Indirect controllability of some linear parabolic systems of two equations with one control involving coupling terms of first order.

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September 1st, 2015

Benasque
Partial differential equations, optimal design and numerics

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Setting

Let $T > 0$, Ω be a bounded domain in $\mathbb{R}^N$ ($N \in \mathbb{N}^*$) of class $\mathcal{C}^2$ and ω be an nonempty open subset of Ω. Consider the system

$$\begin{cases} \partial_t y_1 = \operatorname{div}(d_1 \nabla y_1) + g_{11} \cdot \nabla y_1 + g_{12} \cdot \nabla y_2 + a_{11} y_1 + a_{12} y_2 + \mathbb{1}_\omega u & \text{in } Q_T, \\ \partial_t y_2 = \operatorname{div}(d_2 \nabla y_2) + g_{21} \cdot \nabla y_1 + g_{22} \cdot \nabla y_2 + a_{21} y_1 + a_{22} y_2 & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot, 0) = y^0 & \text{in } \Omega, \end{cases} \quad (1)$$

where $y^0 \in L^2(\Omega; \mathbb{R}^2)$, $u \in L^2(Q_T)$, $g_{ij} \in L^\infty(Q_T; \mathbb{R}^N)$, $a_{ij} \in L^\infty(Q_T)$ for all $i, j \in \{1, 2\}$, $Q_T := \Omega \times (0, T)$, $\Sigma_T := (0, T) \times \partial\Omega$ and

$$\begin{cases} d_l^{ij} \in W^1_\infty(Q_T), \\ d_l^{ij} = d_l^{ji} \text{ in } Q_T, \end{cases} \qquad \sum_{i,j=1}^N d_l^{ij} \xi_i \xi_j \geqslant d_0 |\xi|^2 \text{ in } Q_T, \ \forall \xi \in \mathbb{R}^N.$$

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$$\begin{cases} d_l^{ij} \in W^1_\infty(Q_T), \\ d_l^{ij} = d_l^{ji} \text{ in } Q_T, \end{cases} \qquad \sum_{i,j=1}^{N} d_l^{ij} \xi_i \xi_j \geqslant d_0 |\xi|^2 \text{ in } Q_T, \ \forall \xi \in \mathbb{R}^N.$$

Cascade condition

THEOREM (M. González-Burgos, L. De Teresa, 2010)

Let us suppose that

$$g_{21} \equiv 0 \text{ in } q_T$$
$$\text{and}$$
$$(a_{21} > a_0 \text{ in } q_T \quad \text{or} \quad a_{21} < -a_0 \text{ in } q_T),$$

for a constant $a_0 > 0$ *and* $q_T := \omega \times (0, T)$.
Then System (1) is null controllable at time T.

Condition on the dimension

THEOREM (S. Guerrero, 2007)

Let $N = 1$, $d_1,\ d_2,\ a_{11},\ g_{11},\ a_{22},\ g_{22}$ be constant and suppose that

$$g_{21} \cdot \nabla + a_{21} = P_1 \circ \theta \ \text{ in } \ \Omega \times (0, T),$$

where $\theta \in \mathcal{C}^2(\overline{\Omega})$ with $|\theta| > C$ in $\omega_0 \subset \omega$ and

$$P_1 := m_0 \cdot \nabla + m_1,$$

for $m_0, m_1 \in \mathbb{R}$. Moreover, assume that

$$m_0 \neq 0.$$

Then System (1) is null controllable at time T.

Boundary condition

THEOREM (A. Benabdallah, M. Cristofol, P. Gaitan, L. De Teresa, 2014)

Assume that $a_{ij} \in \mathcal{C}^4(\overline{Q}_T)$, $g_{ij} \in \mathcal{C}^3(\overline{Q}_T)^N$, $d_i \in \mathcal{C}^3(\overline{Q}_T)^{N^2}$ for all $i, j \in \{1, 2\}$ and

$$\left\{ \begin{array}{l} \exists \gamma \neq \varnothing \text{ an open subset of } \partial\omega \cap \partial\Omega, \\ \exists x_0 \in \gamma \text{ s.t. } g_{21}(t, x_0) \cdot \nu(x_0) \neq 0 \text{ for all } t \in [0, T], \end{array} \right.$$

where ν is the exterior normal unit vector to $\partial\Omega$.
Then System (1) is null controllable at time T.

