance of energies in the case of symmetrically coupled systems.
- this property is lost for cascade systems, however we proved that the two-level energy method can be extended to handle this case.

Positive results on cascade or symmetric systems have also been obtained by:

• A.-B. SICON 2003 for symmetric coupled systems of two abstract wave equations in case of coercive coupling operators, with different diffusion operators

• Rosier and de Teresa 2011 for a one dimensional system of two wave equations coupled in cascade and for Schrödinger cascade system in a $n-$dimensional torus.

• A.-B. and Léautaud JMPA 2012 for symmetrically coupled systems of two abstract wave equations in case of partially coercive couplings and the same diffusion operators

• Dehman Le Rousseau and Léautaud (2012) for the case of 2-coupled cascade systems in a $\mathcal{C}^\infty$ compact connected riemannian manifold without boundary with an implicit characterization of the minimal control time via micro-local analysis. They prove a "coupled" observability inequality.

Indeed we can observe that

Remark

- *Symmetric systems are "more coupled" than cascade systems:*

  Indeed when the initial data for U_1 is vanishing in the dual cascade system, then $U_1 \equiv 0$ for all times so that the system reduces to a scalar wave equation. This property does not hold true for symmetric systems. This also tells than the study of cascade systems is somehow easier *than the study of symmetric (or even more general coupled systems).*

- Decoupled *versus* Coupled *observability estimates:*

  It also explains why we can obtain decoupled *observability inequalities for the dual cascade system, and respectively a* coupled *observability inequality for the dual symmetric system.*

Concerning methods:

One can note that the methods based on micro-local analysis or Carleman estimates, neglect in general lower order terms, since they are absorbed by dominant terms (high frequencies or large values of parameters).

Here the positive controllability/observability results are based on the zero order terms due to the coupling and their coercivity properties. This is a main point in the two-level energy method.

The way to "measure" this positive effect is to work in a bigger space $H \times H_{-1}$ for the unobserved component. That is why it is important to work with two *different levels of energies*.

The two-level energy method is constructive so that it allows us to obtain quantitative results.

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We consider the scalar wave equation with a locally distributed control v (the location of the control depending on the the support of the coefficient function b):

$$\begin{cases} y_{tt} - \Delta y = \xi + bv \quad \text{in } (0,T) \times \Omega\,, \\ y = 0 \quad \text{in } (0,T) \times \Gamma\,, \\ y(0,.) = y^0 + \tau_0 z^0 \text{ in } \Omega\,, y_t(0,.) = y^1 + \tau_1 z^1 \text{ in } \Omega\,, \end{cases} \tag{2}$$

and the scalar wave equation with a boundary control v (the location of the control depending on the the support of the coefficient function b in Γ):

$$\begin{cases} y_{tt} - \Delta y = \xi \quad \text{in } (0,T) \times \Omega\,, \\ y = bv \quad \text{in } (0,T) \times \Gamma\,, \\ y(0,.) = y^0 + \tau_0 z^0 \text{ in } \Omega\,, y_t(0,.) = y^1 + \tau_1 z^1 \text{ in } \Omega\,, \end{cases} \tag{3}$$

where for both cases, the source term $\xi \in L^2((0,T) \times \Omega)$, the initial data (y^0, y^1) are given known functions in $H_0^1(\Omega) \times L^2(\Omega)$ or in $L^2(\Omega) \times H^{-1}(\Omega)$.

Theorem (A.-B. MCSS 2013)

Assume that $c \geq 0$ *satisfies* $c > 0$ *on* $\overline{O}$ *where* O *satisfies* (GCC)*. Let* $b \geq 0$ *be given such that* $b > 0$ *on* $\overline{\omega}$ *where* ω *satisfies* (GCC)*. Then for any given* $\xi \in L^2((0,T);L^2(\Omega))$ *and* $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$*, there exists an exact control* $v \in L^2((0,T);L^2(\Omega))$ *that drives back the solution of the scalar wave equation*

$$\begin{cases} y_{2,tt} - \Delta y_2 = \xi + bv & \text{in } (0,T) \times \Omega\,, \\ y_2 = 0 & \text{in } (0,T) \times \Gamma\,, \\ (y_2, y_{2,t})_{|t=0} = (y^0, y^1) & \text{in } \Omega\,, \end{cases}$$

to equilibrium, i.e. $y_2(T) = y_{2,t}(T) = 0$ *and insensitizes* Φ *along the solutions.*

