Null controllability of the heat equation by the flatness approach

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Joint works with

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Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth open set, $\Gamma_0 \subset \partial\Omega$ be a (nonempty) open set, and $T > 0$.

We are concerned with the **null controllability problem**:
given θ_0, find a function u s.t. the solution of

$$\begin{aligned} \theta_t - \Delta\theta &= 0 && (t,x) \in (0,T) \times \Omega, \\ \frac{\partial\theta}{\partial\nu} &= 1_{\Gamma_0} u(t,x) && (t,x) \in (0,T) \times \Omega, \\ \theta(0,x) &= \theta_0(x), && x \in \Omega. \end{aligned}$$

satisfies

$$\theta(T,x) = 0 \qquad x \in \Omega$$

Huge literature...

- Duality methods (observability estimate for the adjoint eq.)
  - Fattorini-Russell '71, Luxembourg-Korevarr '71, Dolecki '73 (1D, using biorthogonal families and complex analysis)
  - Imanuvilov-Fursikov 95', Lebeau-Robbiano '95 (ND, $\forall(\Omega, \Gamma_0, T)$, using Carleman estimates)
- Direct methods
  - Jones '77, Littman '78 (construction of a fundamental solution with compact support in time, $\Gamma_0 = \partial\Omega$)
  - Littman-Taylor '07 (solution of ill-posed problems)
  - Laroche-Martin-Rouchon '00 (approximate controllability using a flatness approach)

Here, we shall revisit the flatness approach, deriving the **null controllability**, and show its relevance to numerics.

- Introduced in 1995 by M. Fliess, J. Lévine, Ph. Martin, P. Rouchon for (linear or nonlinear) ODEs; very useful for motion planning of mechanical systems
- The basic idea is as follows. Assume given a smooth control system

$$\dot{x} = f(x, u), \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}^m$$

  together with an **output** $y \in \mathbb{R}^m$ depending on x, u and a finite number of derivatives of u:

$$y = h(x, u, \dot{u}, ..., u^{(r)})$$

- y is a **flat output** if, conversely, for given smooth y, there is a unique solution (x, u) of the control system with output y, and x and u can be expressed as functions of y and a finite number of its derivatives:

$$\begin{aligned} x &= k(y, \dot{y}, ..., y^{(p)}) \\ u &= l(y, \dot{y}, ..., y^{(q)}) \end{aligned}$$

Since x and u are parameterized by y, to solve the control problem

$$\begin{aligned} \dot{x} &= f(x,u) \\ x(0) &= x_0,\ x(T) = x_T \end{aligned}$$

it is sufficient to pick $y \in C^\infty([0,T])$ such that

$$k(y, \dot{y}, ..., y^{(p)})(0) = x(0) = x_0 \qquad (1)$$

$$k(y, \dot{y}, ..., y^{(q)})(0) = x(T) = x_T \qquad (2)$$

(1)-(2) are (in general) easy to satisfy.

The control is then given by

$$u = l(y, \dot{y}, ..., y^{(q)})$$

- Consider the double integrator

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= u\end{aligned}$$

  Then $y = x_1$ is a flat output, as $(x_1, x_2, u) = (y, \dot{y}, \ddot{y})$.
  Note that $z = x_2$ is not a flat output, for $x_1 = \int z(t)dt$

- To steer the system from $x_0 = (0, 0)$ to $x_T = (1, 0)$, we have to find $y \in C^\infty([0, T])$ s.t.

$$y(0) = 0, \quad \dot{y}(0) = 0, \quad y(T) = 1, \quad \dot{y}(T) = 0.$$

  A simple solution is $y(t) = t^2(2T - t)^2/T^4$.

- Method applied by Laroche-Martin-Rouchon to derive the approximate controllability of (i) the 1D heat eq; (ii) the beam equation; (iii) the linearized KdV equation.
- The heat control problem reads:

$$\begin{aligned} \theta_t - \theta_{xx} &= 0, \qquad x \in (0,1) \\ \theta_x(t,0) &= 0, \quad \theta_x(t,1) = u(t), \\ \theta(0,x) &= \theta_0(x). \end{aligned}$$

- They proved in 2000 that for initial data decomposed as

$$\theta_0(x) = \sum_{i \geq 0} y_i \frac{x^{2i}}{(2i)!}$$

where

$$|y_i| \leq C \frac{i!^s}{R^i}, \qquad i \geq 0$$

with $s \in (1,2)$, C, $R > 0$, then the system can be driven to 0 with a control that is Gevrey of order s.

