Universal entanglement of non-smooth surfaces

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Based on

P.B., R. C. Myers and W. Witczak-Krempa, PRL 115, 021602 (2015);
P.B. and R. C. Myers arXiv:1505.07842;
+ work in progress with the gentlemen above

Gravity - New perspectives from strings and higher dimensions – Benasque

July 16th, 2015

Outline

1. Introduction
2. Holographic calculations
3. QFT comparison
4. A universal ratio
5. Generalizations

Corner contribution to entanglement entropy

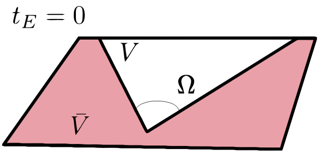
Let us consider a 3d CFT. If V has a corner of opening angle Ω, the entanglement entropy is given by

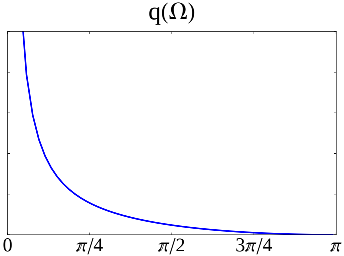
$$S_{EE} = \frac{c_1}{\delta} - q(\Omega)\log\left(\frac{H}{\delta}\right) - 2\pi c_0\,,$$

where $q(\Omega)$ is a positive even function of Ω constrained to behave as Casini, Huerta; Myers, Singh

$$q(\Omega\sim 0) = \kappa/\Omega\,,\quad q(\Omega\sim\pi) = \sigma\cdot(\Omega-\pi)^2\,.$$

κ and σ encode well-defined information about the CFT.





Stress tensor 2-point function

- The stress tensor 2-point function defines a central charge for CFT's in any spacetime dimension.
- In the vacuum, functional form completely fixed by conformal symmetry and energy conservation Erdmenger, Osborn, Petkou

$$\langle\, T_{ab}(x)\, T_{cd}(0)\,\rangle = \frac{C_T}{x^{2d}}\, \mathcal{I}_{ab,cd}(x)\,,$$

where

$$\mathcal{I}_{ab,cd}(x) \equiv \frac{1}{2}\left(I_{ac}(x)\, I_{db}(x) + I_{ad}(x)\, I_{cb}(x)\right) - \frac{1}{d}\,\delta_{ab}\,\delta_{cd}\,, \quad I_{ab}(x) \equiv \delta_{ab} - 2\frac{x_a\, x_b}{x^2}\,.$$

- In 2d CFT's, standard definition of the central charge c: $C_T = c$.
- In 4d CFT's, $C_T = 40\, c/\pi^4$ where c is the coefficient of the Weyl-squared term in the trace anomaly.

Conjecture

P.B., Myers, Witczak-Krempa

Based on strong evidence, we conjecture that σ and C_T are equal up to a numerical factor for general 3d CFT's

$$\boxed{\frac{\sigma}{C_T} = \frac{\pi^2}{24}}$$

Holographic entanglement entropy

Ryu-Takayanagi prescription for the EE of CFT's dual to Einstein gravity

Ryu, Takayanagi

$$S_{EE}(V) = \underset{m \sim V}{\mathrm{ext}} \left[\frac{\mathcal{A}(m)}{4G} \right] .$$

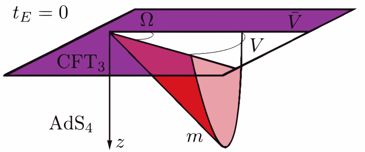
We extremize the area functional $\mathcal{A}(m)$ over all possible bulk surfaces m whose boundary coincides with ∂V.

