Approximation of the controls for the beam equation

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Controlled beam equation

Given any $T > 0$ and initial data $(u^0, u^1) \in H_0^1(0,\pi) \times H^{-1}(0,\pi)$, the exact controllability in time T of the linear beam equation with hinged ends,

$$\begin{cases} u''(t,x) + u_{xxxx}(t,x) = 0, \quad x \in (0,\pi),\ t > 0 \\ u(t,0) = u(t,\pi) = u_{xx}(t,0) = 0, \quad t > 0 \\ u_{xx}(t,\pi) = v(t), \quad t > 0 \\ u(0,x) = u^0(x),\ u'(0,x) = u^1(x), \quad x \in (0,\pi) \end{cases} \tag{1}$$

consists of finding a scalar function $v \in L^2(0,T)$, called control, such that the corresponding solution (u, u') of (1) verifies

$$u(T,\,\cdot\,) = u'(T,\,\cdot\,) = 0. \tag{2}$$

(Many) methods to study the controllability

Several approaches are available for the study of a controllability problem:

- Moment theory

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  - Microlocal Analysis

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- Direct methods
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- Optimization methods (Hilbert Uniqueness Method)
  - Multipliers
  - Carleman estimates
  - Microlocal Analysis

Fattorini H. O. and Russell D. L., Exact controllability theorems for linear parabolic equations in one space dimension, Arch. Rat. Mech. Anal., 4 (1971), 272-292.

Observability inequality

Equation (1) is exactly controllable in time T if the following observability inequality holds:

$$\|(\varphi(0), \varphi'(0))\|^2_{H^1_0 \times H^{-1}} \leq C \int_0^T |\varphi_x(t, \pi)|^2 dt, \tag{3}$$

for any $(\varphi^0, \varphi^1) \in H^1_0(0, \pi) \times H^{-1}(0, \pi)$ and (φ, φ') solution of the adjoint equation

$$\begin{cases} \varphi''(t, x) + \varphi_{xxxx}(t, x) = 0, \quad x \in (0, \pi),\ t > 0 \\ \varphi(t, 0) = \varphi(t, \pi) = \varphi_{xx}(t, 0) = \varphi_{xx}(t, \pi) = 0, \quad t > 0 \\ \varphi(T, x) = \varphi^0(x),\ \varphi'(T, x) = \varphi^1(x), \quad x \in (0, \pi). \end{cases} \tag{4}$$

Moreover, for any initial data $(u^0, u^1) \in H^1_0(0, \pi) \times H^{-1}(0, \pi)$ there exists a control $v \in L^2(0, T)$ with the property

$$\|v\|_{L^2} \leq \sqrt{C} \|(u^0, u^1)\|_{H^1_0 \times H^{-1}}. \tag{5}$$

An old friend: Ingham's inequality

Observability inequality (3) is equivalent to inequality of the form

$$\sum_{n\in\mathbb{Z}^*}|\alpha_n|^2 \le C(T)\int_{-\frac{T}{2}}^{\frac{T}{2}}\left|\sum_{n\in\mathbb{Z}^*}\alpha_n e^{\nu_n t}\right|^2 dt,\ \ (\alpha_n)_{n\in\mathbb{Z}^*}\in\ell^2. \quad (6)$$

Ingham's inequality

For any $T > \frac{2\pi}{\gamma_\infty}$, $\gamma_\infty = \liminf_{n\to\infty}|\nu_{n+1}-\nu_n|$, inequality (6) holds.

A. E. Ingham, Some trigonometric inequalities with applications to the theory of series, Math. Zeits., 41 (1936), 367-379.

J. Ball and M. Slemrod, Nonharmonic Fourier series and the stabilization of distributed semilinear control systems, Comm. Pure Appl. Math., 32 (1979), 555-587.

J. P. Kahane: Pseudo-Périodicité et Séries de Fourier Lacunaires, Ann. Sci. Ecole Norm. Super. 37, 93-95 (1962).

Observability inequality

In our particular case

$$\gamma_\infty = \liminf_{n\to\infty} |\nu_{n+1} - \nu_n| = \infty.$$

Ingham's inequality implies that the observability inequality (3) is verified for any $T > 0$.