Necessary and sufficient condition

THEOREM 1 (M. D., P. Lissy, 2015)

*Let us assume that d_i, g_{ij} and a_{ij} are **constant in space and time** for all $i, j \in \{1, 2\}$.*
Then System (1) is null controllable at time T if and only if

$$g_{21} \neq 0 \text{ or } a_{21} \neq 0.$$

1. The term g_{21} can be different from zero in the control domain
2. Theorem 1 is true for all $N \in \mathbb{N}^*$
3. We have no geometric restriction
4. Concerning the controllability of a system with m equations controlled by $m - 1$ forces, the last condition becomes:

$$\exists i_0 \in \{1, ..., m-1\} \text{ s.t. } g_{mi_0} \neq 0 \text{ or } a_{mi_0} \neq 0.$$

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Additional result

THEOREM 2 (M. D., P. Lissy, 2015)

Let us assume that $\Omega := (0, L)$ *with* $L > 0$.
Then System (1) is null controllable at time T *if for an open subset* $(a,b)\times\mathcal{O} \subseteq q_T$ *one of the following conditions is verified:*
(i) $d_i,\ g_{ij},\ a_{ij} \in \mathcal{C}^1((a,b),\mathcal{C}^2(\mathcal{O}))$ *for* $i,j = 1,2$ *and*

$$\left\{\begin{array}{ll} g_{21} = 0 \text{ and } a_{21} \neq 0 & \text{in } (a,b)\times\mathcal{O}, \\ 1/a_{21} \in L^\infty(\mathcal{O}) & \text{in } (a,b). \end{array}\right.$$

(ii) $d_i,\ g_{ij},\ a_{ij} \in \mathcal{C}^3((a,b),\mathcal{C}^7(\mathcal{O}))$ *for* $i = 1,2$ *and*

$$|\det(H(t,x))| > C \textit{ for every } (t,x) \in (a,b)\times\mathcal{O}, \textit{ where}$$

$$H := \begin{pmatrix} -a_{21} + \partial_x g_{21} & g_{21} & 0 & 0 & 0 & 0 \\ -\partial_x a_{21} + \partial_{xx} g_{21} & -a_{21} + 2\partial_x g_{21} & 0 & g_{21} & 0 & 0 \\ -\partial_t a_{21} + \partial_{tx} g_{21} & \partial_t g_{21} & -a_{21} + \partial_x g_{21} & 0 & g_{21} & 0 \\ -\partial_{xx} a_{21} + \partial_{xxx} g_{21} & -2\partial_x a_{21} + 3\partial_{xx} g_{21} & 0 & -a_{21} + 3\partial_x g_{21} & 0 & g_{21} \\ -a_{22} + \partial_x g_{22} & g_{22} - \partial_x d_2 & -1 & -d_2 & 0 & 0 \\ -\partial_x a_{22} + \partial_{xx} g_{22} & -a_{22} + 2\partial_x g_{22} - \partial_{xx} d_2 & 0 & g_{22} - 2\partial_x d_2 & -1 & -d_2 \end{pmatrix}.$$

Remarks

Even though the last condition seems complicated, it can be simplified in some cases :

1. System (1) is null controllable at time T if there exists an open subset $(a, b) \times \mathcal{O} \subseteq q_T$ such that

$$\begin{cases} g_{21} \equiv \kappa \in \mathbb{R}^* & \text{in } (a,b) \times \mathcal{O}, \\ a_{21} \equiv 0 & \text{in } (a,b) \times \mathcal{O}, \\ \partial_x a_{22} \neq \partial_{xx} g_{22} & \text{in } (a,b) \times \mathcal{O}. \end{cases}$$

2. If the coefficients depend only on the time variable, the condition becomes :

$$\begin{cases} \exists\ (a,b) \subset (0,T) \text{ s.t.:} \\ g_{21}(t)\partial_t a_{21}(t) \neq a_{21}(t)\partial_t g_{21}(t) \quad \text{in } (a,b). \end{cases}$$

A related result

Consider the following system

$$\begin{cases} \partial_t z_1 = \partial_{xx} z_1 + \mathbb{1}_\omega u & \text{in } (0,\pi)\times(0,T), \\ \partial_t z_2 = \partial_{xx} z_2 + p(x)\partial_x z_1 + q(x) z_1 & \text{in } (0,\pi)\times(0,T), \\ z(0,\cdot) = z(\pi,\cdot) = 0 & \text{on } (0,T), \\ z(\cdot,0) = z^0 & \text{in } (0,\pi), \end{cases}$$

where $z^0 \in L^2((0,\pi);\mathbb{R}^2)$, $u \in L^2((0,\pi)\times(0,T))$, $p \in W^1_\infty(0,\pi)$, $q \in L^\infty(0,\pi)$ and $\omega := (a,b) \subset (0,\pi)$.