Result for a corner region: Hirata, Takayanagi

$$S_{EE} = \frac{\tilde{L}^2}{2G} \frac{H}{\delta} - q_E(\Omega) \log\left(\frac{H}{\delta}\right) + \mathcal{O}(\delta^0) ,$$

$$q_E(\Omega) = \frac{\tilde{L}^2}{2G} \int_0^\infty dy \left[1 - \sqrt{\frac{1 + h_0(\Omega)^2(1+y^2)}{2 + h_0(\Omega)^2(1+y^2)}} \right] ,$$

$$\Omega(h_0) = \int_0^{h_0} dh \, \frac{2h^2\sqrt{1+h_0^2}}{\sqrt{1+h^2}\sqrt{(h_0^2 - h^2)(h_0^2 + (1+h_0^2)h^2)}} .$$



$$\Longrightarrow \sigma_E = \frac{\tilde{L}^2}{8\pi G}$$

Holographic entanglement entropy

- $\tilde{L}^2/G \sim \tilde{L}^2/\ell^2_{Planck} \sim \#$ d.o.f. For Einstein gravity this ratio appears everywhere, so we cannot distinguish among charges $\Rightarrow$ higher-derivative terms, which introduce new dimensionless quantities λ_i.
- Higher-derivative terms $\Rightarrow$ Ryu-Takayanagi no longer valid. Replace by Hung, Myers, Smolkin; Dong; Fursaev, Patrushev, Solodukhin; Sarkar, Wall...

$$S_{EE}(V) = \underset{m\sim V}{\text{ext}}\, S_{\text{grav}}(m)\,. \quad \text{Not Wald!}$$

- Corner contribution for the higher-order theory

$$I = \frac{1}{16\pi G}\int d^4x\,\sqrt{g}\left[\frac{6}{L^2} + R + L^2\left(\lambda_1 R^2 + \lambda_2 R_{\mu\nu}R^{\mu\nu} + \lambda_{\text{GB}}\mathcal{X}_4\right)\right.$$
$$\left. + L^4\left(\lambda_{3,0}R^3 + \lambda_{1,1}R\mathcal{X}_4\right) + L^6\left(\lambda_{4,0}R^4 + \lambda_{2,1}R^2\mathcal{X}_4 + \lambda_{0,2}\mathcal{X}_4^2\right)\right].$$

The final expressions take the form P.B., Myers

$$q(\Omega) = \alpha\, q_E(\Omega) \quad \Rightarrow \quad \sigma = \alpha\, \sigma_E\,,$$

$$\alpha = 1 - 24\lambda_1 - 6\lambda_2 + 432\lambda_{3,0} + 24\lambda_{1,1} - 6912\lambda_{4,0} - 576\lambda_{2,1} + \mathcal{O}(\lambda^2)\,.$$

Stress tensor 2-point function

- In holography, the stress tensor is dual to the normalizable mode of the metric Witten; Gubser, Klebanov, Polyakov $\Rightarrow$ metric fluctuations in AdS_4.

$$-\frac{\alpha}{2}\left[\bar{\Box}+\frac{2}{\tilde{L}^2}\right]h_{\mu\nu}-\frac{\lambda_2 L^2}{2}\left[\bar{\Box}+\frac{2}{\tilde{L}^2}\right]^2 h_{\mu\nu}=8\pi G T_{\mu\nu}$$

where again P.B., Myers

$$\alpha=1-24\lambda_1-6\lambda_2+432\lambda_{3,0}+24\lambda_{1,1}-6912\lambda_{4,0}-576\lambda_{2,1}+\mathcal{O}(\lambda^2)\,.$$

$$\Rightarrow \quad C_T=\alpha C_{T,E}=\alpha\frac{3}{\pi^3}\frac{\tilde{L}^2}{G}\,.$$