Consequently, given any $T > 0$, there exists a control $v \in L^2(0,T)$ for each $(u^0, u^1) \in H^{-1}(0,\pi) \times H^{-1}(0,\pi)$.

The control function v is not unique.

Moment problem for the beam equation

The null-controllability of the beam equation is equivalent to solve the following moment problem:

For any $(u^0, u^1) = \left(\sum_{n=1}^{\infty} a_n^0 \sin(nx), \sum_{n=1}^{\infty} a_n^1 \sin(nx)\right)$, find $v \in L^2(0,T)$ such that

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} v\left(t+\frac{T}{2}\right) e^{t\overline{\nu}_n} dt = \frac{(-1)^n e^{-\frac{T}{2}\overline{\nu}_n}}{\sqrt{2}n\pi}\left(\overline{\nu}_n a_n^0 - a_n^1\right) \quad (n \in \mathbb{Z}^*), \tag{7}$$

where $\nu_n = i\, n^2\, \mathrm{sgn}(n)$ are the eigenvalues of the unbounded skew-adjoint differential operator corresponding to (1).

A solution v of the moment problem may be constructed by means of a biorthogonal family to the sequence $(e^{\nu_n t})_{n\in\mathbb{Z}^*}$.

Moment problem for the beam equation

Definition

A family of functions $(\phi_m)_{m\in\mathbb{Z}^*} \subset L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ *with the property*

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \phi_m(t) e^{\overline{\nu}_n t} dt = \delta_{mn} \qquad \forall\, m, n \in \mathbb{Z}^*, \tag{8}$$

is called a biorthogonal sequence to $(e^{\nu_n t})_{n\in\mathbb{Z}^*}$ *in* $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$.

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Once we have a biorthogonal sequence to $(e^{\nu_n t})_{n\in\mathbb{Z}^*}$, a "formal" solution of the moment problem is given by

$$v(t) = \sum_{n\in\mathbb{Z}^*} \frac{(-1)^n e^{-\frac{T}{2}\overline{\nu}_n}}{\sqrt{2}n\pi} \left(\overline{\nu}_n a_n^0 - a_n^1\right) \phi_n\left(t - \frac{T}{2}\right). \tag{9}$$

Ingham's inequality and the existence of a biorthogonal

Consider a Hilbert space H and a family $(f_n)_{n\in\mathbb{Z}^*} \subset H$ such that

$$C_1 \sum_{n\in\mathbb{Z}^*} |a_n|^2 \leq \left\| \sum_{n\in\mathbb{Z}^*} a_n f_n \right\|^2, \quad (a_n)_{n\in\mathbb{Z}^*} \in \ell^2. \tag{10}$$

Inequality (10) implies the existence of a biorthogonal sequence to the family $(f_n)_{n\in\mathbb{Z}^*}$.

- $(f_n)_{n\in\mathbb{Z}^*}$ is minimal i. e.

$$f_m \notin \overline{\text{Span}\left\{(f_n)_{n\in\mathbb{Z}^*\setminus\{m\}}\right\}} \qquad (m \in \mathbb{Z}^*).$$

- Apply Hahn-Banach Theorem to $\{f_m\}$ and $\overline{\text{Span}\left\{(f_n)_{n\in\mathbb{Z}^*\setminus\{m\}}\right\}}$
- There exists $\phi_m \in H$ such that

$$(\phi_m, f_m) = 1 \text{ and } (\phi_m, f_n) = 0 \text{ for any } n \neq m.$$

No Ingham?

If we are in a context in which no Ingham's type inequality is available?