Theorem (M. D., 2015)

Suppose that $p \in W^1_\infty(0,\pi) \cap W^2_\infty(\omega)$, $q \in L^\infty(0,\pi) \cap W^1_\infty(\omega)$ and

$$(\mathrm{Supp}(p) \cup \mathrm{Supp}(q)) \cap \omega \neq \varnothing.$$

Then the above system is null controllable at time T.

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- Example 2 : An application to the controllability of parabolic systems

3 Proof of Theorem 1

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The notion of the algebraic resolvability can be found in:

[1] Gromov, M. Partial differential relations Springer-Verlag, 1986, 9.

And was used for the first time the control theory in:

[2] Coron, J.-M. & Lissy, P. Local null controllability of the three-dimensional Navier-Stokes system with a distributed control having two vanishing components Invent. Math., 2014, 198, 833-880.

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Let $f \in \mathcal{C}_c^\infty([0,1])$. Consider the problem

$$\left\{\begin{array}{l}\text{Find } (w_1, w_2) \in \mathcal{C}_c^\infty([0,1];\mathbb{R}^2) \text{ s. t. :}\\ a_1 w_1 - a_2\partial_x w_1 + a_3\partial_{xx} w_1 + b_1 w_2 - b_2\partial_x w_2 + b_3\partial_{xx} w_2 = f,\end{array}\right. \tag{2}$$

with $a_1,\ a_2,\ a_3,\ b_1,\ b_2,\ b_3 \in \mathbb{R}$.

Problem (2) can be rewritten as follows :

$$\mathcal{L}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = f,$$

where the operator $\mathcal{L}$ is given by

$$\mathcal{L} := (a_1 - a_2\partial_x + a_3\partial_{xx}, b_1 - b_2\partial_x + b_3\partial_{xx}).$$

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with $a_1,\ a_2,\ a_3,\ b_1,\ b_2,\ b_3 \in \mathbb{R}$.
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where the operator $\mathcal{L}$ is given by

$$\mathcal{L} := (a_1 - a_2\partial_x + a_3\partial_{xx}, b_1 - b_2\partial_x + b_3\partial_{xx}).$$

If there exists a differential operator $\mathcal{M} := \sum_{i=0}^{M} m_i \partial_x^i$ with $m_0, ..., m_M \in \mathbb{R}$, $M \in \mathbb{N}$ such that

$$\mathcal{L} \circ \mathcal{M} = Id,$$

then $(w_1, w_2) := \mathcal{M}f$ is a solution to problem (2). The last equality is equivalent to

$$\mathcal{M}^* \circ \mathcal{L}^* = Id,$$

with

$$\mathcal{L}^*(\varphi) = \begin{pmatrix} a_1 + a_2\partial_x + a_3\partial_{xx} \\ b_1 + b_2\partial_x + b_3\partial_{xx} \end{pmatrix}.$$

Consider the operator defined for all $\varphi \in \mathcal{C}_c^\infty([0,1])$ by

$$\begin{pmatrix} \mathcal{L}_1^* \\ \mathcal{L}_2^* \\ \partial_x \mathcal{L}_1^* \\ \partial_x \mathcal{L}_2^* \end{pmatrix} \varphi := \begin{pmatrix} a_1 + a_2\partial_x + a_3\partial_{xx} \\ b_1 + b_2\partial_x + b_3\partial_{xx} \\ a_1\partial_x + a_2\partial_{xx} + a_3\partial_{xxx} \\ b_1\partial_x + b_2\partial_{xx} + b_3\partial_{xxx} \end{pmatrix} \varphi = C \begin{pmatrix} \varphi \\ \partial_x\varphi \\ \partial_{xx}\varphi \\ \partial_{xxx}\varphi \end{pmatrix},$$

where

$$C := \begin{pmatrix} a_1 & a_2 & a_3 & 0 \\ b_1 & b_2 & b_3 & 0 \\ 0 & a_1 & a_2 & a_3 \\ 0 & b_1 & b_2 & b_3 \end{pmatrix}.$$