- Then, for all the holographic theories considered

$$\frac{\sigma}{C_T}=\frac{\sigma_E}{C_{T,E}}=\frac{\pi^2}{24}\,.$$

- This is not the case for other charges, *e.g.*,

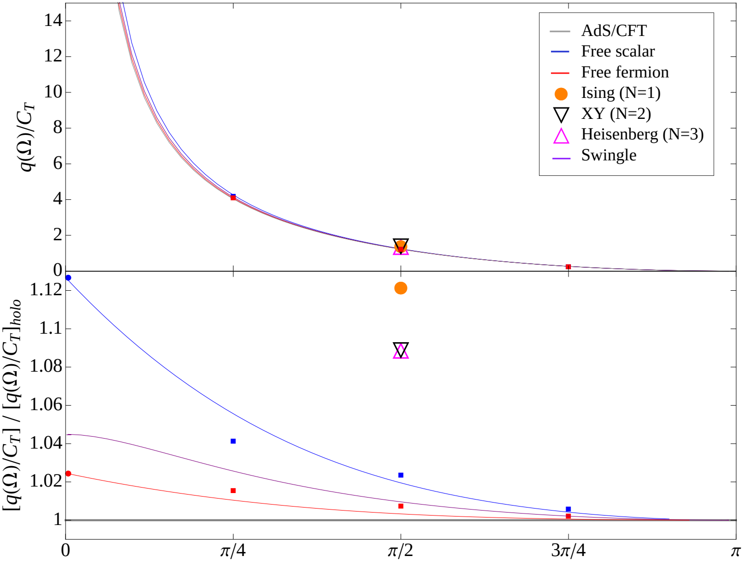
$$S_{EE}^{\text{disk}}=\frac{c_1}{\delta}-F\Rightarrow\frac{\sigma}{F}=\left(1-2\lambda_{\text{GB}}-24\lambda_{1,1}+288\lambda_{2,1}+96\lambda_{0,2}+\mathcal{O}(\lambda^2)\right)\frac{\sigma_E}{F_E}\,,$$

$$s_{\text{th.}}=c_s T^2\Rightarrow\frac{\sigma}{c_s}=\left(1-16\lambda_{0,2}+\mathcal{O}(\lambda^2)\right)\frac{\sigma_E}{c_{s,E}}\,.$$

Comparison with QFT calculations

- Holographic calculations ⇔ strongly coupled 3d CFT's dual to our bulk theories.
- Remarkably, $q(\Omega)$ known for a free scalar and a free Dirac fermion with certain precision. Casini, Huerta
- C_T known exactly since long ago in those cases: Osborn, Petkou $C_T^{\text{fermion}} = 2C_T^{\text{scalar}} = 3/(16\pi^2)$.
- Notably, results for $q(\pi/2)/C_T$ also known for the $N = 1, 2, 3$, $O(N)$ Wilson-Fisher CFT's. Kallin, Stoudenmire, Frendley, Singh, Melko
- Given the different nature of the theories, even a qualitative agreement would look surprising...

Comparison with QFT calculations P.B., Myers, Witczak-Krempa



Comparison with QFT calculations

- Good agreement among all results. $q(\Omega)$ and C_T seem to have the same kind of emergent scaling with the number of d.o.f. In particular, $q^{O(N)}(\pi/2) \approx N\, q^{\text{Ising}}(\pi/2)$ and ${C_T}^{O(N)}(\pi/2) \approx N {C_T}^{\text{Ising}}(\pi/2)$.
- All curves/data points lie above the holographic curve. Does the holographic $q(\Omega)/C_T$ represent some kind of universal bound?
- A striking feature occurs as $\Omega \to \pi$, where $[q(\Omega)/C_T]/[q(\Omega)/C_T]|_{\text{AdS/CFT}}$ seems to tend to 1 both for the free scalar and the free fermion.
- This suggests that $[\sigma/C_T]/[\sigma/C_T]|_{\text{AdS/CFT}} = 1$.

A universal ratio

- σ computed approximately for the free fermion and the free scalar by Casini and Huerta:

$$\sigma_{\text{fermion}} \simeq 0.007813\,, \quad \sigma_{\text{scalar}} \simeq 0.0039063\,,$$

- Using the values for C_T, one finds

$$\frac{\sigma_{\text{fermion}}}{C_T^{\text{fermion}}} \simeq 0.411235\,, \quad \frac{\sigma_{\text{scalar}}}{C_T^{\text{scalar}}} \simeq 0.411235\,.$$

- For all our holographic theories we had found

$$\frac{\sigma}{C_T} = \frac{\sigma_E}{C_{T,E}} = \frac{\pi^2}{24} \simeq 0.411234\,.$$

- All the results agree within $\sim 0.0003\%$!
- Given the extremely different nature of the theories and the calculations, we are led to conjecture that this is a universal quantity for general 3d CFT's.