Take $y = \theta(t, 0)$ as output. It is **flat**, in the sense that the map $\theta \to y$ is a bijection between appropriate spaces of functions.

Seek a formal solution (analytic in x) in the form

$$\theta(t, x) = \sum_{i \geq 0} a_i(t) \frac{x^i}{i!}$$

Plugging this sum in the heat eq. gives $\sum_{i \geq 0} [a_{i+2} - a_i{}'] \frac{x^i}{i!} = 0$, and hence

$$a_{i+2} = a_i', \quad i \geq 0.$$

Since $a_0(t) = \theta(t, 0) = y(t)$ and $a_1(t) = 0$, we arrive to

$$a_{2i+1} = 0, \quad a_{2i} = y^{(i)}, \; i \geq 0,$$

and

$$\theta(t, x) = \sum_{i \geq 0} y^{(i)}(t) \frac{x^{2i}}{(2i)!}, \quad u(t, x) = \sum_{i \geq 1} \frac{y^{(i)}(t)}{(2i-1)!}$$

- Since $\theta(t,x) = \sum_{i\geq 0} y^{(i)}(t)\frac{x^{2i}}{(2i)!}$, it remains to find $y \in C^\infty([0,T])$ s.t. the series converges and

  $$y^{(i)}(0) = y_i, \quad y^{(i)}(T) = 0, \quad i \geq 0.$$

  Impossible to do with an analytic function, but possible with a function Gevrey of order $s > 1$
- $y \in C^\infty([0,T])$ is **Gevrey of order** $s \geq 0$ if there exist $R, C > 0$ such that

  $$|y^{(p)}(t)| \leq C\frac{p!^s}{R^p}, \quad \forall p \in \mathbb{N},\ \forall t \in [0,T]$$

  The larger s, the less regular y is ($s = 1 \iff y \in C^\omega$)
- $\theta \in C^\infty([t_1,t_2]\times[0,1])$ is **Gevrey of order s_1 in x and s_2 in t** if

  $$|\partial_x^{p_1}\partial_t^{p_2}\theta(t,x)| \leq C\frac{(p_1!)^{s_1}(p_2!)^{s_2}}{R_1^{p_1}R_2^{p_2}} \quad \forall p_1,p_2 \in \mathbb{N},\ \forall (t,x) \in [t_1,t_2]\times[0,1]$$

Theorem

Let $\theta_0 \in L^2(0,1)$ and $T > 0$. Pick $\tau \in (0,T)$ and $s \in (1,2)$. There exists $y \in C^\infty([\tau,T])$ Gevrey of order s on $[\tau,T]$ such that, setting

$$u(t) = \begin{cases} 0 & \text{if } 0 \le t \le \tau \\ \sum_{i\ge 0} \frac{y^{(i)}(t)}{(2i-1)!} & \text{if } \tau < t \le T, \end{cases}$$

the solution θ of

$$\begin{aligned} \theta_t - \theta_{xx} &= 0, \qquad x \in (0,1) \\ \theta_x(t,0) = 0,\ \theta_x(t,1) &= u(t), \\ \theta(0,x) &= \theta_0 \end{aligned}$$

satisfies $\theta(T,.) = 0$. Furthermore, u is Gevrey of order s in t on $[0,T]$, and $\theta \in C([0,T],L^2(0,1)) \cap C^\infty((0,T] \times [0,1])$ is Gevrey of order s in t and $s/2$ in x on $[\varepsilon,T] \times [0,1]$ for all $\varepsilon \in (0,T)$.

We apply a null control to smooth the state and reach the class of states for which the result by Laroche-Martin-Rouchon is valid.

Decomposing the initial state θ_0 as a Fourier series of cosines $\theta_0(x) = \sum_{n\geq 0} c_n \sqrt{2}\cos(n\pi x)$, we obtain

$$\theta(\tau, x) = \sum_{n\geq 0} c_n e^{-n^2\pi^2\tau}\sqrt{2}\cos(n\pi x) = \sum_{i\geq 0} y_i \frac{x^{2i}}{(2i)!}$$

where $y_i = \sqrt{2}\sum_{n\geq 0} c_n e^{-n^2\pi^2\tau}(-1)^i(n\pi)^{2i}$

Lemma

$$|y_i| \leq C||\theta_0||_{L^1(0,1)}(1+\tau^{-\frac{1}{2}})\frac{i!}{\tau^i} \qquad \forall i \geq 0$$

for some constant $C > 0$, so that $x \to \theta(\tau, x)$ is Gevrey of order $1/2$.