Theorem (A.-B. MCSS 2013)

Assume that c satisfies the above hypothesis. Let $b \geq 0$ *be given on* Γ *such that* $b > 0$ *on* Γ_1 *where* Γ_1 *satisfies* (GCC)*. Then for any given* $\xi \in L^2((0,T);L^2(\Omega))$ *and* $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$*, there exists an exact control control* $v \in L^2((0,T);L^2(\Gamma))$ *that drives back the solution of the scalar wave equation*

$$\begin{cases} y_{2,tt} - \Delta y_2 = \xi \quad \text{in } (0,T) \times \Omega\,, \\ y_2 = bv \quad \text{in } (0,T) \times \Gamma\,, \\ (y_2, y_{2,t})_{|t=0} = (y^0, y^1) \quad \text{in } \Omega\,, \end{cases}$$

to equilibrium, i.e. $y_2(T) = y_{2,t}(T) = 0$ *and insensitizes* Φ *along the solutions.*

Hence the ability to under-control the cascade system in (y_1, y_2) allows us, via HUM:

to build a control that drives back the solution of the scalar equation to equilibrium,

but also to select among all possible controls, a control that is also robust for the selected measurement, to small perturbations of the initial data.

We can study further generalizations as follows

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We can study further generalizations as follows

Sommaire

We consider the dual bi-diagonal cascade systems

$$\begin{cases} u_1'' + Au_1 = 0\,, \\ u_i'' + Au_i + C_{ii-1}u_{i-1} = 0\,, 2 \leq i \leq n\,, \\ (u_i, u_i')(0) = (u_i^0, u_i^1) \text{ for } i = 1, \dots n\,, \end{cases}$$

In view of applications to simultaneous control of certain systems, we would like to determine sufficient (eventually also necessary) conditions, so that observing only the last component, we can recover the energy of initial data for all the components.

We give a necessary and sufficient condition on the observation operator (of the last component) and on the coupling operators C_{ii-1} for $i = 2, \dots, n$ so that observing only the last component, it is possible to recover suitable initial energies of all the n components of the unknown.

Theorem (A.-B. 2013, Advances in Differential Equations)

Assume that the observation operator $\mathbf{B}_\mathbf{n}^*$ satisfies a suitable admissibility property for the forced scalar wave equation and that the coupling operators on the sub-diagonal are partially coercive as seen for 2-coupled cascade systems and satisfy suitable smoothness properties.

Then the following observability inequality holds for $T > T_n^*$

$$\int_0^T ||\mathbf{B}_\mathbf{n}^* U_n||_{G_n}^2 \, dt \geq C(T)\Big(\sum_{i=1}^n e_{1-n+i}(U_i)(0)\Big),$$

holds if and only if $\mathbf{B}_\mathbf{n}^*$ and the coupling operators C_{ii-1} for $i = 2, \ldots, n$ satisfies an observability property (as given in the case of two equations).

The proof is a generalization of the proof for $n = 2$ (2-coupled cascade systems).

It relies on a tricky induction argument and requires a careful analysis of how the lack of observation of the $n - 1$ first components can be compensated by the coupling terms. The induction argument invokes several estimates with suitable asymptotic estimates for large times.

The multi-levels energy method is a constructive method. It uses the property that one can derive from the original system set in the natural energy space a hierarchy of related systems similar to the original one, but set in weakened energy spaces.

The solutions of these hierarchic systems are linked to each other, and this rich structure allows us to get positive controllability results.

The subclass of cascade bi-diagonal system can be seen as a toy model to understand and capture essential properties which guarantee controllability by a reduced number of controls.

Note that we observe only the last component of the unknown and we want to reconstruct the initial data of *all* the components of the unknown.

We give a necessary and sufficient condition on the observation operator and on the coupling coefficients on the sub-diagonal.

The "price" to pay is that going from the last equation towards the first one, we can reconstruct the initial data of the corresponding component, but in a hierarchy of weaker and weaker energy spaces.