A universal ratio

- The EE of a region with an almost smooth corner would be fully determined by the two point function of the stress tensor.
- We can use our conjecture to **predict** the exact values of σ_{scalar} and σ_{fermion}

$$\sigma_{\text{scalar}} = \frac{1}{256}\,, \qquad \sigma_{\text{fermion}} = \frac{1}{128}\,.$$

- We are using holography to improve free field theory results!
- But we can do even better...

A universal ratio

Let us revisit the original free field computations... Casini, Huerta.

$$
\begin{aligned}
\sigma_{\text{scalar}} &= -2\pi \int_{1/2}^{+\infty} dm \int_0^{+\infty} db\; \mu\, H\, a(1-a)\, m\, \mathrm{sech}^2(\pi b)\,,\\
\sigma_{\text{fermion}} &= -4\pi \int_{1/2}^{+\infty} dm \int_0^{+\infty} db \left[\mu\, H\, a(1-a) - \frac{F}{4\pi}\right] m\, \mathrm{cosech}^2(\pi b)\,,
\end{aligned}
$$

$$
\begin{aligned}
H &\equiv -\frac{c}{2h} X_1 T - \frac{1}{2c} X_2 T + \frac{1}{16\pi a(a-1)}\,, \quad h \equiv \frac{2\left(a(a-1)+m^2\right)\sin^2(\pi a)}{m^2\left(\cos(2\pi a)+\cos\left(\pi\sqrt{1-4m^2}\right)\right)}\,,\\
c &\equiv \frac{2^{2a-1}\pi a(1-a)\sec\left(\frac{\pi}{2}\left(2a+\sqrt{1-4m^2}\right)\right)\Gamma\left(\frac{3}{2}-a+\frac{1}{2}\sqrt{1-4m^2}\right)}{m\,\Gamma(2-a)^2\,\Gamma\left(a-\frac{1}{2}+\frac{1}{2}\sqrt{1-4m^2}\right)}\,,\\
X_1 &\equiv -\frac{\Gamma(-a)\left[\pi\sinh\left(\frac{\pi\mu}{2}\right)+i\cosh\left(\frac{\pi\mu}{2}\right)\left(\psi^{(0)}\left(a+\frac{i\mu}{2}+\frac{1}{2}\right)-\psi^{(0)}\left(a-\frac{i\mu}{2}+\frac{1}{2}\right)\right)\right]}{2^{2a+1}\mu\,\Gamma(a+1)\,\Gamma\left(-a-\frac{i\mu}{2}+\frac{1}{2}\right)\Gamma\left(-a+\frac{i\mu}{2}+\frac{1}{2}\right)(\cos(2\pi a)+\cosh(\pi\mu))}\,,\\
X_2 &\equiv \text{“}X_1\text{” with } a \text{ replaced by } (1-a), \quad T \equiv \sqrt{h(a^2-a+(h+1)m^2)}\,,\\
F &\equiv -\frac{F_1}{F_2}\,, \quad F_1 \equiv a^2\left(8\pi c^2\left(m^2+1\right)X_1 T + 8\pi h\left(m^2+1\right)X_2 T - ch\right) - 16\pi a^3 T\left(c^2 X_1 + hX_2\right)\\
&+ a\left(-8\pi c^2 m^2 X_1 T - 8\pi h m^2 X_2 T + ch\right) + 8\pi a^4 T\left(c^2 X_1 + hX_2\right) - ch(h+1)m^2\,,\\
F_2 &\equiv \frac{8c\,h\left(a^2-a+m^2\right)^2}{(2a-1)\mu}\,, \quad \mu \equiv \sqrt{4m^2-1}\,, \quad a \equiv \begin{cases} i\,b+\frac{1}{2} & \text{for the scalar}\,,\\ i\,b & \text{for the fermion}\,,\end{cases}
\end{aligned}
$$

A universal ratio

We can improve the accuracy of the numerical results:

$$\sigma_{\text{scalar}} \simeq 0.00390625000000(5)\,,$$
$$\sigma_{\text{fermion}} \simeq 0.00781250000000(7)\,,$$

These agree with our predictions

$$\sigma_{\text{scalar}} = \frac{1}{256} = 0.00390625\,, \qquad \sigma_{\text{fermion}} = \frac{1}{128} = 0.0078125\,,$$

with a precision of 1 part in 10^{12} (the precision at which we computed the integrals).