Proposition

(Flatness property) *Let* $s \in (1,2)$ *and* $y \in C^\infty([t_1, t_2])$ $(-\infty < t_1 < t_2 < \infty)$ *be Gevrey of order* s *on* $[t_1, t_2]$. *Let*

$$\theta(t,x) := \sum_{i\geq 0} \frac{x^{(2i)}}{(2i)!} y^{(i)}(t).$$

Then θ *is Gevrey of order* s *in* t *and* $s/2$ *in* x *on* $[t_1, t_2] \times [0,1]$ *and it solves the ill-posed problem*

$$\begin{aligned} \theta_t - \theta_{xx} &= 0, \qquad (t,x) \in [t_1, t_2] \times [0,1], \\ \theta(t,0) = y(t),\ \theta_x(t,0) &= 0. \end{aligned}$$

Thus $u(t) = \theta_x(t,1) = \sum_{i\geq 1} \frac{y^{(i)}(t)}{(2i-1)!}$ *is Gevrey of order* s *on* $[t_1, t_2]$.

It remains to design a function $y \in C^\infty([\tau, T])$ Gevrey of order $s \in (1,2)$ such that

$$y^{(i)}(\tau) = y_i, \qquad y^{(i)}(T) = 0, \qquad \forall i \geq 0.$$

- For any $s \in (1,2)$, we introduce the "step function"

$$\phi_s(t) = \begin{cases} 1 & \text{if } t \leq 0 \\ \frac{e^{-(1-t)^{-\kappa}}}{e^{-(1-t)^{-\kappa}} + e^{-t^{-\kappa}}} & \text{if } 0 < t < 1 \\ 0 & \text{if } t \geq 1. \end{cases}$$

  where $\kappa = (s-1)^{-1}$. Then ϕ_s is Gevrey of order s on $[-T, T]$ for all $T > 0$.

- Let

$$\bar{y}(t) = \sum_{i \geq 0} y_i \frac{(t-\tau)^i}{i!}$$

  Since $|y_i| \leq Ci!/\tau^i$, $\bar{y}$ is Gevrey 1 (analytic) on $[\tau, \tau + R]$ if $R < \tau$. Actually, we noticed that $\bar{y}$ can be extended to $(0, +\infty)$ as an analytic function: indeed, since $y_i = \sqrt{2} \sum_{n \geq 0} c_n e^{-n^2\pi^2\tau} (-1)^i (n\pi)^{2i}$, we have

$$\bar{y}(t) = \sqrt{2} \sum_{n \geq 0} c_n e^{-n^2\pi^2 t}$$

- For y, it is sufficient to pick $s \in (1,2)$, $0 < R \leq T - \tau$ (where $0 < \tau < T$), and to set

$$y(t) := \phi_s\left(\frac{t-\tau}{R}\right)\bar{y}(t), \qquad t \in [\tau, T].$$

- So far, the flatness approach was applied to 1D PDEs (and for radial solutions of 2D problems). The expansion of the solution as an entire series in all the spatial coordinates seems not to work well, even in 2D.
- Here, we shall see that we can deal with the null controllability of the heat equation on a cylinder

$$\Omega = \omega \times (0,1) \subset \mathbb{R}^N$$

where $\omega \subset \mathbb{R}^{N-1}$ is a smooth, bounded open set, and $N \geq 2$. We thus consider the control problem ($x = (x', x_N)$)

$$\begin{array}{rcll} \theta_t - \Delta\theta & = & 0, & (t,x) \in (0,t) \times \Omega \\ \dfrac{\partial\theta}{\partial\nu}(t,x',1) & = & u(t,x'), & (t,x') \in (0,T) \times \omega \\ \dfrac{\partial\theta}{\partial\nu}(t,x) & = & 0 & (t,x) \in (0,T) \times \partial\Omega \setminus \omega \\ \theta(0,x) & = & \theta(x), & x \in \Omega \end{array}$$

- For $N = 3$, this is nothing but the control of the temperature of a metallic rod by the heat flux on one lateral section.