Namely in $H_{1-n+i} \times H_{i-n}$ for the component $U_i = (u_i, u_i')$. Hence, the involved energy of U_i becomes weaker as i goes away from n which is the rank of the observed component.

This observability result leads to a exact controllability result for cascade systems of order n.

We consider the controlled cascade system

$$\begin{cases} y_i'' + Ay_i + C_{i+1i}^* y_{i+1} = 0\,, 1 \le i \le n-1\,, \\ y_n'' + Ay_n = B_n v_n\,, \\ (y_i, y_i')(0) = (y_i^0, y_i^1) \text{ for } i = 1, \ldots, n\,, \end{cases}$$

where either $B_n \in \mathcal{L}(G_n; H)$ (bounded control operator) or $B_n \in \mathcal{L}(G_n, H_2')$ (unbounded control operator).

We set $Y_0 = (y_1^0, \ldots, y_n^0, y_1^1, \ldots, y_n^1)$ and denote by $Y = (y_1, \ldots, y_n, y_1', \ldots, y_n')$ the solution of the above system with initial data Y_0.

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We consider the controlled cascade system

$$\begin{cases} y_i'' + Ay_i + C^*_{i+1i}y_{i+1} = 0\,, 1 \le i \le n-1\,, \\ y_n'' + Ay_n = B_n v_n\,, \\ (y_i, y_i')(0) = (y_i^0, y_i^1) \text{ for } i = 1, \dots, n\,, \end{cases}$$

where either $B_n \in \mathcal{L}(G_n; H)$ (bounded control operator) or $B_n \in \mathcal{L}(G_n, H_2')$ (unbounded control operator).

We set $Y_0 = (y_1^0, \dots, y_n^0, y_1^1, \dots, y_n^1)$ and denote by $Y = (y_1, \dots, y_n, y_1', \dots, y_n')$ the solution of the above system with initial data Y_0.

Theorem

Assume the above hypotheses (for the observability result). We define $T_n^* > 0$ as above.

(i) *Let $\mathbf{B}_n^*(w_n, w_n') = B_n^* w_n'$ with $B_n \in \mathcal{L}(G_n, H)$. We set*

$$X^*_{-(n-1)} = (\Pi_{i=1}^n H_{n-i+1}) \times (\Pi_{i=1}^n H_{n-i}) \,.$$

Then, for all $T > T_n^*$ and all $Y_0 \in X^*_{-(n-1)}$, there exists a control function $v_n \in L^2((0,T); G_n)$ such that the solution Y with initial data Y_0 satisfies $Y(T) = 0$.

(ii) *Let $\mathbf{B}_n^*(w_n, w_n') = B_n^* w_n$ with $B_n \in \mathcal{L}(G_n, H_2')$. We set*

$$X^*_{(n-1)} = (\Pi_{i=1}^n H_{n-i}) \times (\Pi_{i=1}^n H_{n-i-1}) \,.$$

Then, for all $T > T_n^*$ and all $Y_0 \in X^*_{(n-1)}$, there exists a control function $v_n \in L^2((0,T); G_n)$ such that the solution Y with initial data Y_0 satisfies $Y(T) = 0$.

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The above controllability properties can be transferred to controllability properties for heat type systems or Schrödinger type systems

thanks

to the transmutation method Seidman 1984, Phung 2001, Miller 2006, Ervedoza Zuazua 2011

We now consider the following coupled system

$$\begin{cases} e^{i\theta} y_{1,t} - \Delta y_1 + c y_2 = 0 \text{ in } (0,T) \times \Omega\,, \\ e^{i\theta} y_{2,t} - \Delta y_2 = bv \text{ in } (0,T) \times \Omega \\ y_i = 0 \text{ in } (0,T) \times \Gamma \text{ for } i = 1, \dots, 2\,, \\ y_i(0) = y_i^0 \text{ in } \Omega \text{ for } i = 1, \dots, 2\,. \end{cases}$$

We set $Y_0 = (y_1^0, y_2^0)$. We recover 2-coupled heat (resp. Schrödinger) cascade systems when $\theta = 0$ (resp. $\theta = \pm\pi/2$) and diffusive coupled cascade systems when $\theta \in (-\pi/2, \pi/2)$.

We can also consider n-coupled casade heat systems and apply our previous results to get positive null controllability results with a single control acting on the last equation.