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We can take the inverse way:

- Construction of the biorthogonal
  R. E. A. C. Paley and N. Wiener, Fourier Transforms in Complex Domains, AMS Colloq. Publ., Vol. 19, Amer. Math. Soc., New-York, 1934.
- Evaluation of its norm
- Construction of the control

Finite differences for the beam equation

$N \in \mathbb{N}^*$, $h = \frac{\pi}{N+1}$, $x_j = jh$, $0 \leq j \leq N+1$,
$x_{-1} = -h$, $x_{N+2} = \pi + h$.

$$\begin{cases} u_j''(t) = -\frac{u_{j+2}(t)-4u_{j+1}+6u_j(t)-4u_{j-1}(t)+u_{j-2}(t)}{h^4}, \ t > 0 \\ u_0(t) = u_{N+1}(t) = 0, \ u_{-1}(t) = -u_1(t), \ t > 0 \\ u_{N+2} = -u_{N-1} + h^2 v_h(t), \ t > 0 \\ u_j(0) = u_j^0, \ u_j'(0) = u_j^1, \ 1 \leq j \leq N. \end{cases} \tag{11}$$

Discrete controllability problem: given $T > 0$ and $(U_h^0, U_h^1) = (u_j^0, u_j^1)_{1 \leq j \leq N} \in \mathbb{C}^{2N}$, there exists a control function $v_h \in L^2(0,T)$ such that the solution u of (11) satisfies

$$u_j(T) = u_j'(T) = 0, \ \forall j = 1, 2, ..., N. \tag{12}$$

System (11) consists of N linear differential equations with N unknowns $u_1, u_2, ..., u_N$.
$u_j(t) \approx u(t, x_j)$ if $(U_h^0, U_h^1) \approx (u^0, u^1)$.

Discrete controls

- Existence of the discrete control v_h.
- Boundedness of the sequence $(v_h)_{h>0}$ in $L^2(0,T)$.
- Convergence of the sequence $(v_h)_{h>0}$ to a control v of the beam equation (1).

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L. LEON and E. ZUAZUA: Boundary controllability of the finite-difference space semi-discretizations of the beam equation. ESAIM:COCV, A Tribute to Jacques- Louis Lions, Tome 2, 2002, pp. 827-862.

Equivalent vectorial form

System (11) is equivalent to

$$\begin{cases} U_h''(t) + (A_h)^2 U_h(t) = F_h(t) & t \in (0,T) \\ U_h(0) = U_h^0 \\ U_h'(0) = U_h^1, \end{cases} \tag{13}$$

$$A_h = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}, \quad U_h(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_N(t) \end{pmatrix}$$

$$F_h(t) = \frac{1}{h^2}\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -v_h(t) \end{pmatrix}, \quad U_h^0 = \begin{pmatrix} u_1^0 \\ u_2^0 \\ \vdots \\ u_N^0 \end{pmatrix}, \quad U_h^1 = \begin{pmatrix} u_1^1 \\ u_2^1 \\ \vdots \\ u_N^1 \end{pmatrix}.$$

Discrete observability inequality

$$\begin{cases} W_h''(t) + A_h^2 W_h(t) = 0 \quad t \in (0,T) \\ W_h(T) = W_h^0 \in \mathbb{C}^N \\ W_h'(T) = W_h^1 \in \mathbb{C}^N. \end{cases} \tag{14}$$

The energy of (14) is defined by

$$E_h(t) = \frac{1}{2}\left(\langle A_h U_h(t), U_h(t)\rangle + \langle A_h^{-1} U_h'(t), U_h'(t)\rangle\right). \tag{15}$$

The exact controllability in time T of (11) holds if the following discrete observability inequality is true

$$E_h(t) \le C \int_0^T \left|\frac{W_{hN}(t)}{h}\right|^2 dt, \quad (W_h^0, W_h^1) \in \mathbb{C}^{2N} \tag{16}$$

$$C = C(T,h)$$

One or two problems

Eigenvalues: $\nu_n = i\,\mathsf{sgn}\,(n)\,\mu_n$,

$$\mu_n = \frac{4}{h^2}\sin^2\left(\frac{n\pi h}{2}\right), \quad 1 \le |n| \le N.$$

Eigenvectors form an orthogonal basis in $\mathbb{C}^{2N}$:

$$\phi^n = \frac{1}{\sqrt{2\mu_n}}\begin{pmatrix} \varphi^n \\ \\ -\nu_n\,\varphi^n \end{pmatrix}, \quad \varphi^n = \sqrt{2}\begin{pmatrix} \sin(nh\pi) \\ \sin(2n\pi) \\ \vdots \\ \sin(Nn\pi) \end{pmatrix}, 1 \le |n| \le N.$$