- The good way to solve the problem is to consider "hybrid" expansions of θ mixing Fourier decomposition in x' (no control on $\partial\omega$) and analytic decomposition in x_N (control at $x_N = 1$).
- Introduce an orthonormal basis in $L^2(\omega)$, $(e_j)_{j\geq 0}$, constituted of eigenvectors for the Neumann Laplacian in ω, i.e.

$$\begin{aligned} -\Delta' e_j &= \lambda_j e_j && \text{in } \omega \\ \frac{\partial e_j}{\partial \nu'} &= 0 && \text{on } \partial\omega \end{aligned}$$

  where $\Delta' = \partial^2_{x_1} + \cdots \partial^2_{x_{N-1}}$, ν' = outward unit normal to ω, $0 = \lambda_0 < \lambda_1 \leq \lambda_j \leq \lambda_{j+1}$.
- Decompose $\theta(t, x', 0)$ as

$$\theta(t, x', 0) = \sum_{j\geq 0} z_j(t) e_j(x').$$

  We claim that the system is flat, with $(z_j(t))_{j\geq 0}$ as "flat output". Indeed, given a sequence $(z_j(t))_{j\geq 0}$ of smooth functions, we seek a formal solution of the heat equation in the form

$$\theta(t, x', x_N) = \sum_{i\geq 0} \frac{x_N^i}{i!} a_i(t, x')$$

  where the a_i's are still to be defined.

Plugging the formal solution $\theta = \sum_{i\geq 0} \frac{x_N^i}{i!} a_i$ in the heat equation gives

$$\sum_{i\geq 0} \frac{x_N^i}{i!}[a_{i+2}(t,x') - (\partial_t - \Delta')a_i(t,x')] = 0$$

so that $a_{i+2} = (\partial_t - \Delta')a_i$ for all $i \geq 0$. Moreover

$$a_0(t,x') = \theta(t,x',0) = \sum_{j\geq 0} z_j(t)e_j(x'), \qquad a_1(t,x') = 0.$$

Therefore, for all $i \geq 0$

$$\begin{aligned} a_{2i+1} &= 0, \\ a_{2i} &= (\partial_t - \Delta')^i a_0 = \sum_{j\geq 0}(\partial_t - \Delta')^i[z_j(t)e_j(x')] = \sum_{j\geq 0} e_j(x')(\partial_t + \lambda_j)^i z_j(t) \\ &= \sum_{j\geq 0} e_j(x')e^{-\lambda_j t} y_j^{(i)}(t) \end{aligned}$$

where we have set $y_j(t) := e^{\lambda_j t} z_j(t)$. We arrive to

$$\theta(t,x',x_N) = \sum_{j\geq 0} e^{-\lambda_j t} e_j(x') \sum_{i\geq 0} y_j^{(i)}(t) \frac{x_N^{(2i)}}{(2i)!}$$

Proposition

Let $s \in (1,2)$, $-\infty < t_1 < t_2 < \infty$, and let $y = (y_j)_{j\geq 0}$ in $C^\infty([t_1,t_2])$ satisfy for some constants $M, R > 0$

$$|y_j^{(i)}(t)| \leq M \frac{i!^s}{R^i}, \qquad \forall i,j \geq 0,\ \forall t \in [t_1,t_2].$$

Then the function

$$\theta(t,x',x_N) = \sum_{j\geq 0} e^{-\lambda_j t} e_j(x') \sum_{i\geq 0} y_j^{(i)}(t) \frac{x_N^{(2i)}}{(2i)!}$$

is well defined in $[t_1,t_2]\times\overline{\Omega}$, and it is Gevrey of order s in t, $1/2$ in $x_1,\ldots,x_{N-1}$ and $s/2$ in x_N. It solves the ill-posed problem

$$\begin{aligned} \theta_t - \Delta\theta &= 0, \qquad (t,x) \in [t_1,t_2]\times\overline{\Omega}, \\ \theta(t,x',0) &= \sum_{j\geq 0} e^{-\lambda_j t} y_j(t) e_j(x'), \\ \theta_{x_N}(t,x',0) &= 0. \end{aligned}$$

The proof is similar to the one in dimension 1, but more technical (we need Weyl's formula $\lambda_j \sim j^{\frac{2}{N-1}}$).