Then we have the following exact controllability result.

Corollary (A.-B. 2012, 2013)

We assume that the coefficient c satisfies the hypothesis $(A2)$ for some open subset $O \subset \Omega$ that satisfies (GCC). We also assume that the coefficient b satisfies $(A1)$ where the subset ω satisfies (GCC). Then, the following properties hold

- *(i) The case $\theta \in (-\pi/2, \pi/2)$ (Heat type systems). We have for all $T > 0$, and all $Y_0 \in (L^2(\Omega))^2$, there exist a control function $v \in L^2((0,T); L^2(\Omega))$ such that the solution $Y = (y_1, y_2)$ with initial data Y_0 satisfies $Y(T) = 0$.*

- *(ii) The case $\theta = \pm\pi/2$ (Schrödinger systems). We have for all $T > 0$ and all $Y_0 \in H_0^1(\Omega) \times L^2(\Omega)$, there exist a control function $v \in L^2((0,T); L^2(\Omega))$ such that the solution $Y = (y_1, y_2)$ with initial data Y_0 satisfies $Y(T) = 0$.*

Consider now the case of a boundary control:

$$\begin{cases} e^{i\theta} y_{1,t} - \Delta y_1 + c y_2 = 0 \text{ in } (0,T)\times\Omega\,, \\ e^{i\theta} y_{2,t} - \Delta y_2 = 0 \text{ in } (0,T)\times\Omega \\ y_1 = 0 \text{ in } (0,T)\times\Gamma\,, y_2 = bv \text{ in } (0,T)\times\Gamma\,, \\ y_i(0) = y_i^0 \text{ in } \Omega \text{ for } i = 1,\ldots,2\,. \end{cases}$$

We have the following exact controllability result.

Corollary (A.-B. 2012, 2013)

We assume that the coefficients c satisfies the above hypothesis for some open subset $O \subset \Omega$ that satisfies (GCC). We also assume that the coefficients b satisfies the above hypothesis where the subset Γ_1 satisfies (GCC). Then we have

- *(i) The case $\theta \in (-\pi/2, \pi/2)$ (Heat type systems). We have for all $T > 0$, and all $Y_0 \in (H^{-1}(\Omega))^2$, there exist a control function $v \in L^2((0, T); L^2(\Gamma))$ such that the solution $Y = (y_1, y_2)$ with initial data Y_0 satisfies $Y(T) = 0$.*

- *(ii) The case $\theta = \pm\pi/2$ (Schrödinger systems). We have for all $T > 0$ and all $Y_0 \in L^2(\Omega) \times H^{-1}(\Omega)$, there exist a control function $v \in L^2((0, T); L^2(\Omega))$ such that the solution $Y = (y_1, y_2)$ with initial data Y_0 satisfies $Y(T) = 0$.*

Sommaire

- Geometric issues: at the present time, direct methods do not allow to reach situations for empty intersections between the control and coupling regions. Indirect methods (via transmutation), allow us to handle situations for parabolic cascade systems, for empty intersections, under (GCC) type conditions. It is not a natural condition for heat type equations. Is it possible to relax these conditions?
- Different dynamics: for instance different diffusion operators?
- Smoothness assumptions on the coupling coefficients. Our results require smoothness of the coupling coefficients. Rosier and de Teresa results are valid for 1-D wave cascade systems (resp. n-dimensional torus for Schrödinger) for non smooth coupling (characteristic function). Can our smoothness assumptions be weakened, to handle for instance characteristic functions?
- Necessity of the partial coercivity of the coupling, necessity and sufficiency of $(A3)-A5)$ if partial coercivity of the coupling is removed?

What about more complex systems and further questions?

- We can generalize the above results to mixed bi-diagonal and non bi-diagonal cascade systems (with a larger band of non vanishing coupling coefficients). The price is that we require then "more" observation operators. Further generalizations are in progress.
- Extensions to more complex equations (in collaboration with Y. Privat, E. Trélat and J. Valein) . . .
- Extensions to other types of couplings (work in progress).
- Work in progress for inverse problems (with P. Cannarsa and M. Yamamoto).
- Numerics and optimal design (in collaboration with Y. Privat and E. Trélat).
- Nonlinear systems (work in progress).

Thanks for your attention