The observability constant is not uniform in h:

$$(W_h^0, W_h^1) = \phi^N \Rightarrow C(T,h) = \frac{1}{T\cos^2\left(\frac{N\pi h}{2}\right)} \approx \frac{1}{Th^2}.$$

Cures (L. Leon and E. Zuazua, COCV 2002)

- Control the projection of the solution over the space Span$\{\phi^n : 1 \leq |n| \leq \gamma N\}$, with $\gamma \in (0,1)$.

$$\sum_{1\leq |n| \leq \gamma N} |\alpha_n|^2 \leq C \int_{-\frac{T}{2}}^{\frac{T}{2}} \left| \sum_{1\leq |n| \leq \gamma N} \alpha_n e^{\nu_n t} \right|^2 dt. \qquad (17)$$

- Introduce a new control which vanishes in the limit

$$E_h(t) \leq C \left[\int_0^T \left| \frac{W_{hN}(t)}{h} \right|^2 dt + h^2 \int_0^T \left| \frac{W'_{hN}(t)}{h} \right|^2 dt \right]. \qquad (18)$$

$C = C(T) \Rightarrow$ uniform controllability

Numerical vanishing viscosity

Instead of (13) we consider the system

$$\begin{cases} U_h''(t) + (A_h)^2 U_h(t) + \varepsilon A_h U_h'(t) = F_h(t) & t \in (0,T) \\ U_h(0) = U_h^0 \\ U_h'(0) = U_h^1, \end{cases} \tag{19}$$

- $\varepsilon = \varepsilon(h), \quad \lim_{h\to 0} \varepsilon = 0$
- If $F_h = 0$, $\dfrac{dE_h}{dt}(t) = -\varepsilon \langle A_h U_h'(t), U_h'(t) \rangle \leq 0$
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Can we obtain the uniform controllability in any $T > 0$ (without projection or additional controls) using this new discrete scheme?

Spectral analysis. Good news but no Ingham.

Eigenvalues: $\lambda_n = \frac{1}{2}\left(\varepsilon + i\,\mathsf{sgn}\,(n)\sqrt{4-\varepsilon^2}\right)\mu_{|n|}$, $1 \leq |n| \leq N$.

Eigenvectors:

$$\phi^n = \frac{1}{\sqrt{2\mu_n}}\begin{pmatrix} \varphi^n \\ \\ -\lambda_n\,\varphi^n \end{pmatrix}, \quad \varphi^n = \sqrt{2}\begin{pmatrix} \sin(nh\pi) \\ \sin(2n\pi) \\ \vdots \\ \sin(Nn\pi) \end{pmatrix}, 1 \leq |n| \leq N.$$

If $(W_h^0, W_h^1) = \phi^N$ we obtain that

$$C(T,h) = \frac{\int_0^T \left|\frac{W_{hN}(t)}{h}\right|^2 dt}{\|(W_h(0), W_h'(0))\|^2} \approx \frac{1}{\cos^2\left(\frac{N\pi h}{2}\right)}\frac{\Re(\lambda_N)}{e^{2T\Re(\lambda_N)}-1}.$$

To ensure the uniform observability of these initial data we need

$\varepsilon > C\ln\left(\frac{1}{h}\right)h^2$

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To ensure the uniform observability of these initial data we need
$\varepsilon > C\ln\left(\frac{1}{h}\right)h^2 \Rightarrow \Re(\lambda_N) > C\ln\left(\frac{1}{h}\right)$.

Discrete moments problem

Theorem

Let $T > 0$ and $\varepsilon > 0$. System (13) is null-controllable in time T if and only if, for any initial datum $(U_h^0, U_h^1) \in \mathbb{C}^{2N}$ of form

$$(U_h^0, U_h^1) = \left(\sum_{j=1}^{N} a_{jh}^0 \varphi^j, \sum_{j=1}^{N} a_{jh}^1 \varphi^j \right), \tag{20}$$

the exists a control $v_h \in L^2(0,T)$ such that

$$\int_0^T v_h(t) e^{\overline{\lambda}_n t} dt = \frac{(-1)^n h}{\sqrt{2}\sin(|n|\pi h)} \left(-a_{|n|h}^1 + (\overline{\lambda}_n - \varepsilon \mu_{|n|}) a_{|n|h}^0 \right), \tag{21}$$

for any $n \in \mathbb{Z}^*$ such that $|n| \leq N$.