Consider the control system

$$
(S) \qquad \begin{cases} \theta_t - \Delta\theta = 0, & (t,x) \in (0,t) \times \Omega \\ \frac{\partial \theta}{\partial \nu}(t,x',1) = u(t,x'), & (t,x') \in (0,T) \times \omega \\ \frac{\partial \theta}{\partial \nu}(t,x) = 0 & (t,x) \in (0,T) \times \partial\Omega \setminus \omega \\ \theta(0,x) = \theta(x), & x \in \Omega \end{cases}
$$

Theorem

Let $\Omega = \omega \times (0,1) \subset \mathbb{R}^{N-1} \times \mathbb{R}$ be as above, and let $\theta_0 \in L^2(\Omega)$ and $T > 0$ be given. Pick any $\tau \in (0,T)$ and any $s \in (1,2)$. Then there exists a sequence $(y_j)_{j \geq 0}$ of functions in $C^\infty([\tau, T])$ which are Gevrey of order s on $[\tau, T]$ and such that the control input

$$
u(t,x') = \begin{cases} 0 & \text{if } 0 \leq t \leq \tau, \\ \sum_{i,j \geq 0} e^{-\lambda_j t} \frac{y_j^{(i)}(t)}{(2i-1)!} e_j(x') & \text{if } \tau \leq t \leq T, \end{cases}
$$

is Gevrey of order s in t and $1/2$ in $x_1, \ldots, x_{N-1}$ on $[0,T] \times \overline{\omega}$, and the solution θ of (S) satisfies $\theta(T,.) = 0$.
Furthermore, $\theta \in C([0,T], L^2(\Omega)) \cap C^\infty((0,T] \times \overline{\Omega})$, and θ is Gevrey of order s in t, $1/2$ in $x_1, \ldots, x_{N-1}$ and $s/2$ in x_N on $[\epsilon, T] \times \overline{\Omega}$ for all $\epsilon \in (0,T)$.

Assume given $T > 0$, $\tau \in (0, T)$, $s \in (1, 2)$, and $\theta_0 \in L^2(\Omega)$ decomposed as

$$\theta_0(x', x_N) = \sum_{j,n\geq 0} c_{j,n} e_j(x') \sqrt{2} \cos(n\pi x_N).$$

The exact solution θ of the previous control problem such that $\theta(T, .) = 0$ was given as

$$\begin{aligned}
\theta(t, x', x_N) &= \sum_{j,n\geq 0} c_{j,n} e^{-(\lambda_j + n^2\pi^2)t} e_j(x') \sqrt{2} \cos(n\pi x_N), \qquad 0 \leq t \leq \tau, \\
\theta(t, x', x_N) &= \sum_{j\geq 0} e^{-\lambda_j t} e_j(x') \sum_{i\geq 0} y_j^{(i)}(t) \frac{x_N^{2i}}{(2i)!}, \qquad \tau \leq t \leq T,
\end{aligned}$$

where

$$\begin{aligned}
y_j(t) &= \phi(t) \sum_{n\geq 0} c_{j,n} e^{-n^2\pi^2 t}, \qquad \tau \leq t \leq T, \\
\phi(t) &= \phi_s\left(\frac{t-\tau}{T-\tau}\right), \qquad \tau \leq t \leq T.
\end{aligned}$$

In practice, only partial sums can be computed.

Introduce for given $\bar{i}, \bar{j}, \bar{n} \in \mathbb{N}$

$$\overline{\theta}(t, x', x_N) = \sum_{0 \le j \le \bar{j}} \sum_{0 \le n \le \bar{n}} c_{j,n} e^{-(\lambda_j + n^2\pi^2)t} e_j(x') \sqrt{2} \cos(n\pi x_N), \qquad 0 \le t \le \tau.$$

for the free evolution, and

$$\overline{\theta}(t, x', x_N) = \sum_{0 \le j \le \bar{j}} e^{-\lambda_j t} e_j(x') \sum_{0 \le i \le \bar{i}} \overline{y}_j^{(i)}(t) \frac{x_N^{2i}}{(2i)!}, \qquad \tau \le t \le T.$$

with

$$\overline{y}_j(t) = \phi(t) \sum_{0 \le n \le \bar{n}} c_{j,n} e^{-n^2\pi^2 t}, \qquad \tau \le t \le T,$$

for the controlled evolution.