Recently, the integrals have been evaluated analitically Helvang, Hadjiantonis
The results agree with our expectations.

Comments and summary of the 3d EE results

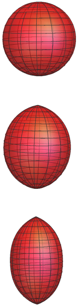
- $q(\Omega)/C_T$ is an *almost universal* ratio for a broad class of 3d CFTs.
- Is the holographic curve $q(\Omega)/C_T$ a universal lower bound (reminiscent of $\eta/s = 1/(4\pi)$)?
- Conjecture: σ/C_T is a universal quantity for general 3d CFTs: very different theories and procedures (holographic vs field theoretical methods), same result.
- This result allowed us to improve the free field theory results for σ.

Generalizations to come soon...

- Higher-dimensions (cones)
- Rényi entropies

- Sharpening spheres: cone coefficients $\sigma^{(d)}$ $\Rightarrow$ for general holographic theories in higher-dimensions (Using Mezei's formula)

$$\sigma^{(d)} = C_T^{(d)} \frac{\pi^{d-1}(d-1)\Gamma[\frac{d-1}{2}]^2}{\Gamma[\frac{d-2}{2}]\Gamma[\frac{d}{2}]\Gamma[d+2]} \begin{cases} (-1)^{\frac{d+2}{2}} & d \text{ even} \\ \pi/4 & d=3 \\ (-1)^{\frac{d+1}{2}}\pi & d=5 \end{cases}$$



- Cones with non-spherical cross-sections?

Corner Rényi entropies P.B., Myers, Witczak-Krempa

- Rényi entropies: $S_n(V) = \frac{1}{1-n} \log \mathrm{Tr}\, \rho_V^n$.
- Analogous corner coefficients σ_n:

$$S_n = B_n \frac{H}{\delta} - q_n(\Omega) \log(H/\delta) + c_n \Rightarrow q_n(\Omega \sim \pi) = \sigma_n (\pi - \Omega)^2 \,.$$

- Twist operators $\tau_n(V)$: line operators (3d) extending over ∂V, $\langle \tau_n \rangle_n = \mathrm{Tr}\, \rho_V^n$. Its scaling dimension h_n, defined by the coefficient of the leading divergence in $\langle T_{\mu\nu} \tau_n \rangle_n$, *e.g.*,

$$\langle T_{ab} \tau_n \rangle_n = -\frac{h_n}{2\pi} \frac{\delta_{ab}}{y^3} \,.$$

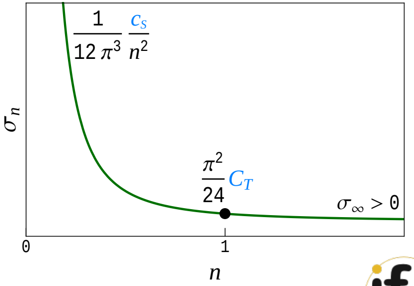
- h_n independent of the geometry of the entangling surface.

Generalized conjecture:

$$\sigma_n = \frac{h_n}{\pi(n-1)}$$

- It reduces to our previous conjecture for $n = 1$.
- It works $\forall$ n for a free scalar and a free fermion!
- Some interesting consequences, *e.g.*,

$$\sigma_{n\to 0} = \frac{1}{12\pi^3}\frac{c_s}{n^2}, \quad s_{\text{thermal}} = c_s T^2$$



The end?

Bonus slide: a heuristic argument

- $\sigma_n \Rightarrow$ universal response of S_n to a small deformation of an originally smooth surface.
- In the case of EE, this response is determined by correlators of the stress tensor and the modular Hamiltonian H. H involves an integral of the stress tensor over $V \Rightarrow$ the calculation involves $\langle TT \rangle \sim C_T$. Natural to expect $\sigma \sim C_T$.
- In the case $n > 1$, the calculation involves in turn $\langle T \tau_n \rangle \sim h_n$. Natural to expect $\sigma_n \sim h_n$.