Biorthogonal family

If $(\theta_m)_{1\leq|m|\leq N} \subset L^2\left(-\frac{T}{2},\frac{T}{2}\right)$ is a *biorthogonal sequence to the family of exponential functions* $\left(e^{\lambda_n t}\right)_{1\leq|n|\leq N}$ *in* $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$ then a control of (13) will be given by

$$v_h(t) = \sum_{1\leq|n|\leq N} \frac{(-1)^n h e^{-\lambda_n \frac{T}{2}}}{\sqrt{2}\sin(|n|\pi h)} \left(-a^1_{|n|h} + (\overline{\lambda}_n - \varepsilon\mu_{|n|})a^0_{|n|h}\right)\theta_n\left(t - \frac{T}{2}\right).$$

Now the main task in to show that there exists a biorthogonal sequence $(\theta_m)_{1\leq|m|\leq N}$ and to evaluate its L^2-norm in order to estimate the right hand side sum.

Construction of a biorthogonal (I) - The big picture

Suppose that $(\theta_m)_{1\leq |m|\leq N}$ is a biorthogonal sequence to the family of exponential functions $\left(e^{\lambda_n t}\right)_{1\leq |n|\leq N}$ in $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$ and define

$$\Psi_m(z) = \int_{-\frac{T}{2}}^{\frac{T}{2}} \overline{\theta_m(t)} e^{-i\,tz} dt.$$

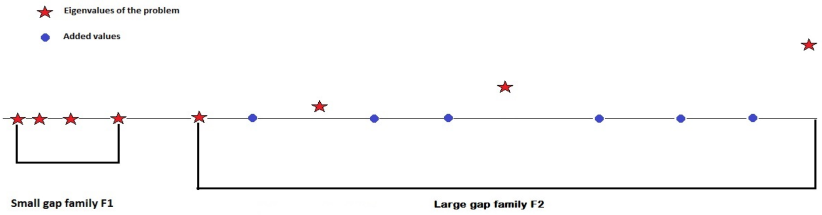
- $\Psi_m(i\lambda_n) = \delta_{nm}$
- Ψ_m is an entire function of exponential type $\frac{T}{2}$
- $\Psi_n \in L^2(\mathbb{R})$

Paley-Wiener Theorem ensures that the reciprocal is true and gives a constructive way to obtain a biorthogonal sequence.

$$\Psi_m(z) = P_m(z) \times E_m(z) = \prod_{n\neq m} \frac{i\lambda_n - z}{i\lambda_n - i\lambda_m} \times E_m(z).$$

P_m and M_m should have small exponential type and good behavior on the real axis.

Construction of a biorthogonal (II) - A small picture



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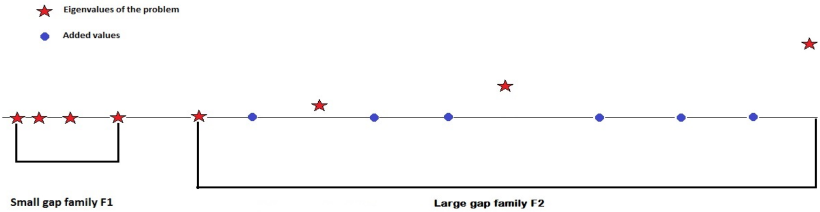

- $(\xi_l^1)_l$ is a biorthogonal to family F_1 which is finite.
- $(\xi_k^2)_k$ is a biorthogonal to family F_2 with good gap properties.
- A biorthogonal $(\theta_m)_m$ to full family $F_1 \cup F_2$ can be constructed by using the Fourier transforms $\widehat{\theta_k^1}$ and $\widehat{\theta_l^2}$.

Construction of a biorthogonal (III) - The model

In the large family F_2 we introduce new values. Why?