Let us suppose that C is invertible, denote by

$$C^{-1} := \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{pmatrix}.$$

Then we have

$$C^{-1} \begin{pmatrix} \mathcal{L}_1^* \\ \mathcal{L}_2^* \\ \partial_x \mathcal{L}_1^* \\ \partial_x \mathcal{L}_2^* \end{pmatrix} \varphi = \begin{pmatrix} \varphi \\ \partial_x \varphi \\ \partial_{xx} \varphi \\ \partial_{xxx} \varphi \end{pmatrix},$$

where the first line is given by

$$c_{11}\mathcal{L}_1^*\varphi + c_{12}\mathcal{L}_2^*\varphi + c_{13}\partial_x\mathcal{L}_1^*\varphi + c_{14}\partial_x\mathcal{L}_2^*\varphi = \varphi.$$

Thus the problem is solved if we define $\mathcal{M}^*$ for all $(\psi_1, \psi_2) \in \mathcal{C}_c^\infty([0,1];\mathbb{R}^2)$ by

$$\mathcal{M}^* \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} := c_{11}\psi_1 + c_{12}\psi_2 + c_{13}\partial_x\psi_1 + c_{14}\partial_x\psi_2.$$

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Kalman condition

Consider the system of n linear parabolic equations controlled by m controls

$$\begin{cases} \partial_t y = \Delta y + Ay + B\mathbb{1}_\omega u & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot, 0) = y^0 & \text{in } \Omega, \end{cases} \tag{3}$$

where $y^0 \in L^2(\Omega; \mathbb{R}^n)$, $u \in L^2(Q_T; \mathbb{R}^m)$, $A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$.

THEOREM (Ammar Khodja F., Benabdallah A., Dupaix C., González-Burgos M., 2009)

System (3) is null controllable on the time interval $(0, T)$, i.e. for all initial condition $y^0 \in L^2(\Omega; \mathbb{R}^n)$ there exists a control $u \in L^2(Q_T; \mathbb{R}^m)$ such that the solution y to system (3) is equal to zero at time T, if and only if

$$\mathrm{Rank}[A|B] = n,$$

where $[A|B] := (B|AB|...|A^{n-1}B)$.

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$$\mathrm{Rank}[A|B] = n,$$

where $[A|B] := (B|AB|...|A^{n-1}B)$.

Let us prove the positive result using the algebraic resolvability :

Analytic Problem:

Find (z, v) with $\mathrm{Supp}(v) \subset \omega \times (\varepsilon, T - \varepsilon)$ such that

$$\begin{cases} \partial_t z = \Delta z + Az + \mathbb{1}_\omega v & \text{in } Q_T, \\ z = 0 & \text{on } \Sigma_T, \\ z(\cdot, 0) = y^0, \ z(\cdot, T) = 0 & \text{in } \Omega. \end{cases}$$

Algebraic Problem:

For $f := \mathbb{1}_\omega v$, find $(\widehat{z}, \widehat{v})$ such that

$$\mathrm{Supp}(\widehat{z}, \widehat{v}) \subset \omega \times (\varepsilon, T - \varepsilon)$$

and

$$\partial_t \widehat{z} = \Delta \widehat{z} + A\widehat{z} + B\widehat{v} + f \ \text{ in } \ Q_T.$$

Conclusion:

The couple $(y, u) := (z - \widehat{z}, -\widehat{v})$ is solution to system (3) in $W(0, T)^n \times L^2(Q_T; \mathbb{R}^m)$ satisfying $y(T) \equiv 0$ in Ω.