The approximation of the free evolution is easily estimated. Let us focus on the approximation of the controlled evolution.

$$\overline{\theta}(t,x',x_N) = \sum_{0\le j\le \bar{j}} e^{-\lambda_j t} e_j(x') \sum_{0\le i\le \bar{i}} \overline{y_j}^{(i)}(t) \frac{x_N^{2i}}{(2i)!}$$
$$\overline{y_j}(t) = \phi(t) \sum_{0\le n\le \bar{n}} c_{j,n} e^{-n^2\pi^2 t}$$

Theorem

Let $N \geq 2$ and let T, τ and s be as above. Then there exist some positive constants C_i, $i = 1, ..., 4$ such that for any $\theta_0 \in L^2(\Omega)$ and any $\bar{i}, \bar{j}, \bar{n} \in \mathbb{N}$, we have for all $t \in [\tau, T]$

$$||\theta(t) - \overline{\theta}(t)||_{L^\infty(\Omega)} \leq C_1 \left(e^{-C_2 \bar{j}^{\frac{2}{N-1}}} + e^{-C_3 \bar{i} \ln \bar{i}} + e^{-C_4 \bar{n}^2} \right) ||\theta_0||_{L^2(\Omega)}.$$

In applications, one would like to apply to the (physical) system the approximate control $\bar{u} = \frac{\partial \bar{\theta}}{\partial x_N}(t, x', 1)$. Let us denote by $\hat{\theta}$ the (real) trajectory associated with θ_0 and $\bar{u}$.

Corollary

Let $N \geq 2$, $T, \tau, s, C_2, C_3, C_4$ be as above. Then there exists some positive constant C_1' such that for any $\theta_0 \in L^2(\Omega)$ and any $\bar{i}, \bar{j}, \bar{n} \in \mathbb{N}$, we have

$$||\theta - \hat{\theta}||_{L^\infty((0,T)\times\Omega)} \leq C_1' \left(e^{-C_2 \bar{j}^{\frac{2}{N-1}}} + e^{-C_3 \bar{i} \ln \bar{i}} + e^{-C_4 \bar{n}^2} \right) ||\theta_0||_{L^2(\Omega)}.$$

Some references about the numerical control of the heat equation

- Carthel-Glowinski-Lions 1994
- Münch-Zuazua 2010
- Micu-Zuazua 2011
- Belgacem-Kaber 2011
- Boyer-Hubert-Le Rousseau 2011
- Fernandez Cara-Münch 2012

Initial state: $\theta_0 := 1_{(1/2,1)}(x) - 1_{(0,1/2)}(x)$
Parameters: $\tau = 0.3$, $R = 0.2$, $T = \tau + R = 0.5$, $s = 1.6$