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$$F(z) = \frac{\sin(\delta z)}{\delta z} = \prod_{n \in \mathbb{Z}^*} \left(1 - \frac{\delta z}{n\pi}\right)$$

- F has small exponential type if δ is small.
- F is L^2 on the real axis.
- The last property is not true if we take a finite part of the product.

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- F is L^2 on the real axis.
- The last property is not true if we take a finite part of the product.

We have added new values to obtain a product P_m similar to F.

Construction of a biorthogonal (IV): The End

Theorem

Let $T > 0$. There exist two positive constants h_0 and ε_0 such that for any $h \in (0, h_0)$ and $\varepsilon \in \left(c_0 h^2 \ln\left(\frac{1}{h}\right), c_0 h\right)$ there exists a biorthogonal $(\theta_m)_m$ to $(e^{\lambda_n t})_n$ and two constants $\alpha < T$ and $C = C(T) > 0$ (independent of ε and h) such that

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \left| \sum_m \alpha_m \theta_m(t) \right|^2 dt \leq C(T) \sum_m |\alpha_m|^2 e^{\alpha |\Re(\lambda_m)|}, \tag{22}$$

for any finite sequence $(\alpha_m)_m$.

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Theorem

Let $T > 0$. There exist two positive constants h_0 and ε_0 such that for any $h \in (0, h_0)$ and $\varepsilon \in \left(c_0 h^2 \ln\left(\frac{1}{h}\right), c_0 h\right)$ there exists a biorthogonal $(\theta_m)_m$ to $(e^{\lambda_n t})_n$ and two constants $\alpha < T$ and $C = C(T) > 0$ (independent of ε and h) such that

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \left| \sum_m \alpha_m \theta_m(t) \right|^2 dt \leq C(T) \sum_m |\alpha_m|^2 e^{\alpha |\Re(\lambda_m)|}, \qquad (22)$$

for any finite sequence $(\alpha_m)_m$.

Since

$$v_h(t) = \sum_{1 \leq |n| \leq N} \frac{(-1)^n h e^{-\frac{T\lambda_n}{2}}}{\sqrt{2}\sin(|n|\pi h)} \left(-a^1_{|n|h} + (\overline{\lambda}_n - \varepsilon \mu_{|n|}) a^0_{|n|h} \right) \theta_n\left(t - \frac{T}{2}\right).$$

we obtain immediately from (22) the uniform boundedness (in h) of the family of controls $(v_h)_{h>0}$.

Sorry! One last remark: return to Ingham's inequality

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \left| \sum_m \alpha_m \theta_m(t) \right|^2 dt \leq C(T) \sum_m |\alpha_m|^2 e^{\alpha |\Re(\lambda_m)|}.$$

$$\sum_m |\alpha_m|^2 e^{-\alpha |\Re(\lambda_m)|} = \int_{-\frac{T}{2}}^{\frac{T}{2}} \left(\sum_m \alpha_m \theta_m(t) e^{-\alpha |\Re(\lambda_n)|} \right) \overline{\left(\sum_m \alpha_m e^{\lambda_m t} \right)} dt \leq$$

$$\leq \sqrt{\int_{-\frac{T}{2}}^{\frac{T}{2}} \left| \sum_m \alpha_m e^{-\alpha |\Re(\lambda_n)|} \theta_m(t) \right|^2 dt} \sqrt{\int_{-\frac{T}{2}}^{\frac{T}{2}} \left| \sum_m \alpha_m e^{\lambda_m t} \right|^2 dt} \leq$$

$$\leq \sqrt{C(T) \sum_m |\alpha_m|^2 e^{-2\alpha |\Re(\lambda_n)|} e^{\alpha |\Re(\lambda_n)|}} \sqrt{\int_{-\frac{T}{2}}^{\frac{T}{2}} \left| \sum_m \alpha_m e^{\lambda_m t} \right|^2 dt}.$$

$$\sum_m |\alpha_m|^2 e^{-\alpha |\Re(\lambda_m)|} \leq C(T) \int_{-\frac{T}{2}}^{\frac{T}{2}} \left| \sum_m \alpha_m e^{\lambda_m t} \right|^2 dt.$$

Now is The End!

Thank you very much for your attention!