Resolution of the analytic problem

THEOREM

For all initial condition $y^0 \in L^2(\Omega;\mathbb{R}^n)$, there exists a control $v \in L^2(Q_T;\mathbb{R}^n)$ such that the solution to

$$\begin{cases} \partial_t z = \Delta z + Az + \mathbb{1}_\omega v & \text{in } Q_T, \\ z = 0 & \text{on } \Sigma_T, \\ z(\cdot,0) = y^0,\ z(\cdot,T) = 0 & \text{in } \Omega, \end{cases}$$

is null at time T. Moreover, we have Supp$(v) \subset \omega \times (\varepsilon, T-\varepsilon)$ and

$$\|v\|_{W_2^{2,1}(Q_T;\mathbb{R}^n)} \leqslant e^{C(1+T+1/T)} \|y^0\|_{L^2(\Omega;\mathbb{R}^n)}.$$

This theorem can be proved using:

1. Method by fictitious control
2. Carleman inequalities

Resolution of the algebraic problem

For $f := \mathbb{1}_\omega v$, let us find $(\widehat{z}, \widehat{v})$ with $\mathrm{Supp}(\widehat{z}, \widehat{v}) \subset \omega \times (\varepsilon, T - \varepsilon)$ s.t.:

$$\mathcal{L}(\widehat{z}, \widehat{v}) = f \quad \text{in} \quad Q_T,$$

where

$$\mathcal{L}(\widehat{z}, \widehat{v}) := \ \partial_t \widehat{z} - \Delta \widehat{z} - A\widehat{z} - B\widehat{v}.$$

It suffice to find a differential operator $\mathcal{M}$ defined on $\mathcal{C}_c^\infty(Q_T, \mathbb{R}^n)$ s.t.

$$\mathcal{L} \circ \mathcal{M} = Id.$$

The last equality is equivalent to

$$\mathcal{M}^* \circ \mathcal{L}^* = Id,$$

where $\mathcal{L}^*$ is given for all $\varphi \in \mathcal{C}^\infty(Q_T, \mathbb{R}^n)$ by

$$\mathcal{L}^* \varphi = \begin{pmatrix} -\partial_t \varphi - \Delta \varphi - A^* \varphi \\ -B^* \varphi \end{pmatrix}.$$

Let $\mathcal{S} := (\mathcal{S}_1, ..., \mathcal{S}_n)$ given for all $(x_1, x_2) \in \mathcal{C}^\infty(\Omega; \mathbb{R}^{n+m})$ by

$$\begin{cases} \mathcal{S}_1(x_1, x_2) := -x_2, \\ \mathcal{S}_2(x_1, x_2) := (\partial_t + \Delta)x_2 - B^* x_1, \\ \mathcal{S}_k(x_1, x_2) := (\partial_t + \Delta)\mathcal{S}_{k-1}(x_1, x_2) - B^*(A^*)^{k-2} x_1, \ \ \forall\, k \in \{3, ..., n\}. \end{cases}$$

Then, we obtain

$$\mathcal{S} \circ \mathcal{L}^* \varphi = \begin{pmatrix} B^* \varphi \\ \vdots \\ B^*(A^*)^{n-1} \varphi \end{pmatrix}.$$

Since the rank of $K := (B|AB|...|A^{n-1}B)$ is equal to n, there exists $L \in \mathcal{M}_{n,nm}(\mathbb{R})$ such that $LK^* = I_n$. The operator

$$\mathcal{M} := \mathcal{S}^* L^*$$

is of order 2 in space and 1 in time and is a solution of our problem.

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4 Comments

Let us recall system (1) and Theorem 1. Consider the system of two parabolic equations controlled by one force

$$\begin{cases} \partial_t y = \mathrm{div}(D\nabla y) + G\cdot\nabla y + Ay + e_1 \mathbb{1}_\omega u & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot,0) = y^0 & \text{in } \Omega, \end{cases} \quad (1)$$

where $y^0 \in L^2(\Omega;\mathbb{R}^2)$, $D := (d_1, d_2) \in \mathcal{L}(\mathbb{R}^{2N},\mathbb{R}^{2N})$
$G := (g_{ij}) \in \mathcal{L}(\mathbb{R}^{2N},\mathbb{R}^2)$, $A := (a_{ij}) \in \mathcal{L}(\mathbb{R}^2)$.