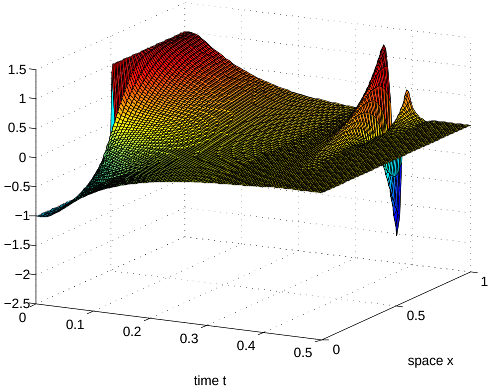


Fig.1. $\bar{\theta}(t, x)$

Computations by Philippe Martin

Initial state: $\theta_0 := 1_{(1/2,1)}(x) - 1_{(0,1/2)}(x)$

Parameters: $\tau = 0.3$, $R = 0.2$, $T = \tau + R = 0.5$, $s = 1.6$

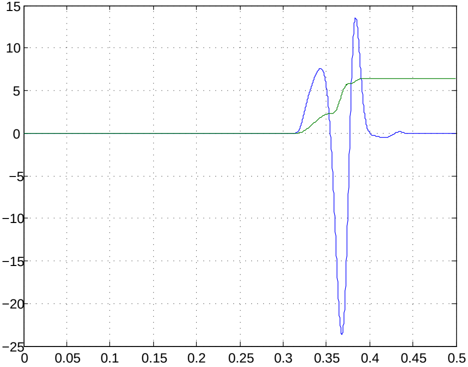
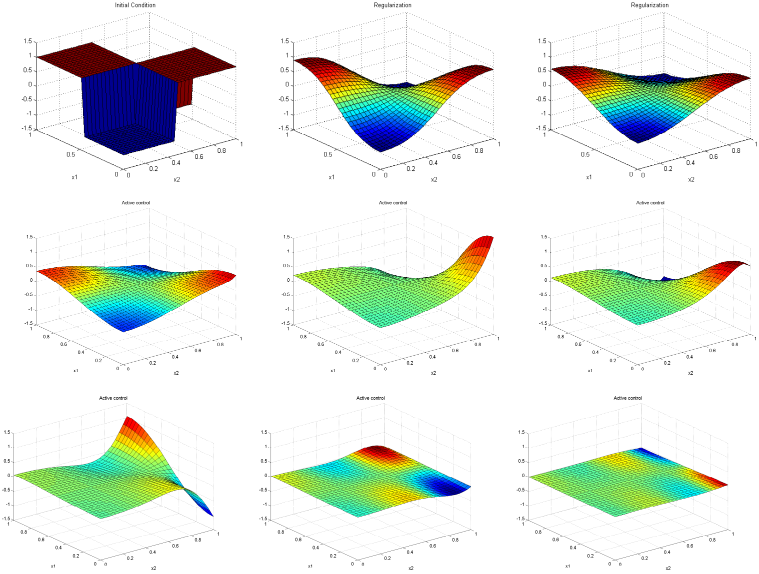


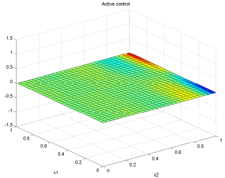
Fig. 2. $\bar{u}(t)$ (blue) and $||\bar{u}(t)||_{L^2(0,t)}$ (green)

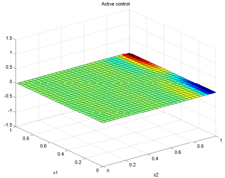
Initial state: $\theta_0 := \left(1_{(1/2,1)}(x_1) - 1_{(0,1/2)}(x_1)\right)\left(1_{(0,1/2)}(x_2) - 1_{(1/2,1)}(x_2)\right)$

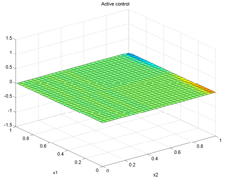
Parameters: $\tau = 0.05$, $R = 0.25$, $T = \tau + R = 0.3$, $s = 1.65$



Initial state: $\theta_0 := \left(1_{(1/2,1)}(x_1) - 1_{(0,1/2)}(x_1)\right)\left(1_{(0,1/2)}(x_2) - 1_{(1/2,1)}(x_2)\right)$
Parameters: $\tau = 0.05$, $R = 0.25$, $T = \tau + R = 0.3$, $s = 1.65$







Computations by Philippe Martin

Initial state: $\theta_0 := \left(1_{(1/2,1)}(x_1) - 1_{(0,1/2)}(x_1)\right)\left(1_{(0,1/2)}(x_2) - 1_{(1/2,1)}(x_2)\right)$
Parameters: $\tau = 0.05$, $R = 0.25$, $T = \tau + R = 0.3$, $s = 1.65$

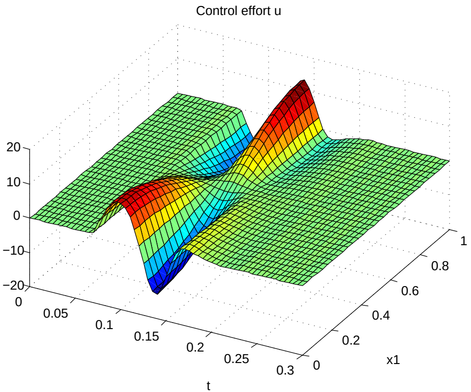


Fig. 4. $\bar{u}(t, x_1)$

- The flatness approach allows to recover the null controllability of the heat equation in cylinders, with explicit controls and trajectories easy to approximate
- Similar results have been obtained for the control of Schrödinger equation. Smoothing effect (step 1) obtained in a different way
- Future direction of research:
  - Extension to any pair (Ω, Γ_0) (challenging)
  - Determination of the space of reachable states (for those Gevrey controls)
  - Other linear/nonlinear equations
  - Numerical investigation of the cost of the control in terms of the parameters τ (free evolution), R (active control), s (Gevrey regularity)
  - Numerical cost for the computation of derivatives

Thank you for your attention!!