THEOREM 1 (M. D., P. Lissy, 2015)

*Let us assume that d_i, $g_{i,j}$ and $a_{i,j}$ are **constant in space and time** for all $i,j \in \{1,2\}$.*
Then System (1) is null controllable at time T if and only if

$$g_{2,1} \neq 0 \ \text{ or } \ a_{2,1} \neq 0.$$

Strategy

Analytic problem:
Find (z, v) with $\mathrm{Supp}(v) \subset \omega \times (\varepsilon, T-\varepsilon)$ s.t. :

$$\begin{cases} \partial_t z = \mathrm{div}(D\nabla z) + G\cdot\nabla z + Az + \mathcal{N}(\mathbb{1}_\omega v) & \text{in } Q_T, \\ z = 0 & \text{on } \Sigma_T, \\ z(0,\cdot) = y^0,\ z(T,\cdot) = 0 & \text{in } \Omega, \end{cases}$$

where the operator $\mathcal{N}$ is well chosen.
Difficulties: • the control is in the range of the operator $\mathcal{N}$,
• v has to be regular enough.

Algebraic problem:
Find $(\widehat{z}, \widehat{v})$ with $\mathrm{Supp}(\widehat{z}, \widehat{v}) \subset \omega \times (\varepsilon, T-\varepsilon)$ s.t. :

$$\partial_t \widehat{z} = \mathrm{div}(D\nabla \widehat{z}) + G\cdot\nabla \widehat{z} + A\widehat{z} + B\widehat{v} + \mathcal{N}f \text{ in } Q_T,$$

Conclusion:
The couple $(y, u) := (z - \widehat{z}, -\widehat{v})$ will be a solution to system (1) and will satisfy $y(T) \equiv 0$ in Ω

Resolution of the algebraic problem

The algebraic problem can be rewritten as :
For $f := \mathbb{1}_\omega v$, find $(\widehat{z}, \widehat{v})$ with Supp$(\widehat{z}, \widehat{v}) \subset \omega \times (\varepsilon, T - \varepsilon)$ such that

$$\mathcal{L}(\widehat{z}, \widehat{v}) = \mathcal{N} f,$$

where

$$\mathcal{L}(\widehat{z}, \widehat{v}) := \quad \partial_t \widehat{z} - \mathrm{div}(D \nabla \widehat{z}) - G \cdot \nabla \widehat{z} - A\widehat{z} - B\widehat{v}.$$

This problem is solved if we can find a differential operator $\mathcal{M}$ such that

$$\mathcal{L} \circ \mathcal{M} = \mathcal{N}.$$

The last equality is equivalent to

$$\mathcal{M}^* \circ \mathcal{L}^* = \mathcal{N}^*,$$

where $\mathcal{L}^*$ is given for all $\varphi \in \mathcal{C}_c^\infty(Q_T)^2$ by

$$\mathcal{L}^*\varphi := \begin{pmatrix} \mathcal{L}_1^*\varphi \\ \mathcal{L}_2^*\varphi \\ \mathcal{L}_3^*\varphi \end{pmatrix} = \begin{pmatrix} -\partial_t\varphi_1 - \mathrm{div}(d_1\nabla\varphi_1) + \sum_{j=1}^2\{g_{j1}\cdot\nabla\varphi_j - a_{j1}\varphi_j\} \\ -\partial_t\varphi_2 - \mathrm{div}(d_1\nabla\varphi_2) + \sum_{j=1}^2\{g_{j2}\cdot\nabla\varphi_j - a_{j2}\varphi_j\} \\ \varphi_1 \end{pmatrix}.$$

We remark that

$$\begin{pmatrix} (g_{21}\cdot\nabla - a_{21})\mathcal{L}_1^*\varphi \\ \mathcal{L}_1^*\varphi + (\partial_t + \mathrm{div}(d_1\nabla) - g_{11}\cdot\nabla + a_{11})\mathcal{L}_3^*\varphi \end{pmatrix} = \mathcal{N}^*\varphi,$$

where

$$\mathcal{N}^* := g_{21}\cdot\nabla + a_{21}.$$

Thus the algebraic problem is solved by taking

$$\mathcal{M}^*\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} := \begin{pmatrix} (g_{21}\cdot\nabla - a_{21})\psi_3 \\ \psi_1 + (\partial_t + \mathrm{div}(d_1\nabla) - g_{11}\cdot\nabla + a_{11})\psi_3 \end{pmatrix}.$$

Resolution of the analytic problem: sketch of the proof

The system

$$\begin{cases} \partial_t z = \operatorname{div}(D\nabla z) + G\cdot\nabla z + Az + \mathcal{N}(\mathbb{1}_\omega v) & \text{in } Q_T, \\ z = 0 & \text{on } \Sigma_T, \\ z(0,\cdot) = y^0,\ z(T,\cdot) = 0 & \text{in } \Omega, \end{cases}$$

is null controllable if for all $\psi^0 \in L^2(\Omega;\mathbb{R}^2)$ the solution of the adjoint system

$$\begin{cases} -\partial_t \psi = \operatorname{div}(D^*\nabla \psi) - G^*\cdot\nabla \psi + A^*\psi & \text{in } Q_T, \\ \psi = 0 & \text{on } \Sigma_T, \\ \psi(T,\cdot) = \psi^0 & \text{in } \Omega. \end{cases}$$

satisfies the following inequality of observability

$$\int_\Omega |\psi(0,x)|^2 dx \leqslant C_{obs} \int_0^T \int_\omega \rho_0 |\mathcal{N}^*\psi(t,x)|^2\, dxdt,$$

where ρ_0 can be chosen to be exponentially decreasing at times $t=0$ and $t=T$.

This inequality is obtain by applying the differential operator

$$\nabla\nabla\mathcal{N}^* = \nabla\nabla(-a_{21} + g_{21}\cdot\nabla)$$

to adjoint system. More precisely we study the solution $\phi_{ij} := \partial_i\partial_j\mathcal{N}^*\psi$ of the system

$$\begin{cases} -\partial_t\phi_{ij} = D\Delta\phi_{ij} - G^*\cdot\nabla\phi_{ij} + A^*\phi_{ij} & \text{in } Q_T, \\ \frac{\partial\phi}{\partial n} = \frac{\partial(\nabla\nabla\mathcal{N}^*\psi)}{\partial n} & \text{on } \Sigma_T, \\ \phi(T,\cdot) = \nabla\nabla\mathcal{N}^*\psi^0 & \text{in } \Omega. \end{cases}$$

Remark : We need that $\nabla\nabla\mathcal{N}^*$ commutes with the other operators of the system, what is possible with some constant coefficients.

Following the ideas of Barbu developed in [1], the control is chosen as the solution of the problem

$$\left\{ \begin{array}{l} \text{minimize } J_k(v) := \dfrac{1}{2}\displaystyle\int_{Q_T} \rho_0^{-1}|v|^2 dxdt + \dfrac{k}{2}\displaystyle\int_{\Omega} |z(T)|^2 dx, \\ v \in L^2(Q_T, \rho^{-1/2})^2. \end{array} \right. \tag{4}$$

We obtain the regularity of the control with the help of the below Carleman inequality

$$\sum_{k=1}^{3} \iint_{Q_T} \rho_k |\nabla^k \psi|^2 dxdt \leqslant C_T \int_0^T \int_{\omega} \rho_0 |\mathcal{N}^* \psi(t,x)|^2 \, dxdt,$$

where ρ_k are some appropriate weight functions.

[1] Barbu, V. Exact controllability of the superlinear heat equation Appl. Math. Optim., 2000, 42, 73-89

1 Introduction

2 Presentation of the used method : Algebraic resolvability
- Example 1: Inversion of a differential operator
- Example 2 : An application to the controllability of parabolic systems

3 Proof of Theorem 1

4 Comments

Comments

1. We remark that each condition implies in particular that

$$|g_{21}| > C \text{ in } (a,b)\times\mathcal{O} \ \text{ or } \ |a_{21}| > C \text{ in } (a,b)\times\mathcal{O}. \tag{5}$$

Our conjecture is that, Condition (5) is sufficient as soon as we restrict to the class of coupling terms that intersect the control region (Work in progress with P. Lissy).

2. Consider the system of n equations controlled by m forces

$$\begin{cases} \partial_t y = \Delta y + G\cdot\nabla y + Ay + B\mathbb{1}_\omega u & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot,0) = y^0 & \text{in } \Omega, \end{cases}$$

where $y^0 \in L^2(\Omega;\mathbb{R}^n)$, $G \in L^\infty(Q_T;\mathcal{L}(\mathbb{R}^{nN},\mathbb{R}^n))$, $A \in L^\infty(Q_T;\mathcal{L}(\mathbb{R}^n))$, $B \in L^\infty(Q_T;\mathcal{L}(\mathbb{R}^m,\mathbb{R}^n))$ and $u \in L^2(Q_T;\mathbb{R}^m)$. Which kind of general condition can we hope ?

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Thank you for your attention !